Statistical Mechanics
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Lecture - 64
Discontinuity in the Specific Heat of a Bose Gas - Part 02

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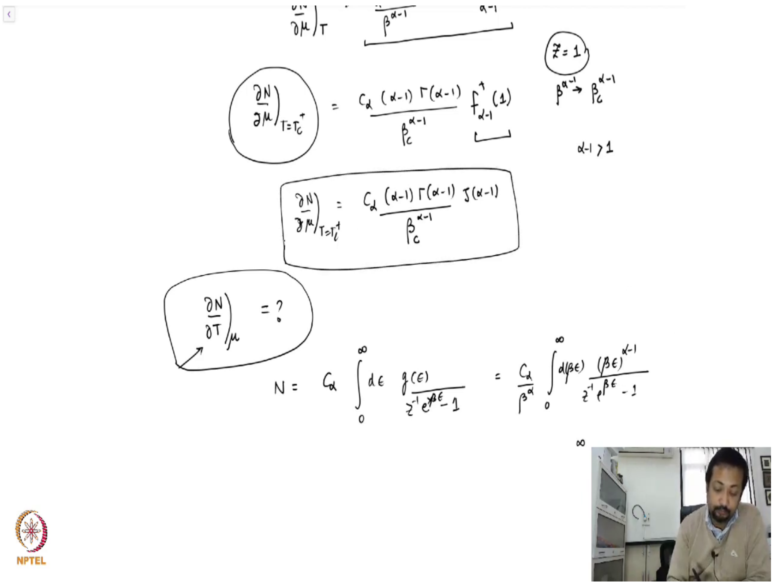

So, we start off with the expression for N, which was C alpha integral d epsa g epsa over z inverse e to the power x minus sorry, e to the power beta epsa minus 1. And we know how to convert this, because this g epsa goes as epsa to the power alpha minus 1.

So, d of beta epsilon times beta epsilon raised to the power alpha minus 1 divided by z inverse e to the power beta epsilon minus 1 divided by beta to the power alpha. Now, this is a repeat, this is a repetitive statement, but it is just to ensure that if you have forgotten you can start off from the scratch.

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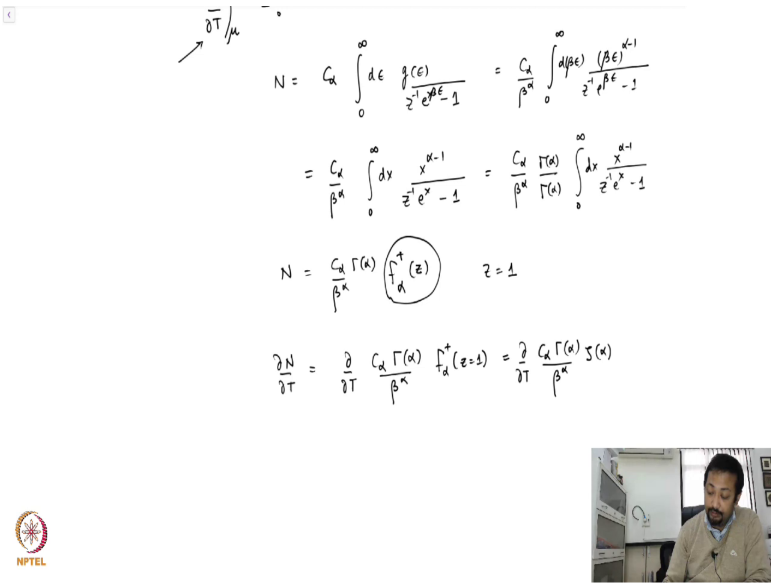

So, C alpha beta alpha and then you see that, this integral is now 0 to infinity, 0 to infinity, 0 to infinity dx x to the power alpha minus 1 z inverse e to the power x minus 1. So, that this gives me C alpha over beta alpha and I need gamma alpha divided by gamma alpha, because I need alpha minus 1 factorial to relate it to the Bose integrals that we have defined, dx x to the power alpha minus 1 z inverse e to the power x minus 1. And this is C alpha for beta alpha times gamma alpha F plus alpha of z.

Now, I want to take a derivative with respect to temperature, chemical potential being held constant, but this also has to be evaluated very close to T C. So, to make my life simple, I mean 1 can simply say that look, I can replace T C everywhere, but then my derivative is going to be identically 0. So, what we will do is we will replace this, by saying that I have z equal to 1.

So, that del N del T is going to be say, del del T of C alpha gamma alpha raised to the power beta alpha F alpha plus 1 evaluated at z equal to 1 extremely close to T C and this becomes del del T of C alpha gamma alpha raised to the power beta sorry, divided by beta raised to the power alpha and then I have zeta of alpha, the Riemann zeta value.

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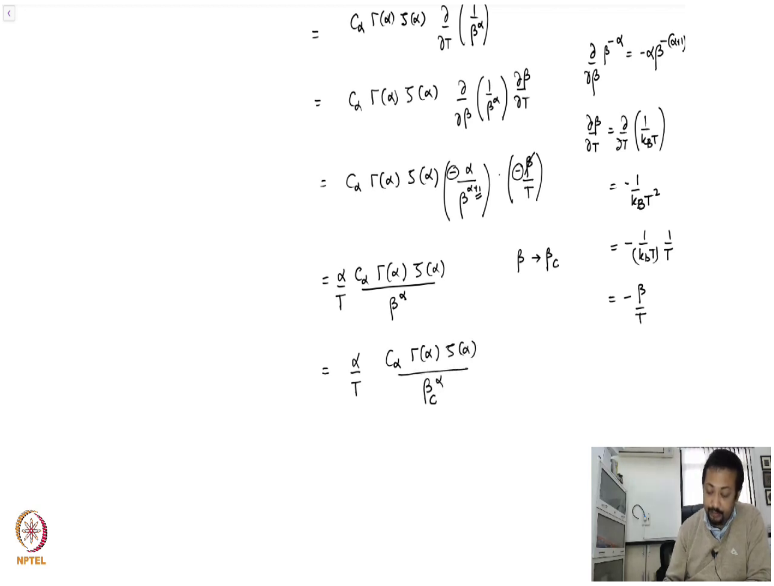

This quantity then becomes C alpha gamma alpha times zeta of alpha and then, I have derivative of del del T of 1 over beta raised to the power alpha, which is C alpha gamma alpha zeta of alpha del del beta of 1 over beta to the power alpha times del beta del T. This is zeta alpha.

Now, del beta del alpha is sorry, del del beta of beta to the power minus alpha is minus alpha beta to the power minus alpha plus 1. And del beta del T is going to be del del T of 1 over K B T, which is going to be minus 1 over K B T square and which I can write, as minus 1 over K B T times 1 over t. So, that this becomes minus beta over T.

So, I have minus alpha beta raised to the power alpha plus 1 times minus beta divided by T and you see, that 1 factor of beta cancels with this 1 factor sitting over here and the two minus 2 negative signs gives me a plus.

So, this becomes C alpha gamma alpha zeta alpha alpha by T and here, in the place of beta raised to the power alpha, since I am extremely close to T C, I am now, going to substitute this as beta C. So, beta now, goes to beta C only after you have taken the derivative not before that. So, that alpha T C alpha, gamma alpha, zeta alpha divided by beta C alpha.

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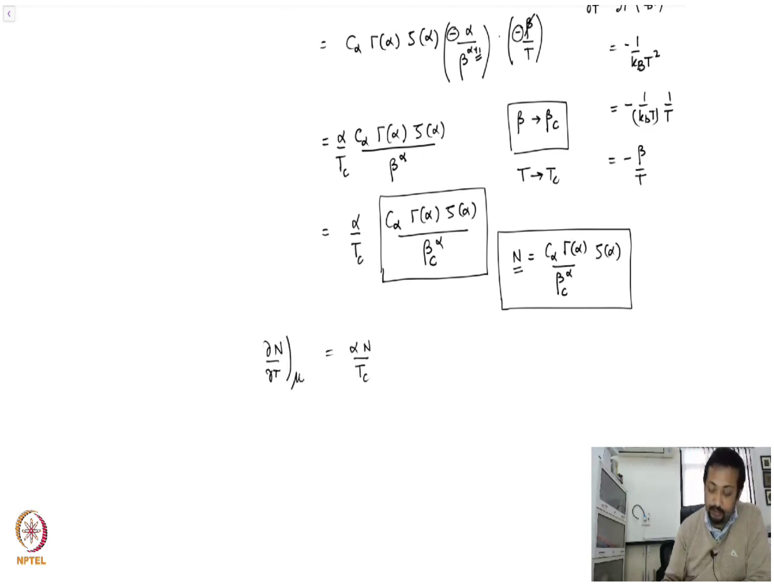

Recall that this C alpha, gamma alpha divided by beta C raised to the power alpha times zeta alpha; the condition for the condensation to happen at beta C is given by this. So, you realize that this quantity that you see over here is nothing but N.

So, I have alpha N divided by T C. So, here also I have T C, because I have replaced beta by beta C, which effectively also means that I have replaced T by T C. So, del N del T mu being held constant is given by this. So, now, we have expression for all the partial derivatives we need we have this value, we have this value.

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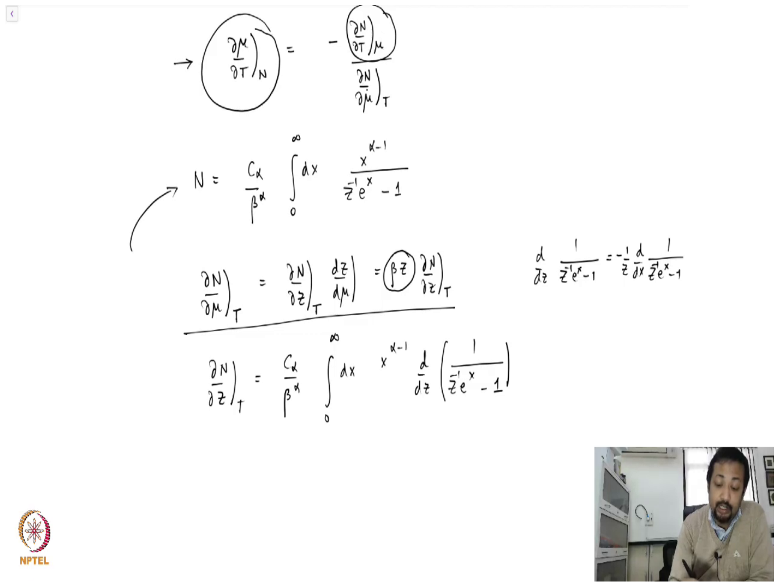

And the quantity that we are looking for, is essentially del mu del T times N, which is del N del T mu held constant, del N del mu T held constant with a minus sign in front of it.

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$$\left.\frac{\partial \mu}{\partial T}\right)_N = -\frac{\left.\frac{\partial N}{\partial T}\right)_\mu}{\left.\frac{\partial N}{\partial \mu}\right)_T} = -\frac{\alpha N}{T_c}\left[\frac{C_\alpha(\alpha-1)\Gamma(\alpha-1)\zeta(\alpha-1)}{\beta_c^{\alpha-1}}\right]^{-1}$$

$$= -\frac{\alpha N}{T_c}\frac{\beta_c^{\alpha-1}}{C_\alpha(\alpha-1)\Gamma(\alpha-1)\zeta(\alpha-1)}$$

$$= -\frac{\alpha N k_B}{k_B T_c}\frac{\beta_c^{\alpha-1}}{C_\alpha(\alpha-1)\Gamma(\alpha-1)\zeta(\alpha-1)}$$

So, that I have del mu del T N held constant is minus del N del T mu held constant divided by del N del mu T held constant. And this is minus alpha N over T C and then we go back and look up the expression for del N del mu evaluated at T is equal to T C plus as this.

So, I have; so, this and then I have the expression C alpha, alpha minus 1, gamma alpha minus 1, zeta of alpha minus 1 divided by beta C raised to the power alpha minus 1 whole minus 1. So, that this is minus alpha N over T C and now, I want to see so, I have C alpha alpha minus 1 times gamma of alpha minus 1 times zeta of alpha minus 1 and then, I have beta C raised to the power alpha minus 1.

So, let us bring in a K B factor in the numerator and write down this as K B T C beta C raised to the power alpha minus 1 C alpha alpha minus 1 gamma of alpha minus 1 zeta of alpha minus 1.

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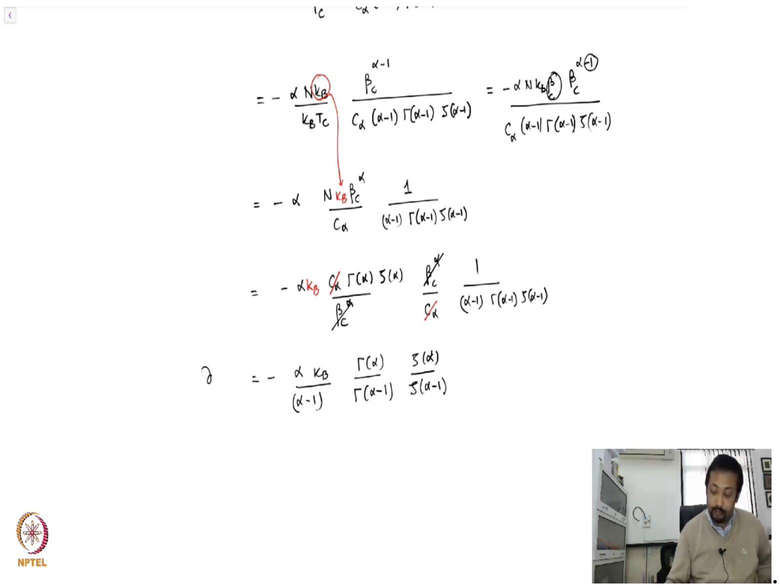

So, that this becomes minus alpha N K B times beta C times beta C raised to the power minus alpha minus 1, then I have C alpha alpha minus 1 gamma alpha minus 1 zeta of alpha minus 1. Looks complicated, but it we are going to simplify this now, minus alpha this factor of beta C cancels with the minus 1 factor.

So, I have N times beta C raised to the power alpha divided by C alpha and then, I have alpha minus 1 gamma of alpha minus 1 zeta of alpha minus 1. One over this, but go back and I know that N is C alpha gamma alpha beta C raise to the power alpha times zeta alpha. So, let us substitute. So, minus alpha C alpha gamma alpha divided times zeta of alpha beta C raised to the power alpha is N.

And then I have beta C raised to the power alpha C alpha 1 over alpha minus 1, gamma alpha minus 1, zeta of alpha minus one. So, things cancel out very nicely, C alpha C alpha cancels out, except that I have missed out the factor K B, which should have been sitting over here. So, I have alpha K B beta C beta C cancels out. So, that I have minus alpha K B over alpha minus 1 and then I have gamma of alpha divided by gamma alpha minus 1, I have zeta of alpha divided by zeta of alpha minus 1.

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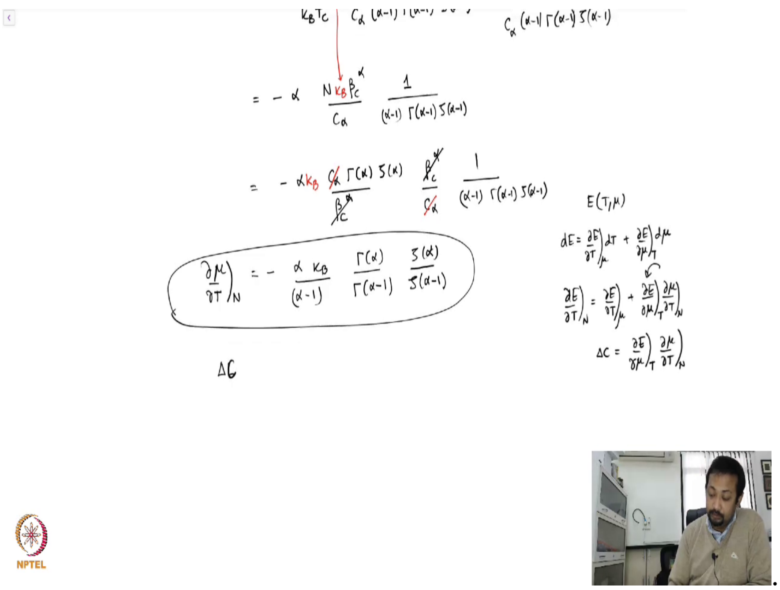

So, this is going to be my del mu del T N held fixed. Now, recall the difference in specific heat, we wanted to evaluate and here we said that this is E; we started off with E as a function of T comma mu. So, that dE was del E del T mu held constant times dT plus del E del mu T held constant times d mu and then we said look, I am interested in evaluating this derivative del E del T N held constant.

So, that that is going to be del E del T mu held constant plus del E del mu T held constant, del mu del T N held constant and this is the part, which I have evaluated. Now, I want to figure out well we already have figured out del E del T, I del E del mu T held constant and the difference in the specific heat is just this term is going to be del e del mu T held constant del mu del T N held constant.

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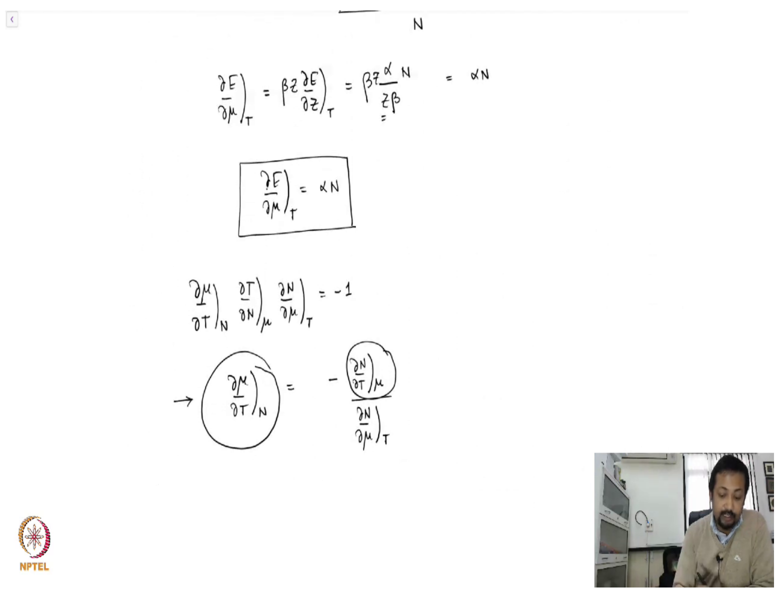

So, we were looking for the term now, del mu del T and we have del E del mu is equal to alpha N.

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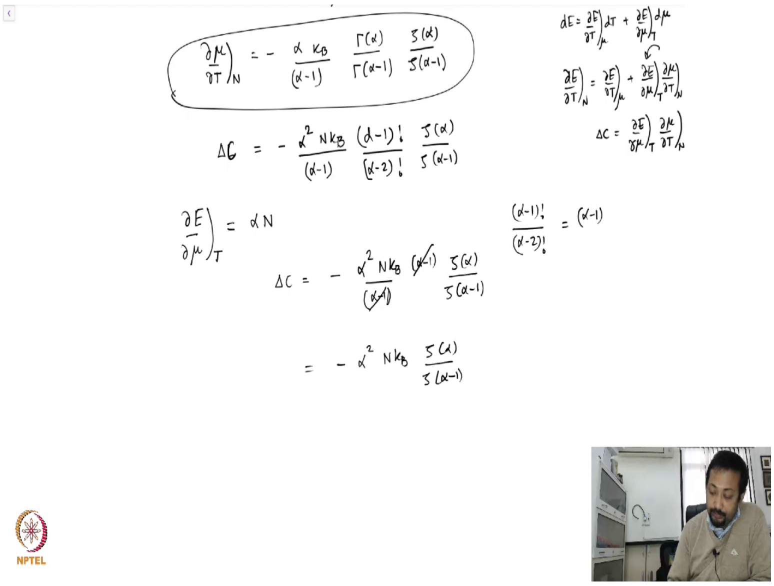

So, I have therefore, delta C as minus alpha square N K B divided by alpha minus 1. Gamma alpha is alpha minus 1 factorial and gamma alpha minus 1 is alpha minus 2 factorial, zeta alpha divided by zeta of alpha minus 1. Here we have used the result, that del e del mu T held constant plus alpha n.

So, that delta C becomes minus alpha square N K B; now, you immediately see that if you take the ratio alpha minus 1 factorial divided by alpha minus 2 factorial and this ratio is nothing but alpha minus 1. So, that this alpha minus 1 is going to cancel with the denominator and then I have zeta alpha divided by zeta of alpha minus 1. So, things looks very nice and I have minus alpha square N K B times zeta alpha divided by zeta alpha minus 1.

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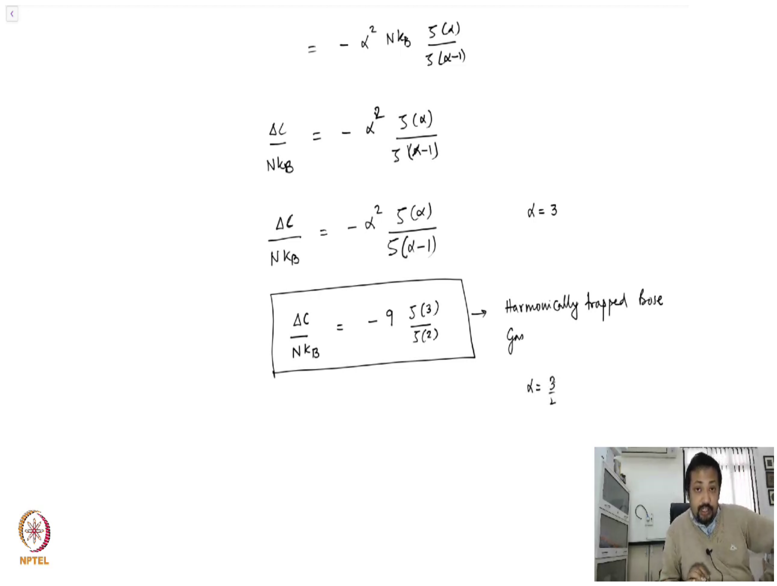

So, that the difference the discontinuity in specific heat is delta C, we write down in a non dimensional way is going to be minus alpha square zeta alpha divided by zeta of alpha minus 1. So, once I have this, looks is minus alpha square zeta of alpha zeta of; so, there is a something problem with this pen alpha minus 1.

Let us validate alpha is equal to 3 and then you realize, that delta C over N K B is going to be minus 9 zeta 3 over zeta 2. This is exactly the expression that we got, when we looked at a harmonically trapped Bose gas and this was the difference in the specific heat. For alpha is equal to 3 half there is no discontinuity.

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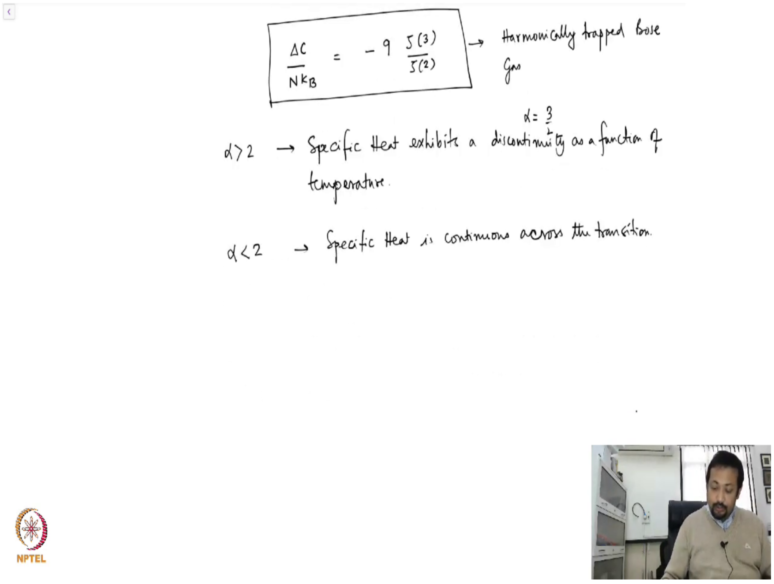

So, again let us reiterate that, that for alpha greater than 2, specific heat exhibits a discontinuity as a function of temperature, for alpha less than 2 specific heat is continuous across the transition.

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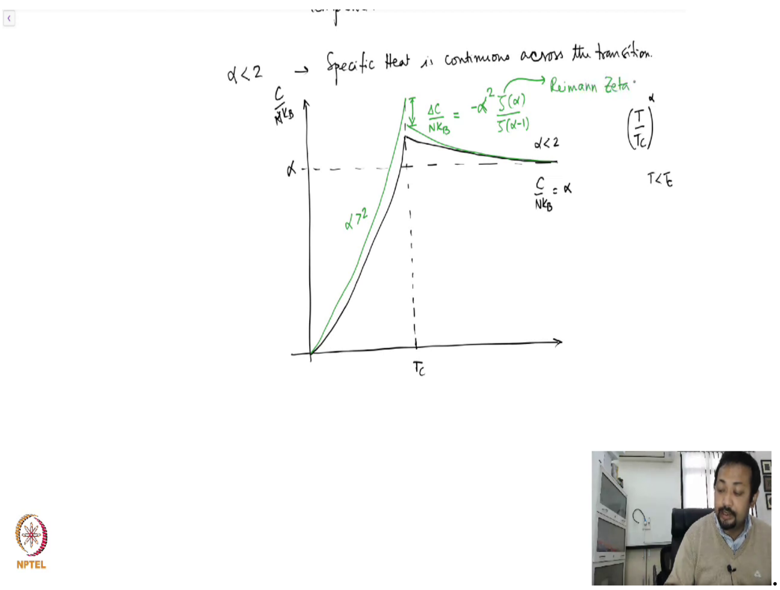

So, if you want to plot, C over N K B and this is your transition temperature. So, let us draw a dotted line, that essentially marks this and let us draw the classical reiterates out like this, which is alpha, then we have the result that it must go as T over T C raised to the power alpha; for T less than T C that is what we saw.

So, that it goes all the way up to here. For alpha less than 2, we also saw alpha less than 2 it is continuous not only that the specific heat at high temperatures, approaches the line the classical result C over N K B is equal to alpha from top not from below. So, therefore, it is continuous and therefore, we can always write down like this way. In contrast; so, let us just to say that this is going to be alpha less than 2 and then I can do I can take the case, when alpha is it is a very poor schematics now.

So, lets it goes all the way up to here, for alpha greater than 2 again it goes behaves like this way, but there is a discontinuity and the discontinuity will somehow come like this way. Across the transition the function, the specific heat as a function of temperature exhibits a discontinuity and this is the amount of discontinuity that you get, which is delta C over N K B and this expression is minus alpha alpha square zeta of alpha divided by zeta of alpha minus 1, where this is your Riemann zeta function.