Statistical Mechanics
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Lecture - 63
Discontinuity in the Specific Heat of a Bose Gas - Part 01

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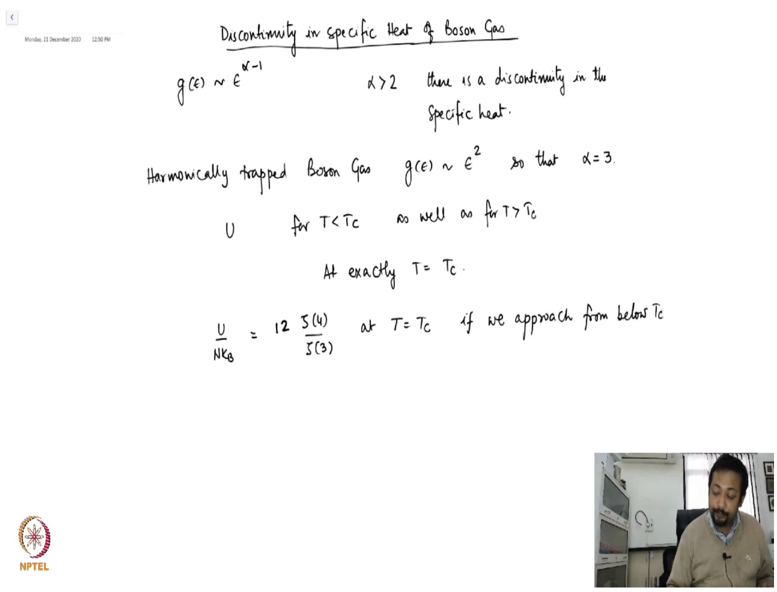

We want to discuss the Discontinuity in Specific Heat of a Boson Gas. We know that the general density of state goes as epsilon raised to the power alpha minus 1. For alpha greater than 2, there is a discontinuity in the specific heat and this we have seen. We looked at the case of a harmonically trapped gas Boson Gas. And here, we worked out the density of state to go and we saw that g epsa behaves as epsa square; so that, we immediately identified alpha is equal to 3.

Consequently, it we calculated the specific heat both using by determining the internal at an energy U or E; we will use this notion notation interchangeably for T less than T c as well as for T greater than T c. And we saw that the 2 results are not the same at exactly T is equal to T c. U over NK B was sorry 12 times zeta 4 over zeta 3 at T is equal to T c.

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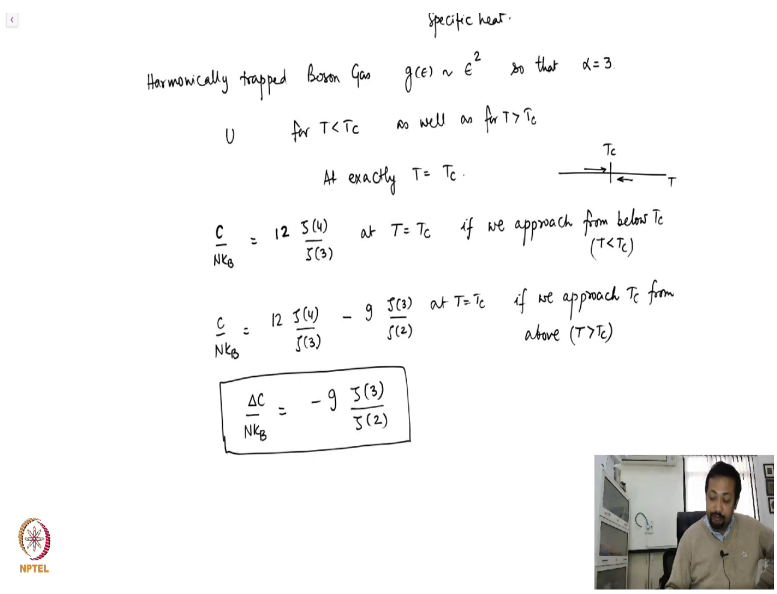

If we approach from below T c. In contrast U over NK B, was 12 over zeta 4 divided by zeta 3 minus 9 over zeta 3 divided by zeta 2 at T equal to T c, if we approach T c from above; which means from temperatures T greater than T c and this means temperature below T less than T c.

So, you have the critical temperature over here and you can approach the critical temperature from temperatures below T c as well as from above. And you see that there is oops this is not going to be U, this is going to be the specific heat if I call. And consequently, there is a difference in specific heat which is minus 9 zeta of 3 divided by zeta of 2 in this case.

Now, the question is we want to try to figure out what exactly this discontinuity this difference in amount is going to be for any general alpha.

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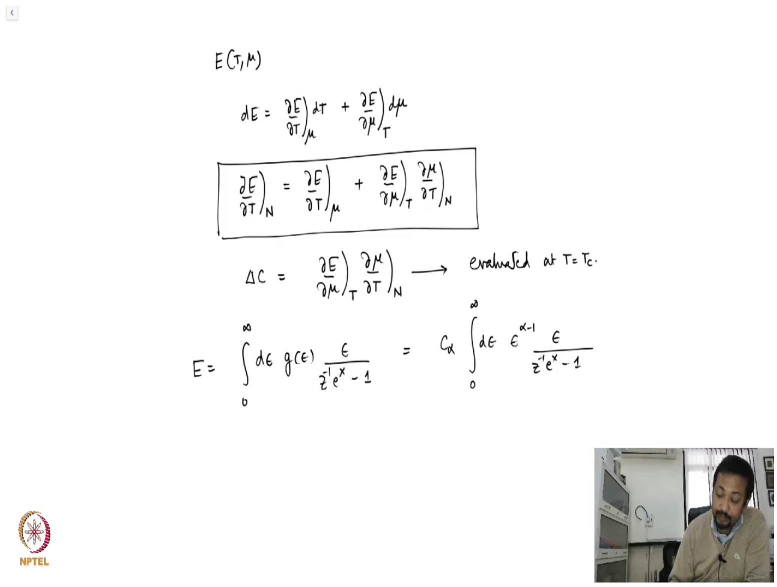

For that, our starting point is to write down E as a function of T comma mu so that; the differential d E is going to be del E del T mu held constant d T plus del E del mu T held constant d mu. Consequently, I have del E del T; this is the specific heat that I am interested in is going to be del E with the N held constant del E del T mu held constant plus del E del mu T held constant del mu del N sorry, del mu del T N held constant.

Now, the first term is going to be the same whether you are approaching from below temperatures below T c or whether you are approaching from temperatures above T c. Therefore, the difference in specific heat is going to be del E del mu temperature held constant del mu del T N held constant and this quantity evaluated at T is equal to T c. Our purpose is to figure out what these things are going to be.

So, the energy we again start is integration d epsilon g epsilon. So, here again it is, because starting from scratch.

So, that; things are not very complicated for you to figure out is going to be this. And which I have as C alpha integration d m epsilon 0 to infinity epsa to the power alpha minus 1, then I have an epsa divided by z inverse e to the power X minus 1.

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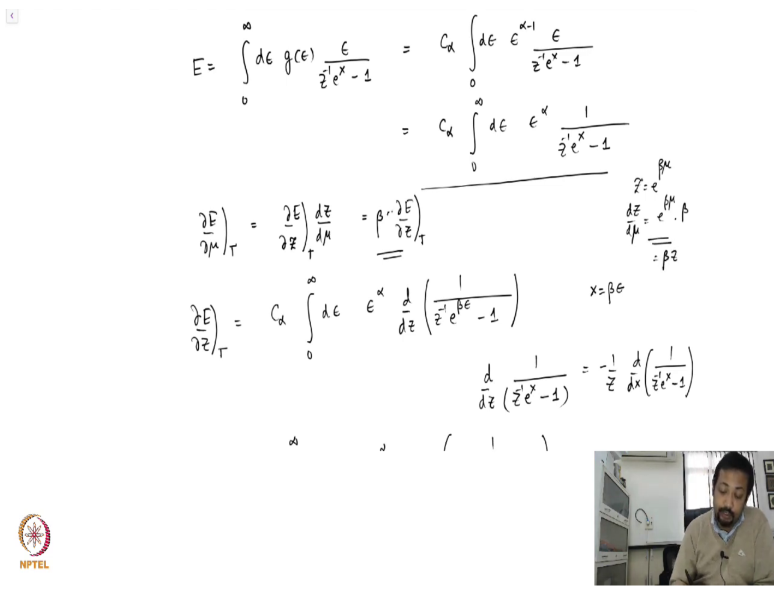

Which I have C alpha 0 to infinity d epsa, epsa to the power alpha 1 over Z inverse e to the power X minus 1. So that, del E del mu temperature held constant I node that is going to be del del x.

So, del del Z of e times del dZ d mu and dZ d mu since Z is equal to e to the power beta mu dZ of d mu is going to be e to the power beta mu times beta. So, I have beta times del e del Z temperature held constant and we go back to the expression that we have written down over here. So, that del E del Z temperature held constant is going to be C alpha 0 to infinity d epsilon to the power alpha d dZ of 1 over Z inverse e to the power X minus 1.

Now, life is a little bit simpler, because I have seen that d dZ of 1 over Z inverse e to the power X minus 1 is just going to be minus 1 over Z d dX of 1 over Z inverse e to the power X minus 1.

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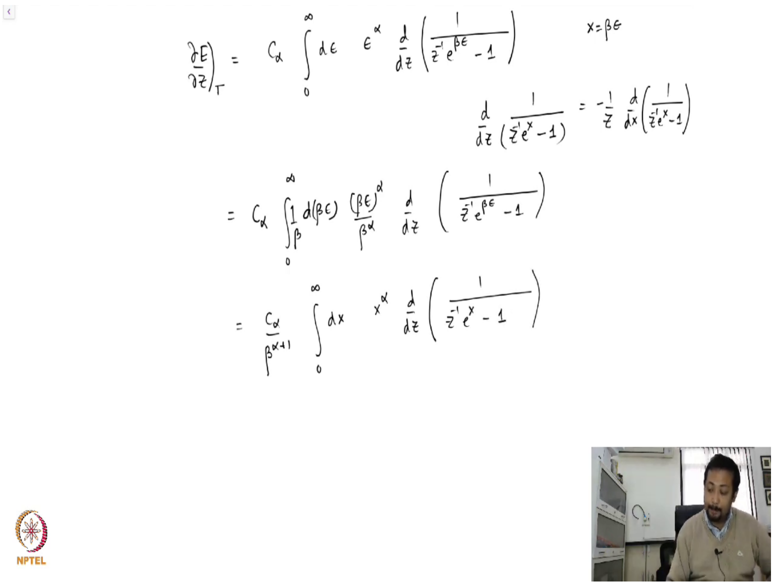

So that, this relation then, I have is going to be C alpha integration 0 to infinity. Now, look, I have used e to the power X and I have used e to the power X which is not the right thing to do. So, we have beta epsilon over here, did we keep it as beta epsilon yeah. So, unnecessarily, I have complete.

Anyway, so since this is beta X is equal to beta epsilon I can straightforward substitute this as d beta epsilon 1 over beta, beta epsilon raised to the power alpha beta to the power alpha d dZ of 1 over Z inverse e to the power beta epsilon minus 1 and this becomes C alpha raised to the power beta alpha plus 1 0 to infinity dX X to the power alpha d dZ of 1 over Z inverse e to the power X minus 1.

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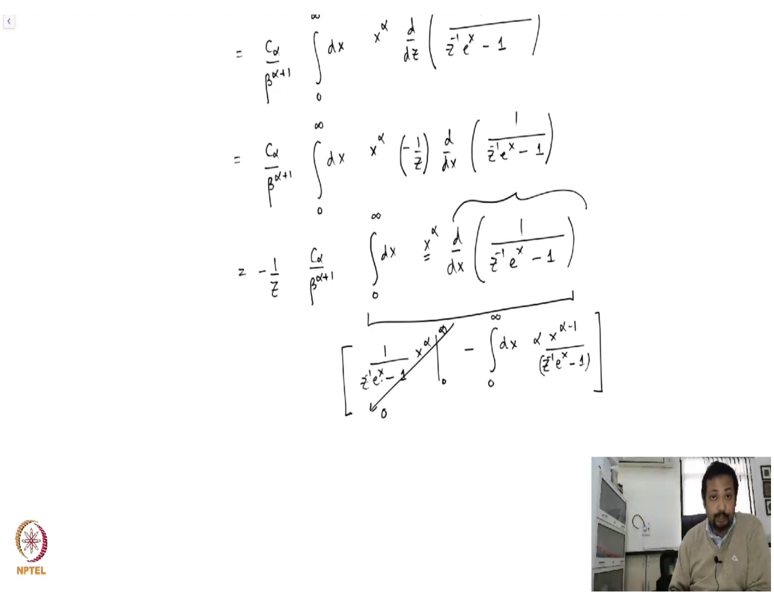

So, that I have C alpha beta raised to the power alpha plus 1 0 to infinity dX X to the power alpha and d dZis minus 1 over Z d dX of 1 over Z inverse e to the power X minus 1. So, this becomes minus 1 over Z C alpha beta alpha plus 1 0 to infinity dX X to the power alpha d dX of 1 over z inverse e to the power X minus 1.

Now, we are familiar with such integrals. We have done it several times in the past and the trick is to integrate by parts taking this as the first function. If I take this as the first function, then it follows that I have 1 over Z inverse e to the power X minus 1 X to the power alpha 0 to infinity minus 0 to infinity dx; a derivative of this is alpha X to the power minus X to the power alpha minus 1 and then, I have Z inverse e to the power X minus 1.

This vanishes in both the limits when X is equal to 0 you have it identically 0. When X is equal to infinity the exponential blows up and you have this also vanishes.

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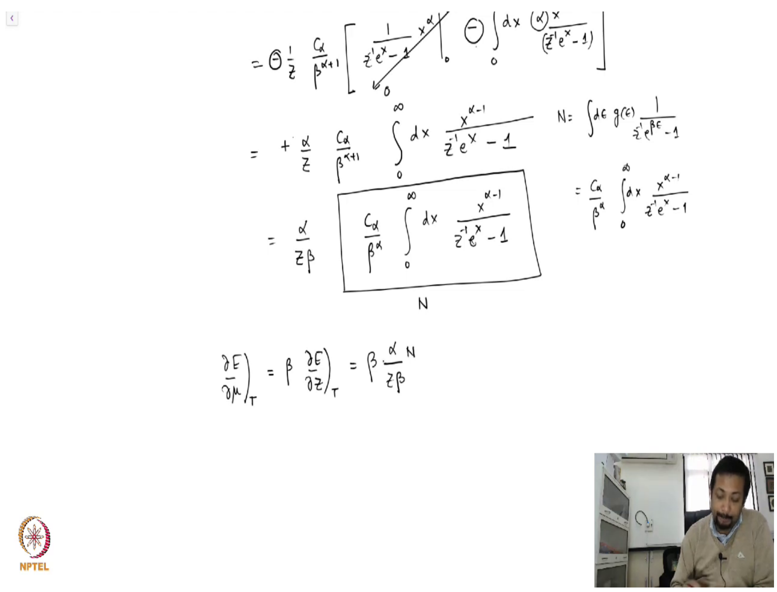

So, in the 2 limits the first term vanishes. So, that the alpha I can bring it in front and I have minus alpha over Z C alpha beta alpha plus 1. Also, note that I have a minus over here and here, I have minus 1 by C, C alpha beta raised to the power alpha minus plus 1. So that, this minus and this minus gives me a plus I have 0 to infinity dX X to the power alpha minus 1 Z inverse e to the power X minus 1. Let us rewrite this expression as this. Alpha divided by Z times beta.

So, that; I have C alpha beta alpha 0 to infinity dX X to the power alpha minus 1 Z inverse e to the power X minus 1. And if you recall, then this is exactly the particle number N. So, particle number N was integration d epsilon g epsilon 1 over Z inverse e to the power beta epsa minus 1 and if you substitute for epsa to the power alpha minus 1, you are going to get C alpha divided by beta alpha dX X to the power alpha minus 1 Z inverse e to the power X minus 1 0 to infinity.

So, that del E del mu temperature held constant is beta times del E del Z temperature health constant which is alpha over sorry beta times alpha over Z beta times N. So, this looks slightly suspicious, because in the denominator I have a Z. So, we go back and see where we have done the mistake and it is over here.

We did the derivative correctly. So, that this is going to be beta times Z we have to include a beta times Z here. So, that this expression is beta times Z and this is going to be alpha times N.

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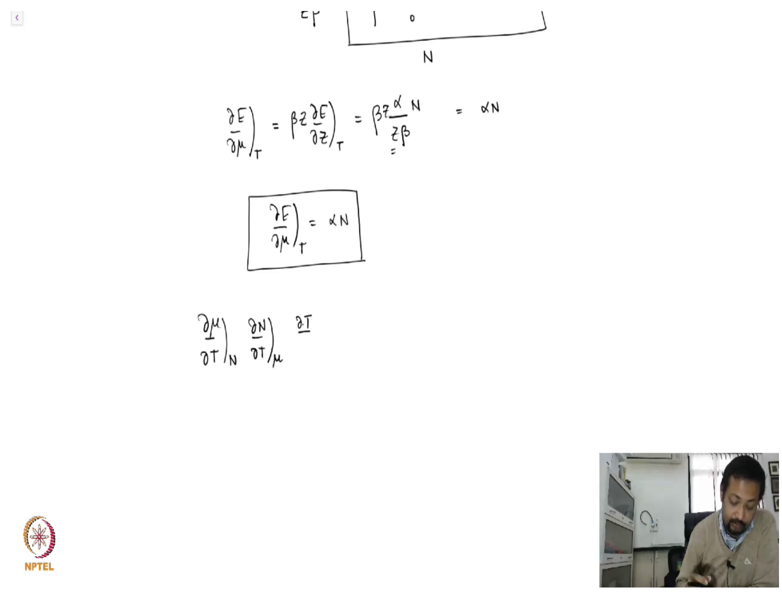

So, that del E del mu comes out to be a very very simple and elegant expression alpha times mu. Now, in this expression I have calculated the first term. So, this is done I have to look at this quantity del mu del T N held constant. So, del mu del T N held constant; to evaluate this I am going to use the cyclic identity that we learned in thermodynamics, that is essentially del N del T mu held constant and then, I have del T.

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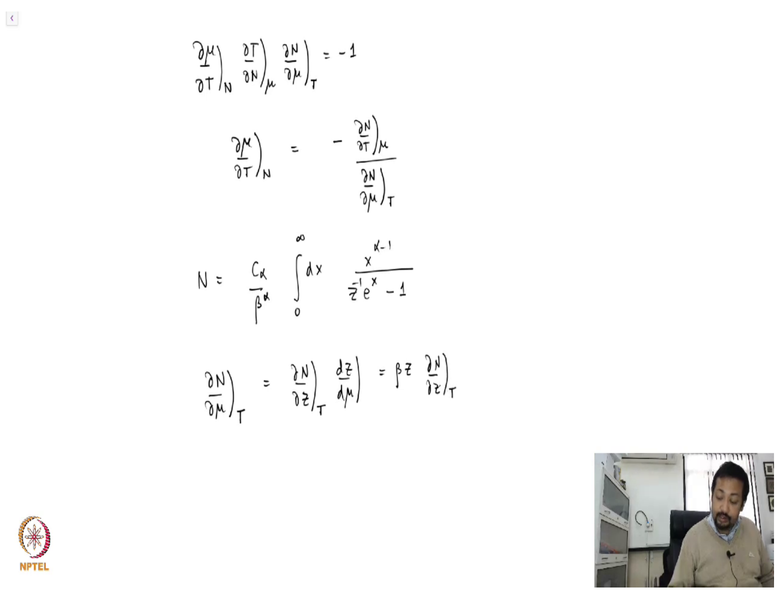

So, no this is not right this is going to be del T del N mu held constant del N del mu T held constant is going to be minus 1. Therefore, I have del mu del T N held constant is going to be minus del N del T mu held constant divided by del N del mu T held constant. So, these are the two partial derivatives which I have to evaluate, but I know the expression for N. N was C alpha raised to the power beta alpha 0 to infinity dX X to the power alpha minus 1 Z inverse e to the power X minus 1 del N del T.

So, the easier one is del N del mu T held constant and that is going to be again del N del Z T held constant dZ d mu which is going to be tell me sorry, dZ d mu is again beta times Z del N del Z T held constant.

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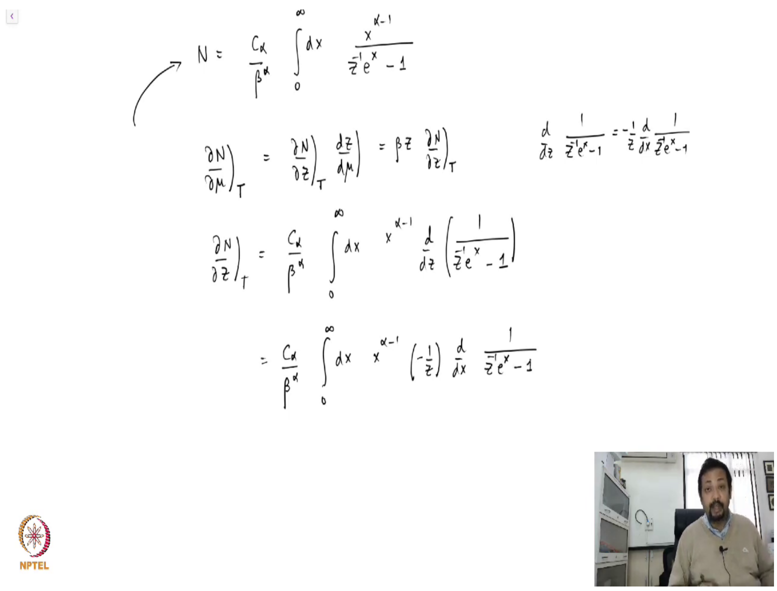

So, let us look take this expression for N and open figure out the derivative del N del Z T held constant. This is going to be C alpha beta raised to the power alpha 0 to infinity d of x, X to the power alpha minus 1 d dZ of 1 over Z inverse e to the power X minus 1. Again, I know the standard trick how to do this. I know that d dZ of 1 over Z inverse e to the power X minus 1 is minus 1 by Z d dX of 1 over Z inverse e to the power X minus 1.

So, that I have C alpha divided by beta alpha this is 0 to infinity d of X x to the power alpha minus 1 minus 1 over Z d dX of 1 over Z inverse e to the power X minus 1.

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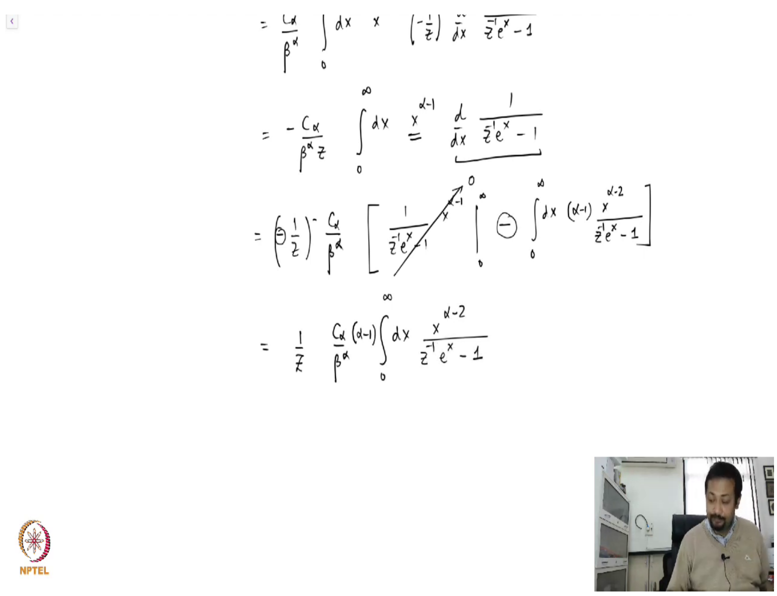

The minus 1 by Z I can bring out. C alpha beta raise to the power alpha divided by time Z 0 to infinity dx x to the power alpha minus 1 d dx of 1 over Z inverse e to the power x minus 1. Now, this integral again I know how to handle and it we do just integration by parts.

So, minus we will just separate this out as minus 1 over Z C alpha beta alpha and then, this becomes my first function; so that, you have Z inverse e to the power x minus 1 x to the power alpha minus 1 0 to infinity minus integration 0 to infinity dX; the derivative of the second function which is going to be alpha minus 1 X to the power alpha minus 2 and the integral of the first function which is just going to be Z inverse e to the power X minus 1 looks very nice and note that this vanishes in both the limits.

So that, I have the minus and this minus makes it a plus. So, I have 1 over Z C alpha beta alpha 0 to infinity; there is a alpha minus 1 dX X to the power alpha minus 2 Z inverse e to the power X minus 1. Let us go back and see I originally want del N del mu T held constant and that is just beta times Z.

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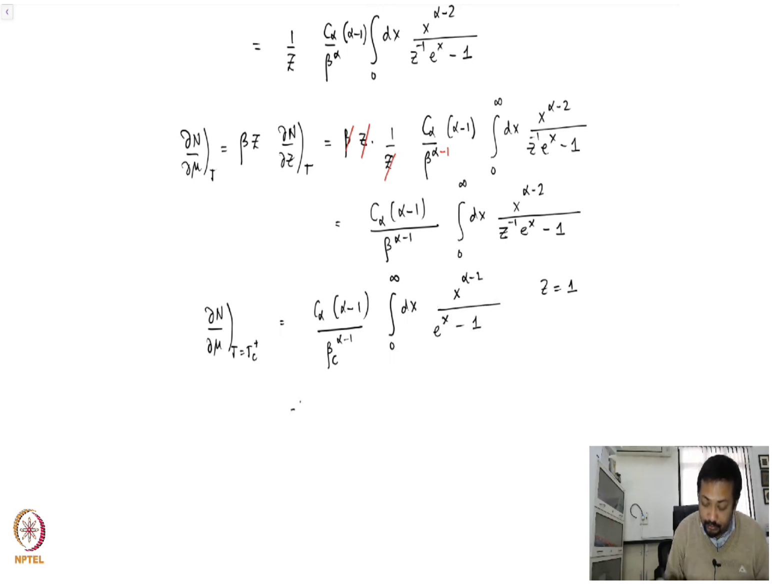

So that; del N del mu T held constant is beta Z times del N del Z T held constant which is going to be beta Z times 1 over Z C alpha over beta alpha. I have alpha minus 1 and then, I have 0 to infinity dX X to the power alpha minus 2 divided by Z inverse e to the power X minus 1 the not the b.

So, the cancellation means that Z cancels out and beta cancels with one factor of beta raised to the power alpha. So, that this becomes beta alpha minus 1 and the answer that I have is C alpha alpha minus 1 divided by beta alpha minus 1 raised to the power alpha minus 1 0 to infinity dX X to the power alpha minus 2 Z inverse e to the power X minus 1. Now, I have to evaluate this derivative. If you recall we originally started off with this derivative and this is the derivative that I have to evaluate at T c.

So, which means; I can replace Z is equal to 1. Since, I am evaluating this Z very close to T c. So, that this becomes del N del mu T is equal to T c is C alpha alpha minus 1. Well, you can also have T c plus 1 T c plus. So, that you are approaching from the high temperature side beta C raised to the power alpha minus 1. Here, Z is equal to 1.

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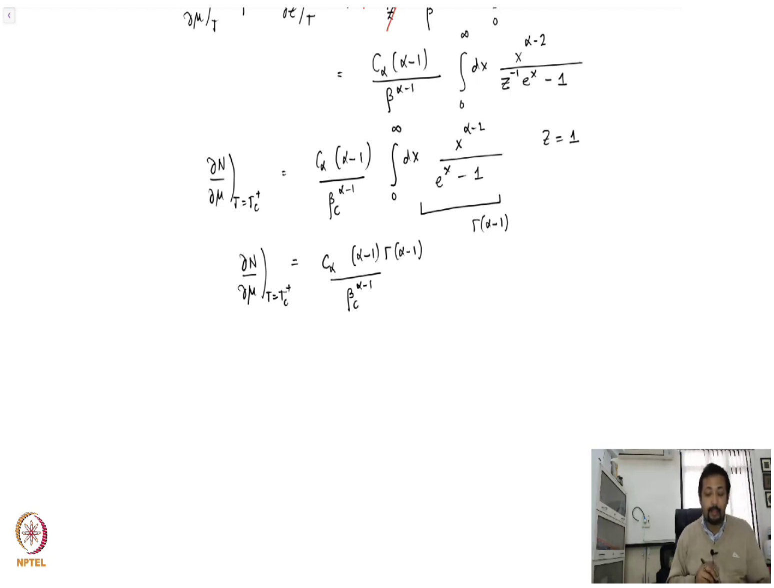

And you have integral 0 to infinity d of X x to the power alpha minus 2 divided by e to the power X minus 1 which is going to be C alpha alpha minus 1 beta C alpha minus 1 and this you see is gamma of alpha minus 1 this is the gamma function. So, I have gamma of alpha minus 1 is del N del mu evaluated at T equal to T c plus.

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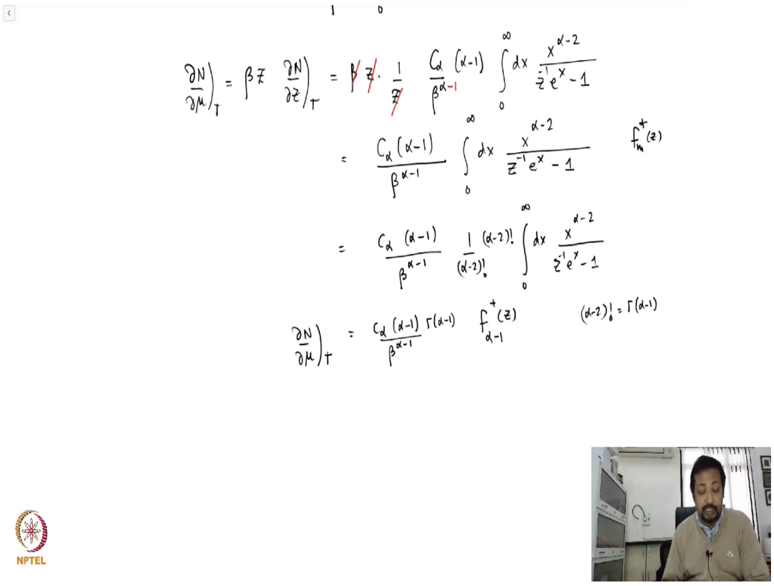

So, this expression now, I can write down as C alpha alpha minus 1 beta raised to the power alpha minus 1. And I see that this I can write down in terms of the integrals f of m eta well f of m plus of Z by identifying the fact that I have a missing 1 over alpha minus 2 factorial over here.

So, for that I introduce 1 over alpha minus 2 factorial in the numerator and alpha minus 2 factorial in the in sorry 1 minus alpha minus 2 1 by alpha minus 2 factorial in the denominator and alpha minus 2 factorial in the numerator and I write down this as dX X to the power alpha minus 2 divided by Z inverse e to the power X minus 1.

This quantity alpha minus 2 factorial I know is gamma of alpha minus 1. So, that I have C alpha alpha minus 1 divided by beta alpha minus 1 and then I have gamma of alpha minus 1 f of alpha minus 1 plus of Z. Now, I need to evaluate this. So, this is the derivative. Let us write down, del N del mu T held constant. Now, this is the derivative that I am going to use, but this derivative I am going to evaluate very close to T c.

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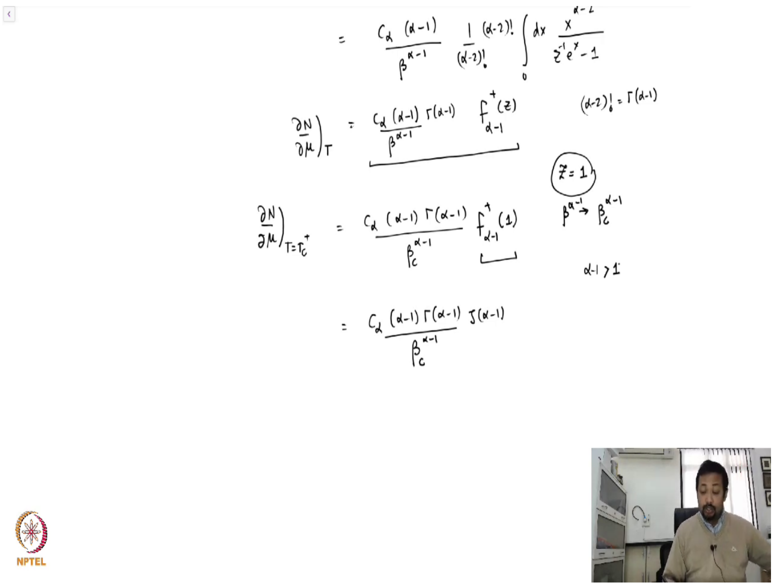

So, I have del N del mu. What I require is del N del mu T is equal to T c plus extremely close to T c. For that; since I want this. I look at this expression on top and I immediately say that oh I am going to put Z equal to 1 over here. So, that this becomes C alpha alpha minus 1 gamma of alpha minus 1 raised to the power beta C alpha minus 1.

So, the 2 substitutions I made. I made Z equal to 1 that is a not the 2 substitutions I put 2 values Z equal to 1 and beta to the power alpha minus 1 goes to beta C raised to the power alpha minus 1 and this becomes alpha minus 1 of 1. So, this function is now, C alpha alpha minus 1 gamma of alpha minus 1 divided by beta C of alpha minus 1 times zeta of alpha minus 1. Of course, we are considering alpha great alpha minus 1 is greater than 1 right.

So, once I have this expression now; all I am left to evaluate now is del N del T mu held constant.

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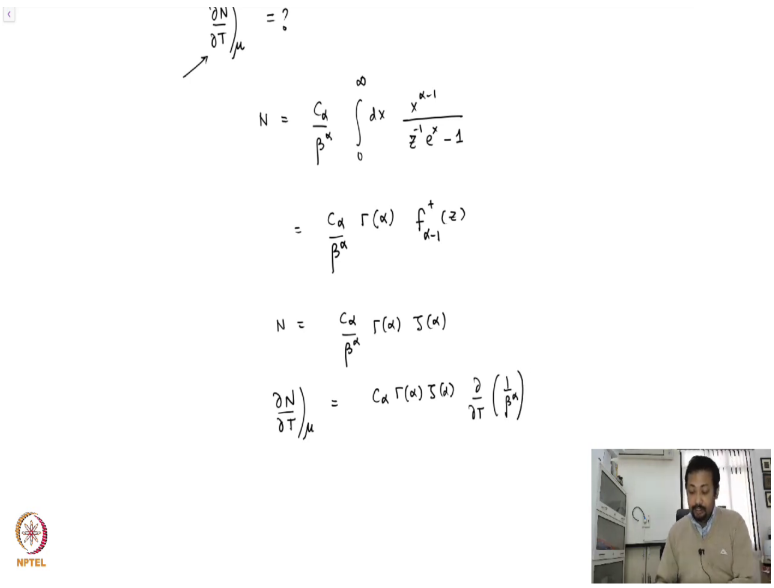

So, I have to evaluate del N del T mu held constant. Now, N is again given by C alpha beta raised to the power alpha 0 to infinity dX X to the power alpha minus 1 divided by Z inverse e to the power X minus 1. And this derivative again, I am going to use very very close to T c. So, that this becomes C alpha beta alpha raised to the power sorry beta raised to the power alpha. I am going to bring in a factor of gamma alpha and this equation becomes f of alpha minus 1 plus of set.

And close to T c; I have N as C alpha beta alpha gamma alpha zeta of alpha very close to T c; so that, del N del T mu held constant becomes C alpha gamma alpha zeta alpha del del T of 1 over beta alpha.

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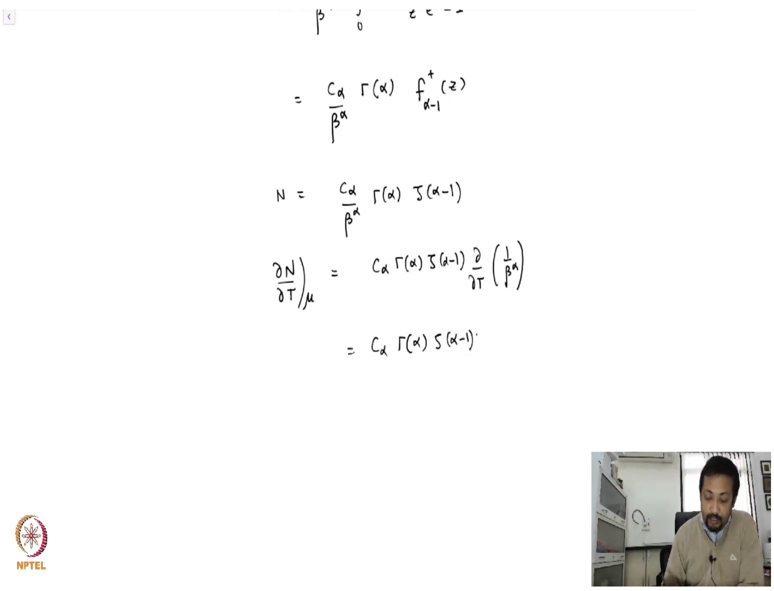

This I rewrite as C alpha gamma alpha zeta alpha. I think I have made a mistake over here. So, this is going to be alpha minus zeta of alpha minus 1. This is also going to be zeta of alpha minus 1 and this is going to be zeta of alpha minus 1. I hope in the earlier one I have written it correctly. This is zeta of alpha minus 1 yeah. So, this is correct. So, this was originally correct this has to be zeta of alpha.

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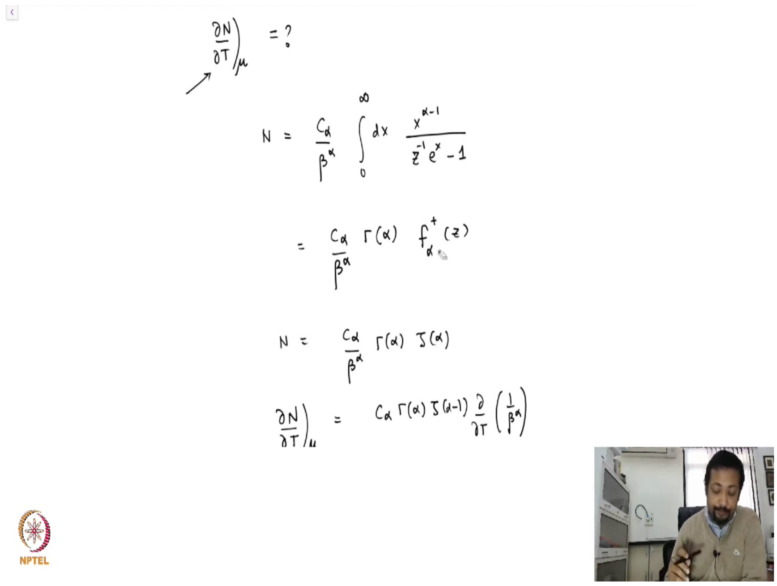

And I had X to the power alpha minus 1. So, this has to be f of alpha.