**Statistical Mechanics**
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**Lecture - 62**
**General Treatment of a Bose gas – Part 02**

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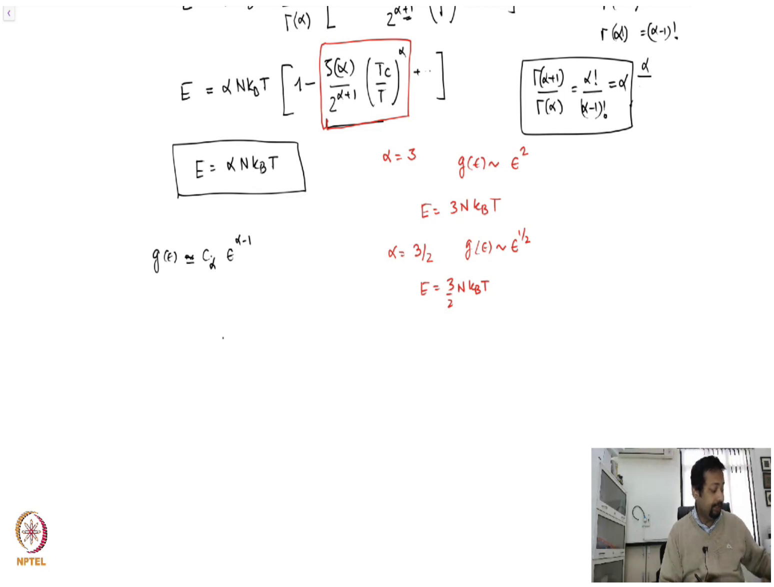


So, let us continue now this is the relation that we got when we wrote down the energy and the particle number, when my density of state g epsilon was the general form, C alpha is equal to C alpha epsa to the power alpha minus 1.

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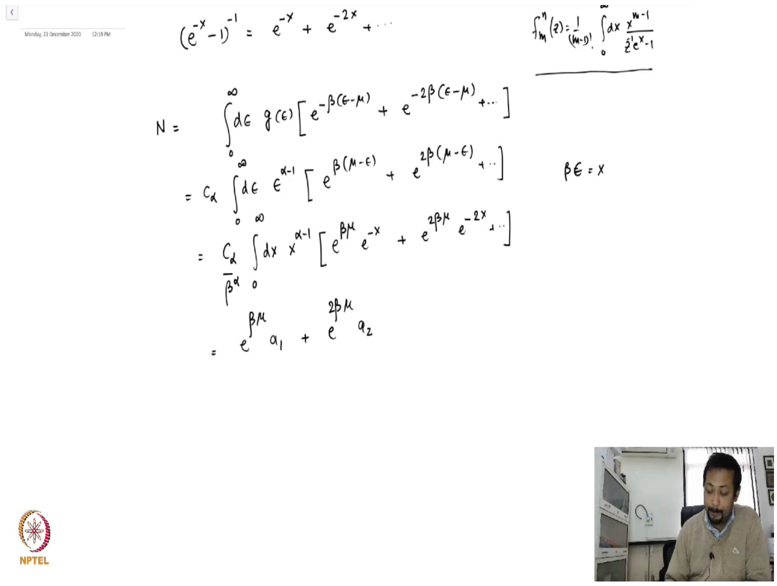


The idea is that if you have forgotten that you can express it in terms of the functions f of m eta of Z, which was 1 by m minus 1 factorial 0 to infinity dx x to the power m minus 1 Z inverse e to the power x minus 1, we had expressed our particle number as well as our energy the energy of the system in terms of these integrals and we got the result in the previous lecture.

We can take the following approach also; we note that at high temperatures I can expand e to the power minus x minus 1 as e to the power minus x plus e to the power minus 2x plus higher order terms. With this result I have N, which is going to be C alpha integral d epsilon g epsilon e to the power minus beta epsilon minus mu plus e to the power minus 2 beta epsilon minus mu plus higher order terms.

So that, sorry the C alpha does not come here and then we write down this as C alpha integration d epsa, epsa raised to the power alpha minus 1 e to the power beta mu minus epsilon plus e to the power twice beta mu minus epsilon. To the leading order we write this as C alpha ok.

First we convert this integral by substituting beta sorry beta epsilon is equal to x and this gives me beta to the power alpha dx the limit is 0 to infinity x to the power alpha minus 1 e to the power beta mu, e to the power minus x, plus e to the power 2 beta mu, e to the power minus 2x plus higher order terms, which we are going to write down as e to the power of beta mu times a 1 plus e to the power beta mu 2 beta mu times a 2.

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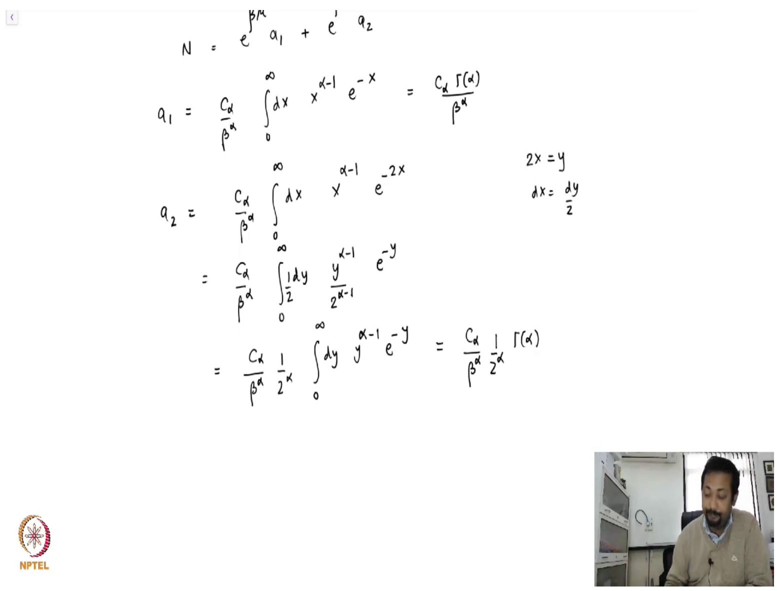


Where, the term this is the particle number a 1 is going to be given by C alpha beta alpha 0 to infinity dx x to the power alpha minus 1 e to the power minus x. Now, this integral is well known and this integral is gamma alpha. So, that I have a 1 as C alpha gamma alpha raised to the power beta alpha, what about a 2? a 2 is C alpha over beta raised to the power alpha 0 to infinity dx x to the power alpha minus 1 e to the power minus 2x.

Once again we make the substitution 2x equal to y over here, in which case dx is going to be dy by 2 and I have C alpha raised to the power beta alpha 0 to infinity dy with a half factor in front of it, x is y raised to the power alpha minus 1, 2 to the power alpha minus 1, e to the power minus y so, that I have C alpha beta alpha 1 over 2 to the power alpha 0 to infinity dy y to the power alpha minus 1 e to the power minus y; and this gives me C alpha beta alpha 1 over 2 to the power alpha times gamma alpha.

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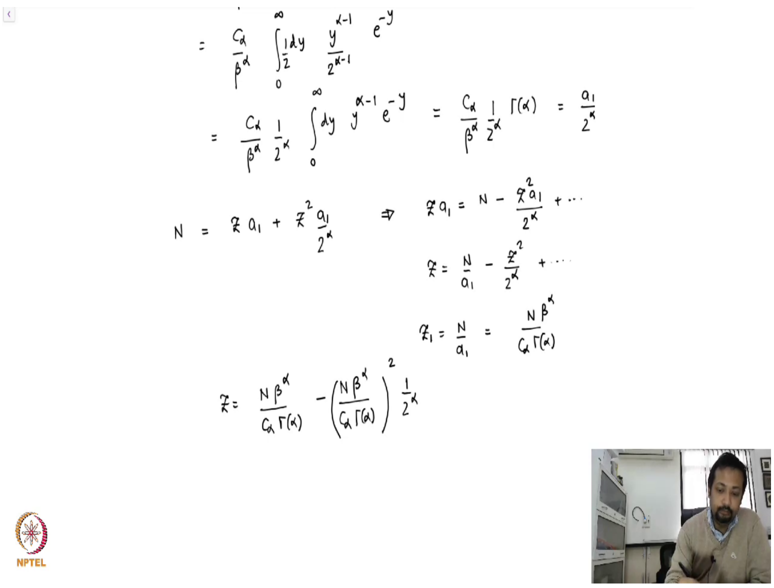


Which I observe that this is a 1 divided by 2 to the power alpha so, that N is going to be Z times a 1 plus Z square times a 1 over 2 to the power alpha, and this implies I can write down

Z times a 1 is equal to N minus Z square times a 1 divided by 2 to the power alpha. So, that Z is going to be N over a 1 minus Z square over 2 to the power alpha, to the leading plus of course there are higher order terms.

So, that to the leading order I have Z 1 as N over a 1, which means this I have N a 1 was C alpha, gamma alpha, raised to the power beta alpha. This is the same result that we obtained in the earlier lecture so, that my Z becomes N beta alpha, over C alpha, gamma alpha, minus N beta alpha, C alpha, gamma alpha whole square 1 over 2 to the power alpha.

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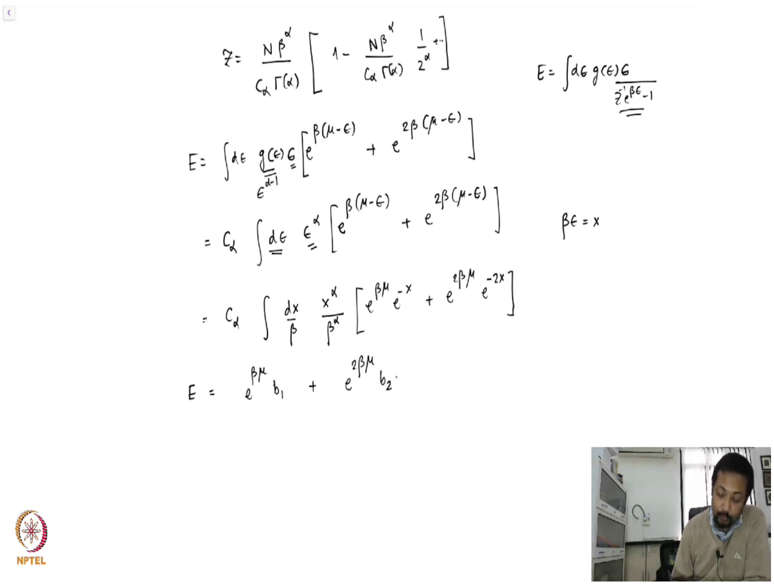


So, this expression is now N raised to the power beta alpha, C alpha, gamma alpha 1 minus N beta raised to the power alpha, C alpha, gamma alpha 1 over 2 to the power alpha, plus higher order terms right ok. Now, the energy E is integration d epsa g epsilon e to the power beta mu minus epsilon plus there is an epsilon factor plus e to the power 2 beta mu minus epsilon.

And this epsilon factor is because I am taking the average of the energy which is d epsa g epsa times epsa times Z inverse e to the power beta epsa minus 1, and this is the part which we have essentially expanded. So, that this becomes C alpha d epsilon epsilon to the power alpha e to the power beta mu minus epsilon plus e to the power twice beta mu minus epsilon, there are higher order terms which we will ignore here.

Now, again the standard trick is to put beta epsilon equal to x and you have C alpha you will have one by beta that will comes from the measure. So, that this becomes dx over beta x to the power alpha over beta to the power alpha. Since g epsa goes as epsa to the power alpha minus 1 and I have a factor of epsilon sitting so, that I this becomes epsa to the power alpha, e to the power beta mu, e to the power minus x, e to the power 2 beta mu, e to the power minus 2x and this again we will write down as e to the power beta mu. Let us say, b 1 prime no so, we will write this as b 1 plus e to the power 2 beta mu b 2.

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Recall e to the power beta mu is going to be Z, b 1 is C alpha raised to the power beta alpha plus 1 dx x to the power alpha e to the power minus x and that is given by gamma alpha plus 1 divided by beta raised to the power alpha plus 1. Similarly, b 2 is going to be C alpha raised to the power beta alpha plus 1 dx all the limits are between 0 to infinity.

So, let us not forget that. 0 to infinity x to the power alpha e to the power minus 2 x and if you substitute 2 x is equal to y you will come up with the relation as in the previous case beta raised to the power alpha plus 1; and I will have 1 over 2 to the power alpha plus 1 and I am going to have gamma alpha plus 1, which is nothing but b 1 divided by 2 to the power alpha plus 1.

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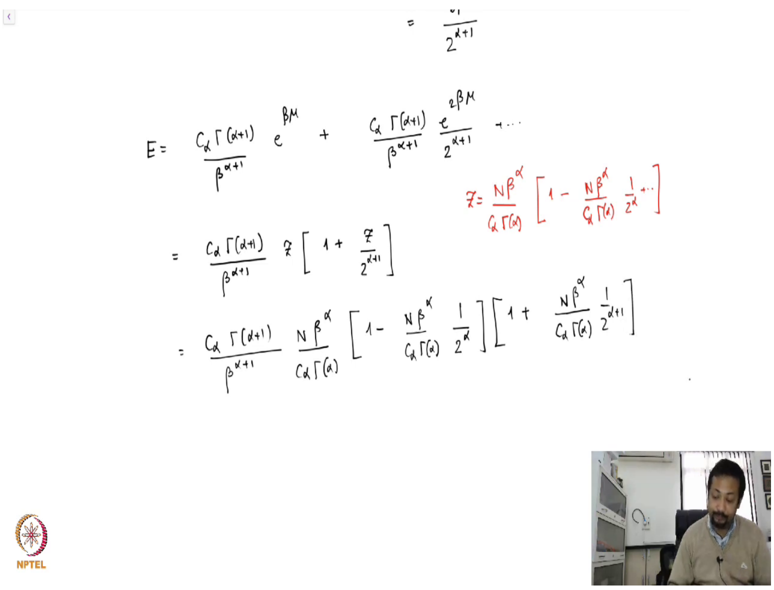


So, the average energy now is going to be z. So, let us say I have C alpha gamma alpha plus 1 divided by beta alpha plus 1 e to the power beta mu plus C alpha gamma alpha plus 1, beta raised to the power alpha plus 1, e to the power 2 beta mu divided by 2 alpha plus 1 plus higher order terms which is C alpha gamma alpha plus 1 divided by beta alpha plus 1, I have I can take Z common e to the power beta mu is Z.

So, that Z 1 plus Z over 2 to the power alpha plus 1; and here again I will substitute Z as this quantity relation. So, this is the relation that I bring down over here there is a too much of a difference. So, let us just write down Z ,which we had obtained above N beta raised to the power alpha divided by C alpha times gamma alpha 1 plus sorry 1 minus N beta raised to the power alpha C alpha gamma alpha 1 over 2 to the power alpha higher order terms.

So, that this becomes C alpha gamma alpha plus 1 beta alpha plus 1, N beta raised to the power alpha C alpha gamma alpha 1 minus N beta raised to the power alpha C alpha gamma alpha. And then, I have 2 to the power alpha times 1 plus N raised to the beta raised to the power alpha C alpha gamma alpha, 1 over 2 to the power alpha plus 1.

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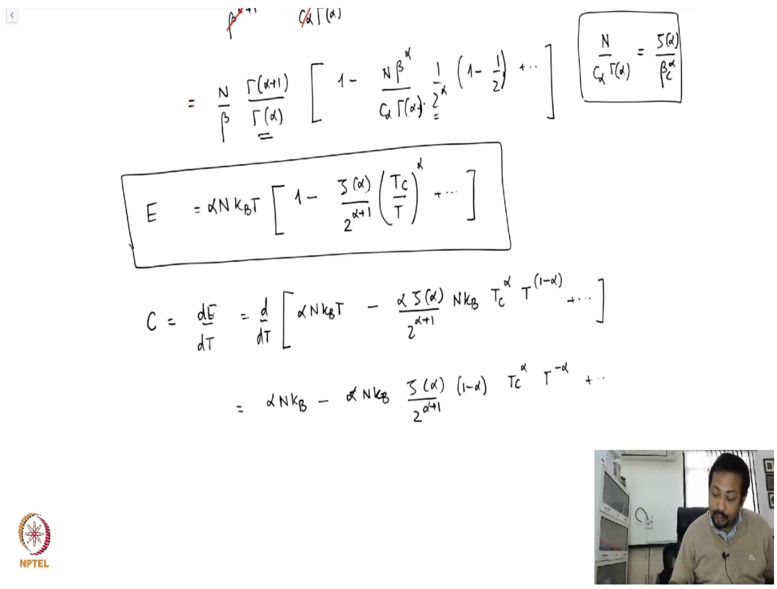


Their cancellations as you can see this this gives you this the beta alpha cancels with this and you are going to have N times gamma alpha plus 1, N over beta divided by gamma alpha and once again this is going to be N beta alpha C alpha gamma alpha 1 over 2 to the power alpha 1 minus 2 to the half.

So, 2 to the power alpha 2 to the power alpha is common, it comes out and then you have the higher order terms and again you identify that N over C alpha gamma alpha was zeta alpha divided by beta C alpha the critical temperature. So, this we have done in the earlier lecture.

So, this becomes N k B T and this is alpha alpha N k B T and you have 1 minus zeta alpha raised to the power alpha 2 to the power alpha plus 1 T C over T raised to the power alpha plus higher order times.

So, that this is your energy, which you obtained earlier also and it gives you the same expression. Except here we have started off from a different using a different approach by expanding the denominator, you could have equally done it using this form using the high temperature expansion of this integrals f integrals.

But, if you do not know that or if you have forgotten somehow then you can easily go between just expand the denominator that you have over here and write it down and do the analysis in the same way. So, that the specific heat is dE dT and which we are going to explicitly write down now d dT as alpha N k B T minus alpha zeta alpha 2 to the power alpha plus 1 N k B T C to the power alpha and I have T divided by T to the power alpha so, T raised to the power 1 minus alpha higher order terms.

So, the first term is alpha N k B and the second term is minus alpha N k B zeta alpha 2 to the power alpha plus 1 1 minus alpha T C alpha T to the power minus alpha higher order terms.

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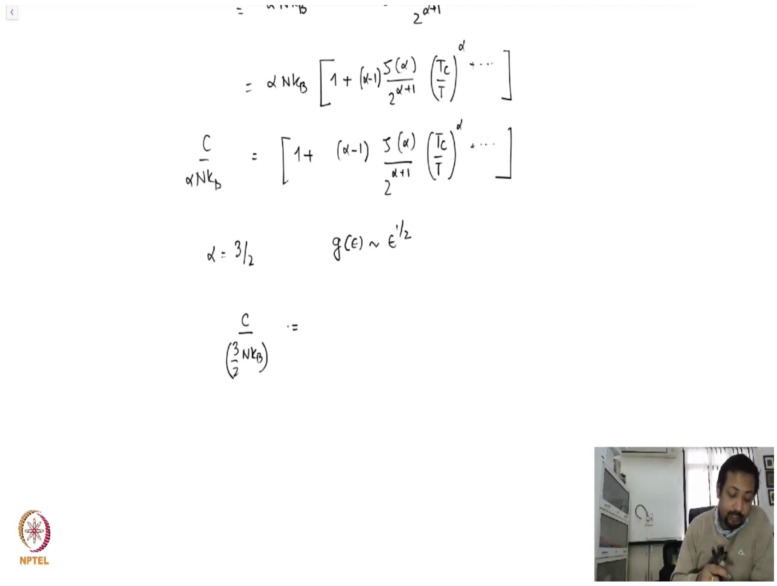


So, that I can take alpha N k B common and this gives me 1 minus zeta alpha 2 to the power alpha plus 1, I have 1 minus alpha that I am going to write as alpha minus 1 and bring me this one as a plus and I have T C over T raised to the power alpha plus higher order terms.

So, that C over N k B well alpha N k B is going to be 1 plus alpha minus 1 zeta alpha 2 to the power alpha plus 1 T C over T raised to the power alpha higher order terms. Now, good so, when alpha is equal to 3 half so that, your g epsilon goes as epsa to the power half, this is the case for an ideal Bose gas without any confinement then, we see that C over 3 half N k B this is the classic Dulong -Petits law is going to be well, you can take the alpha here.

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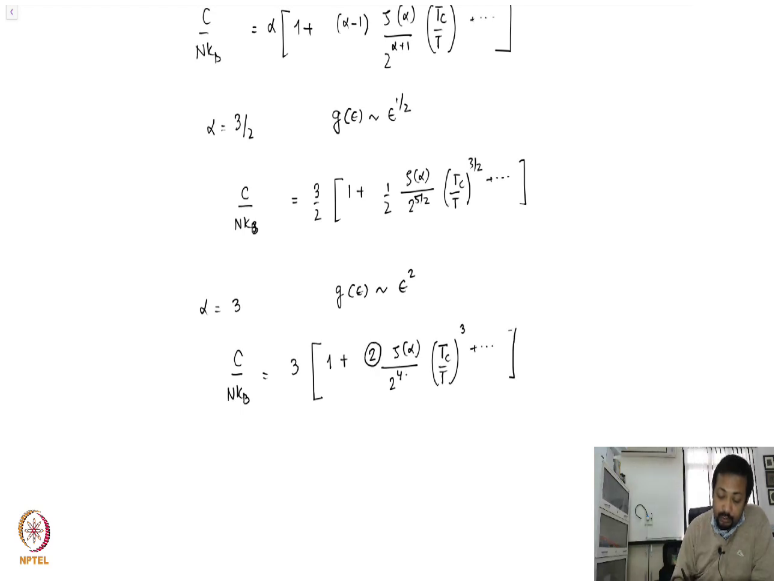


So, that will keep the alpha here, then this is not the left hand side of this is not so ugly looking in expression I have C over N k B is going to be 3 half 1 plus half zeta half 2 to the power 5 by 2 plus T C over T sorry times T C over T raised to the power 3 by 2 and higher order terms.

What is interesting to note ok, first let us do the other case when alpha is equal to 3 and g epsilon goes as epsa square and this is the case when your Bose gas we are looking at an ideal Bose gas, which is now confined in a harmonic potential in which case you have C over N k B is going to be 3. This is the classical Dulong Petits law, 1 plus 2 zeta alpha raised to the power 2 to the power 4 T C over T whole cube higher order terms. The 2 cancels with 1 power of 2 and we can simplify this and rewrite this expression as 2 cube.

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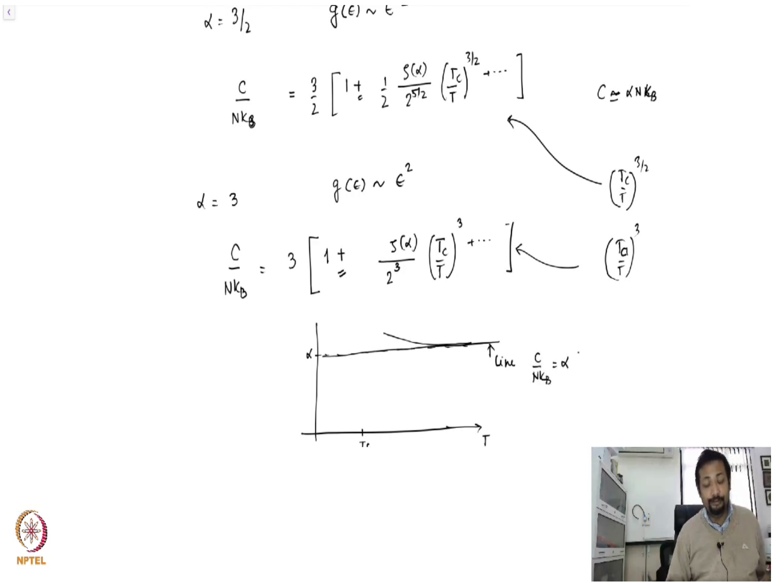

Now, the interesting thing that you should know that there is a plus sign here that there is a plus sign here, which means in the high temperature limit this is your critical temperature and you are far away somewhere over here.

And this is the classical result alpha. You see it approaches C over N k B which is C the specific it approaches the classical result alpha N k B from above this line in both the cases. In one case this is it approaches as T C over T raised to the power 3 half and in this case it approaches T C over T whole cube. But, in both the cases it is going to approach the classical limit C over N k B from above this line, which is C over N k B is equal to alpha.