Statistical Mechanics
Prof. Dipanjan Chakraborty
Department of Physical Sciences
Indian Institute of Science Education and Research, Mohali

Lecture - 61
General Treatment of a Bose gas - Part 01

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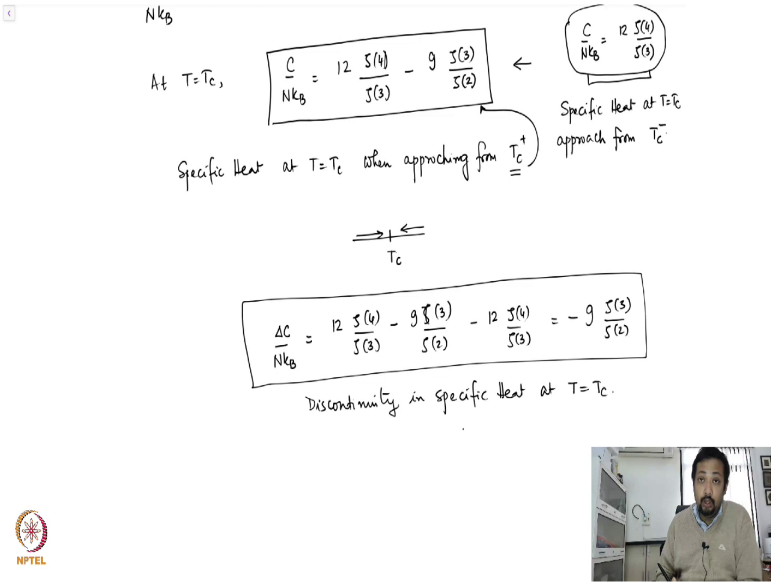

Welcome back. So, we were looking at the discontinuity in the specific heat for a harmonically trapped Bose gas. We will come back to this issue little later, but we want to consider now.

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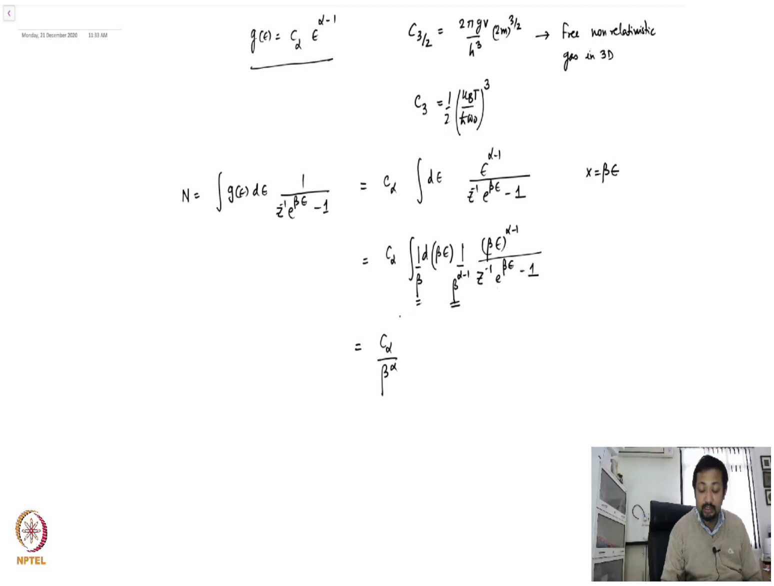

We now want to look at when the density of state goes as C alpha epsa alpha to the power minus 1. This is a slight change in notation because recall in earlier lecture I had Z said that the density of state goes as epsa alpha by 2 sorry epsa alpha by 2 minus 1, but it is better that to be consistent with the degrees of freedom one writes g epsilon as C epsa C alpha epsilon to the power alpha minus 1 and we know of two particular cases when C 3 by 2 is going to be 2 pi g v over h cube twice m 3 half.

And this is the case for a free non relativistic gas in 3 dimension and we also know for C 3 which is the case we are looking at harmonically trapped gas and in this case this was k B T over h bar omega naught whole cube half of this. So, idea is to take this general structure of the density of states where it depends on some exponent alpha right to work our way through.

So, then the particle number is integral g epsa d epsa Z inverse e to the power beta epsa minus 1 and this gives me C alpha integral d epsa epsa alpha minus 1 Z inverse e to the power beta epsa minus 1. In this case again we use the transformation of variables which means we substitute x is equal to beta epsilon to give me d of beta epsilon and then I have beta epsilon raised to the power alpha minus 1 I have Z inverse e to the power beta epsa minus 1 and I have beta to the power alpha minus 1 and 1 over beta.

So, that the expression takes the form beta to the power alpha, one beta cancels with the minus 1 beta to the power minus 1 that you see over here, this beta cancels with this and you are left out with C alpha divided by beta 2 to the power alpha.

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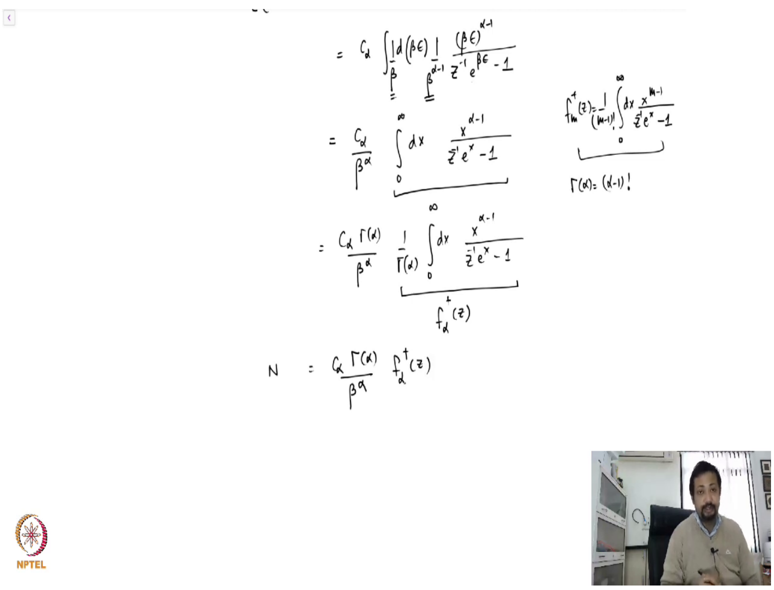

And your integral is dx, x to the power alpha minus 1 Z inverse e to the power x minus 1. Now, see this integral is related to my integrals that we had defined f m plus of Z which is 1 over m minus 1 factorial 0 to infinity dx x to the power m minus 1 Z inverse e to the power x minus 1.

Here if you compare these two integrals then you see that at factor of 1 by m minus 1 is missing, but that is easily remedied. I know that gamma alpha is alpha minus 1 factorial. So, I am going to bring in a gramma alpha in the numerator and a gamma alpha in the denominator which makes it like this and this is now f alpha plus of Z. So, that particle number is given by C alpha gamma alpha beta alpha f alpha plus of Z.

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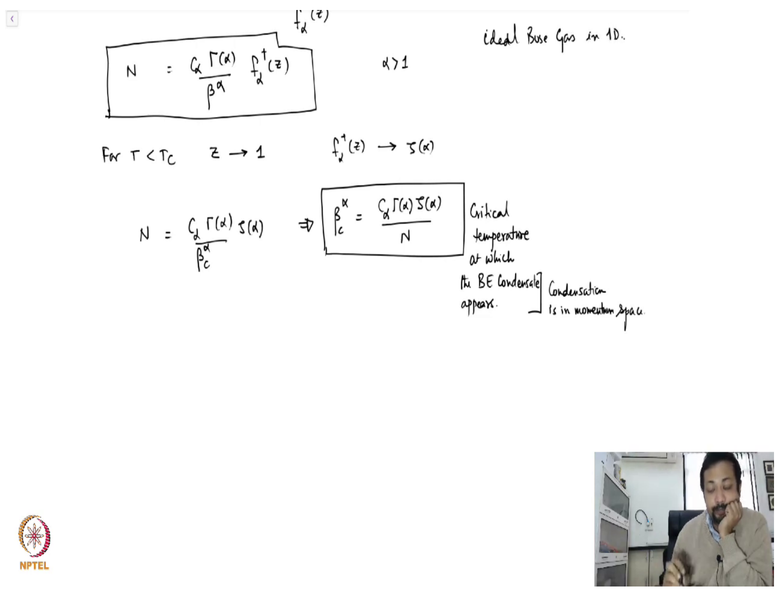

Now, of course, it is a standard it is just two steps now. So, if I now say that alpha is greater than 1 then it follows that for T less than T c means, fugacity hits the value 1, in which case your f alpha plus Z goes to zeta of alpha.

So, that I can use this to determine the critical temperature at which the condensate is going to appear and that is given by beta C alpha raised to the power alpha zeta alpha, which implies we will use this later on beta C alpha is going to be C alpha gamma alpha zeta alpha divided by N.

So, this defines the critical temperature at which the condensate the BE condensate appears and again this condensation is in momentum space not in the real space. Now, before we go further it is very interesting now that for a ideal Bose gas in 1D. So, let us just discuss this very briefly.

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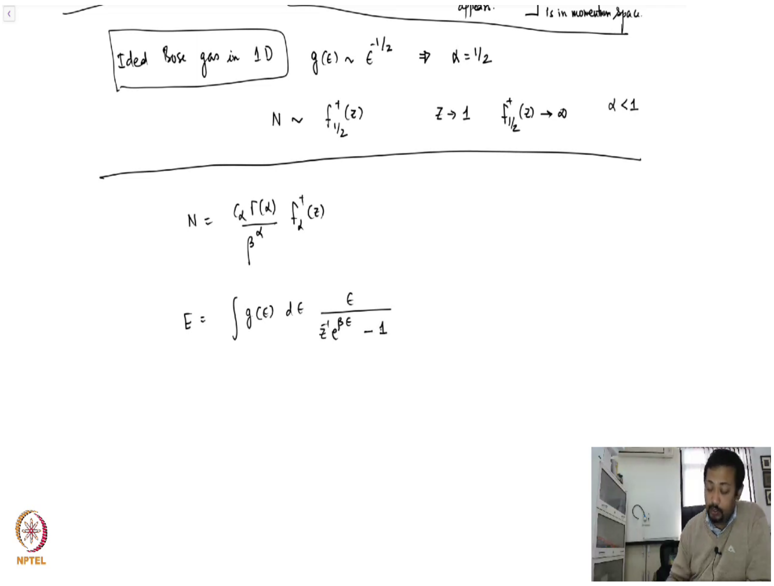

For an ideal Bose gas in 1D my g epsilon goes as epsa to the power minus half which means and this implies that alpha is going to be half in our description that we have written down over here. Therefore, this means that N is proportional to f half of Z, but the curious thing is that as Z tends to 1, f plus half of Z tends to infinity the series does not converge because alpha is less than 1.

And therefore, for a 1D ideal Bose gas you cannot have a Bose Einstein condensation. Now, let us quickly continue with this. So, this is my N, N is C alpha gamma alpha over beta raised to the power alpha and then I have f alpha plus Z. The energy is integral g epsa d epsa epsilon divided by Z e to the power Z inverse e to the power beta e minus 1.

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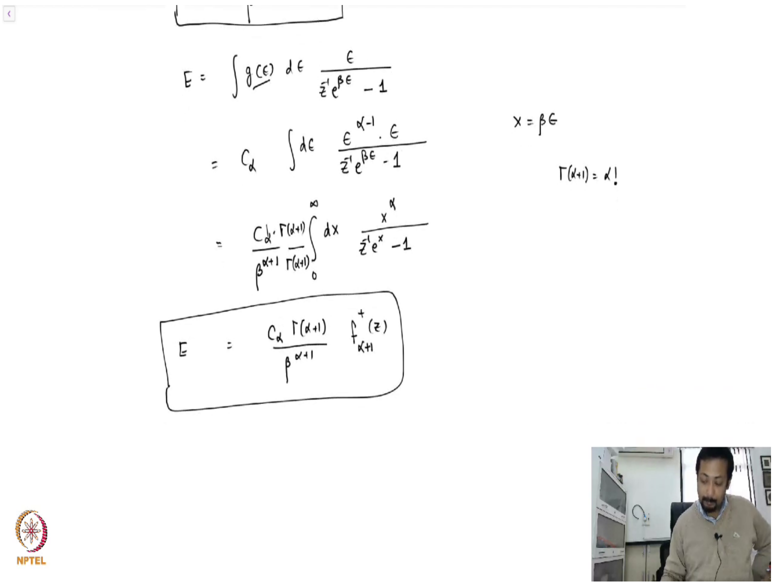

Which we will write down again as C alpha integral d epsa epsa to the power alpha minus 1 times an epsilon that comes epsa to the power alpha minus 1 comes from the g epsa and the epsilon is already there because it is an average of the energy and I have this. If I now substitute x equal to beta epsa then I see that this answer is going to be C alpha is going to be beta raised to the power alpha plus 1 0 to infinity dx x to the power alpha divided by Z inverse e to the power x minus 1.

So, again comparing with the integrals that we had f plus integrals the Boson Bose integrals that we had defined I am missing a factor of 1 by alpha factor alpha factorial. So, gamma of alpha plus 1 is going to be alpha factorial. I bringing a gamma of alpha plus 1 and I bringing a gamma of alpha plus 1.

And that gives me that the energy is going to be C alpha gamma alpha plus 1 divided by beta raised to the power alpha plus 1 f of alpha plus Z right. So, I have two relations. One is the one this and sorry this has to be alpha plus 1, terribly sorry and I have this. The high temperature limit is the one which you are going interested in.

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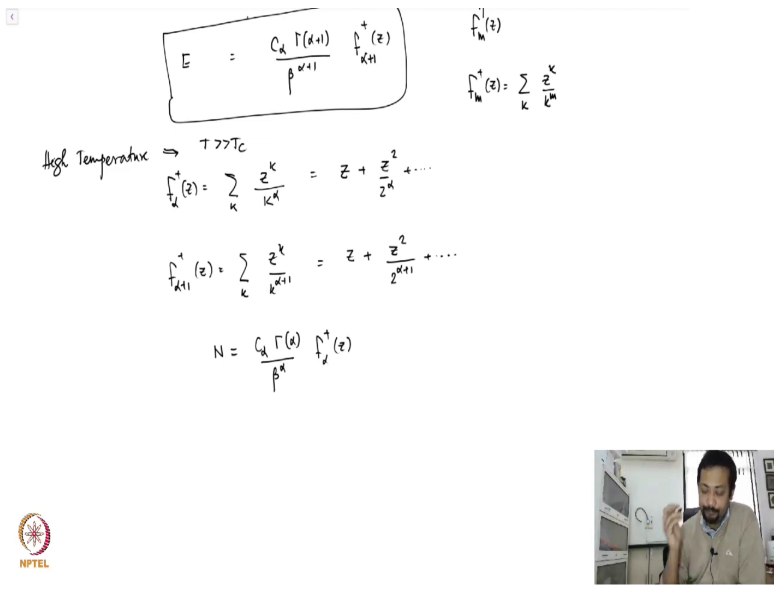

And for that I am going to use the high temperature expansion of the functions f eta of Z, f eta m of Z and in this particular case this becomes sum over k Z to the power k, k to the power m, so that f alpha plus Z high temperatures which of course, always means that T is much much larger than T c.

You are far away from the critical temperature and you have the expansion as sum over k Z to the power k alpha to the power k, sorry k to the power alpha which is Z plus Z square over 2 to the power alpha plus higher order terms. And f of alpha plus 1 plus of Z is going to be sum over k Z to the power k, k to the power alpha plus 1 which is going to be Z plus Z square over 2 to the power alpha plus 1 plus higher order terms.

So, starting again from the expression for the particle number which I had C alpha gamma alpha raised to the power beta to the power alpha f alpha plus of Z and the standard trick is that the fugacity must be determined from this relation.

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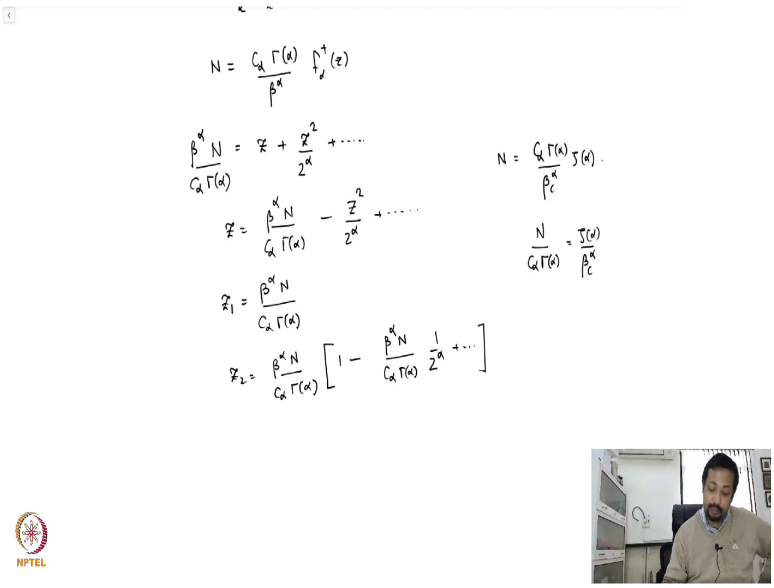

So, in order to determine the fugacity since I am at high temperatures, temperature is far away from my critical temperature I write this as beta alpha C alpha gamma alpha Z plus Z square raised to the power 2 to the power alpha plus higher order term. Now, a perturbative expansion of the fugacity can be done. So, that to the lowest order I have Z 1 as beta to the power alpha and C alpha gamma alpha.

So, let us write down the equation that I am going to use. That means, I can write down Z as beta to the power alpha N C alpha gamma alpha minus Z square over 2 to the power alpha plus higher order terms. And you can immediately see if I use again if I use the perturbative expansion then to the lowest order I ignore all higher orders of Z, I have beta to the power alpha times N C alpha gamma alpha.

And Z 2 is beta to the power alpha times N C alpha gamma alpha 1 minus beta alpha N C alpha gamma alpha times 2 to the power alpha plus higher order terms. Now, you see C alpha, gamma alpha are all these quantities which are essentially a little bit tricky to measure. I mean I do not know I can only have an indirect idea or I can start from a model and I can try to figure out, but in principle here I can make my life easy and I can replace this over here.

So, C alpha gamma alpha divided by N, the condition for the condensation to happen at the temperature was given by C alpha gamma alpha beta C alpha times zeta alpha. So, this means, N by C alpha gamma alpha is going to be zeta alpha divided by beta C alpha.

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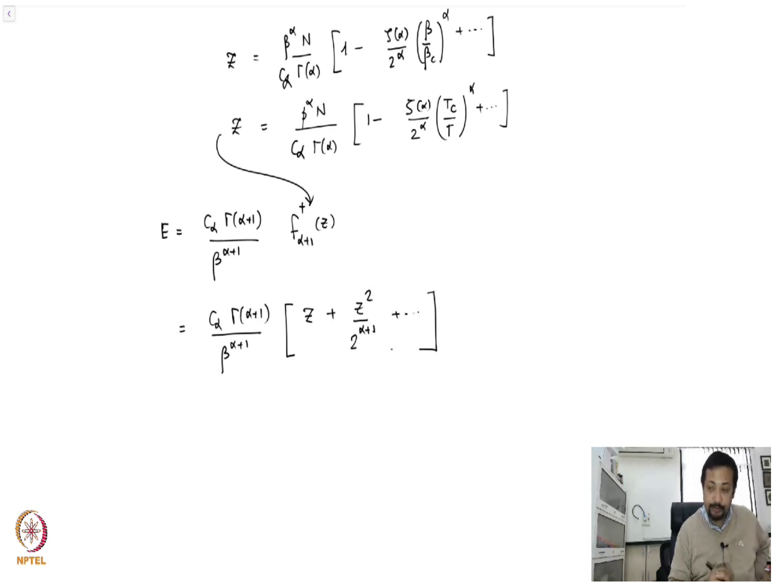

So, I will just write down this. C alpha gamma alpha times 1 minus zeta alpha divided by 2 to the alpha beta over beta C raised to the power alpha plus higher order terms, which gives me beta alpha N divided by C alpha times gamma of alpha 1 minus zeta alpha 2 to the power alpha T c over T raised to the power alpha plus higher order terms.

So, please note that you have T in the denominator and again you might have the desire to put T to 0 here and say claim that this goes to infinity, but we are in the high temperature limit. So, therefore, T actually you cannot take the limit of T to 0, but rather you should take the limit of T to infinity in which case of course, we recover the first result.

What about the energy? Energy was C alpha gamma alpha plus 1 beta raised to the power alpha plus 1 and I had f of alpha plus 1 plus of Z. And again since so, I had determined the fugacity from my equation for the particle number and that Z I want to substitute over here. C alpha gamma alpha plus 1 beta of alpha plus 1 and then if I expand this I have Z plus Z square divided by 2 to the power alpha plus 1 plus higher order terms.

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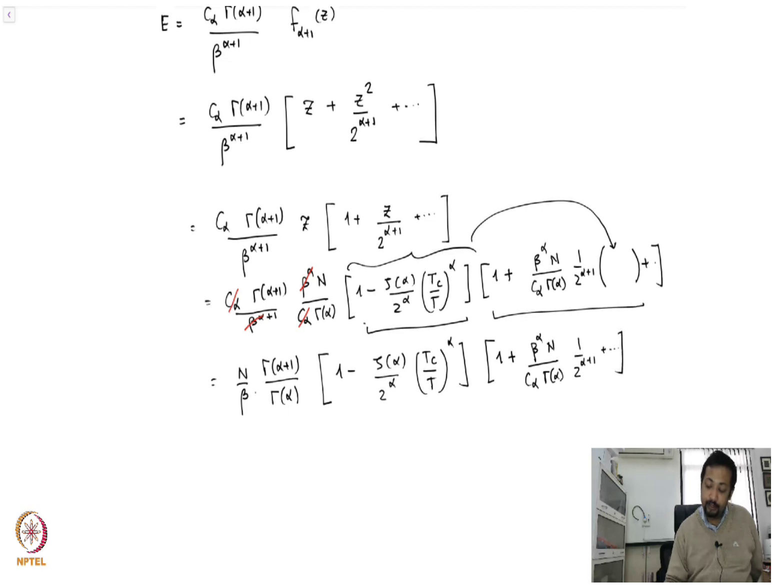

So, this is C alpha gamma alpha plus 1 raised to the power beta alpha plus 1 times Z 1 plus Z raised to the power 2 to the power alpha plus 1 and all the higher orders. Z now I use from this relation. See then Z is given by C alpha gamma alpha plus 1 divided by beta raised to the power alpha and beta raised to the power alpha sorry, the denominator has to be alpha plus 1. The numerator is beta raised to the power alpha then I have a N then I have a C alpha and they have I have a gamma alpha.

The next is zeta alpha over 2 to the power alpha T c over T raised to the power alpha. I hope I had a gap times 1 plus beta alpha N divided by C alpha gamma alpha 1 over 2 to the power alpha plus 1 and of course, you will have multiplying this plus higher order term.

So, this this expression is going to again enter over here because that is a value of this, but we will ignore because after all it is a product of this term and this term. So, we will write this as C alpha, lot of things cancel out. So, first let us look at the cancellations here. C alpha, C alpha cancels out.

A beta alpha cancels with beta alpha and then I am left out with N over beta times gamma alpha plus 1 divided by gamma alpha 1 minus zeta alpha raise to the power 2 to the power alpha T c over T raised to the power alpha 1 plus beta alpha beta to the power alpha N C alpha gamma alpha 1 over 2 to the power alpha plus 1 and higher order terms.

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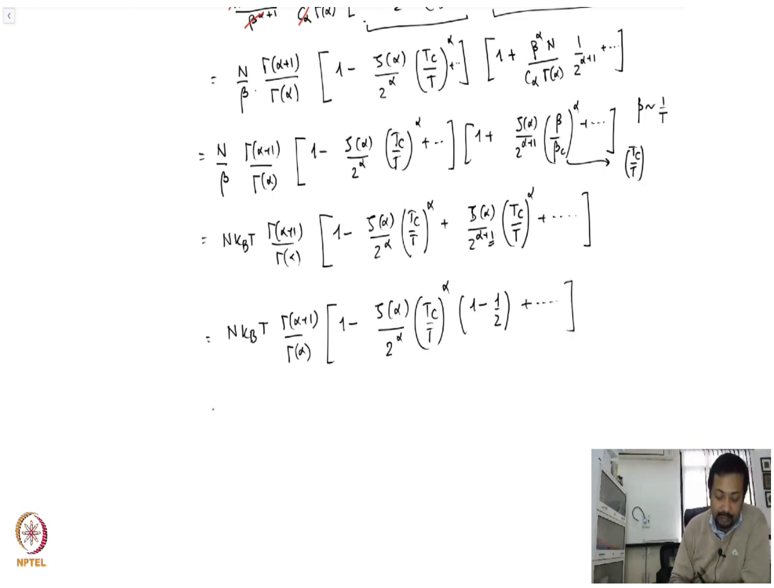

N over beta alpha plus 1 divided by gamma alpha 1 minus zeta alpha 2 to the power alpha T c over T raised to the power alpha plus higher order terms which we always have and should not forget. And then again you see that N is going to be C alpha gamma alpha beta C alpha zeta alpha. So, that N over C alpha gamma alpha is going to be 1 plus zeta alpha 2 to the power alpha plus 1 beta over beta C raised to the power alpha plus higher order terms.

Now, N by beta is N k B T gamma alpha plus 1 over gamma alpha. Let us do this. 1 minus zeta alpha 2 to the power alpha T c over T c over T raised to the power alpha and then you have plus 1 zeta alpha over 2 to the power alpha plus 1 T c over T raised to the power alpha plus higher order term because beta over beta C is essentially T c over T since beta goes as 1 over T.

Therefore, I have N k B T gamma alpha plus 1 over gamma alpha is going to be 1 minus zeta alpha 2 to the power alpha T c over T raised to the power alpha 1 minus half of because of this additional factor.

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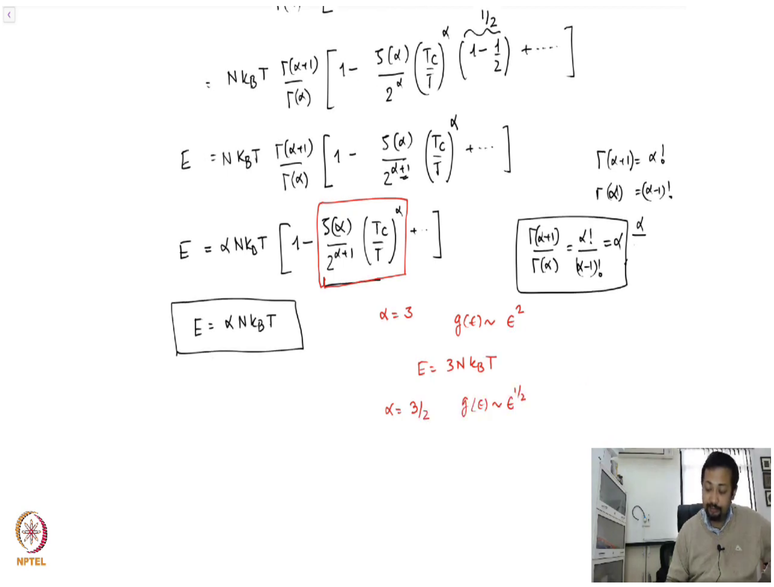

And this gives me the energy as N k B T gamma alpha plus 1 over gamma alpha 1 minus zeta alpha over 2 to the power alpha plus 1 T c over T raised to the power alpha. Since this is half, I get a factor alpha plus 1 plus higher order terms. Now, gamma alpha plus 1 is alpha factorial, gamma alpha is alpha minus 1 factorial. So, alpha, so, this ratio of gamma alpha plus 1 divided by gamma alpha is alpha factorial divided by alpha minus 1 factorial which is just plain alpha.

So, that E over; so, first we write down this as E is going to be alpha N k B T times 1 minus zeta of alpha 2 to the power alpha plus 1 T c over T raised to the power alpha plus has higher order terms. So, at really high temperatures you recover the classical result of E is alpha N k B T. And this term, the term that you see over here is the first order correction as you come down in temperature. This is the first correction that you will notice right

So, for example, if alpha is equal to 3 g epsa will go as epsa square which is our case for a harmonically trapped Bose gas and in which case we saw that the energy was 3 N k B T.

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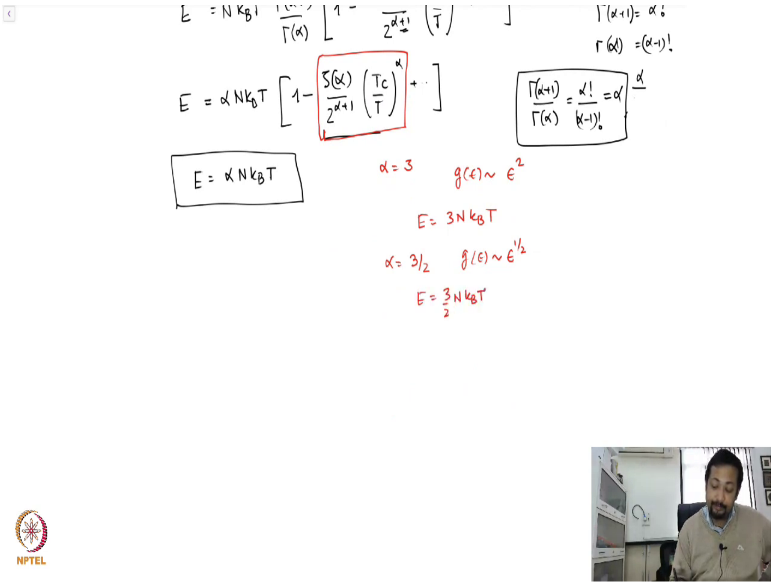

For alpha is equal to 3 half g epsa goes as epsa to the power half and E becomes 3 half N k B T.