Statistical Mechanics
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Lecture - 60
Specific Heat of a Harmonically Trapped Bose Gas

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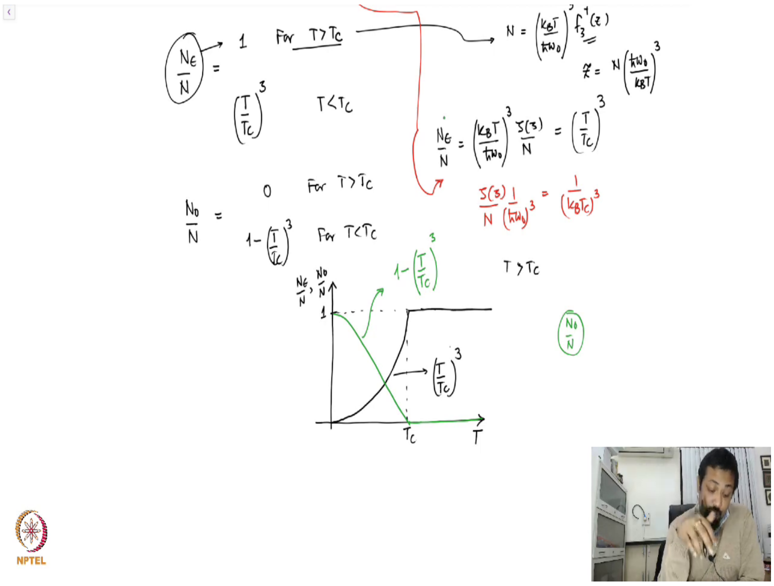

So, we have seen how the fraction behave N epsilon over n and N 0 over N behaves. We now want to look at the energy.

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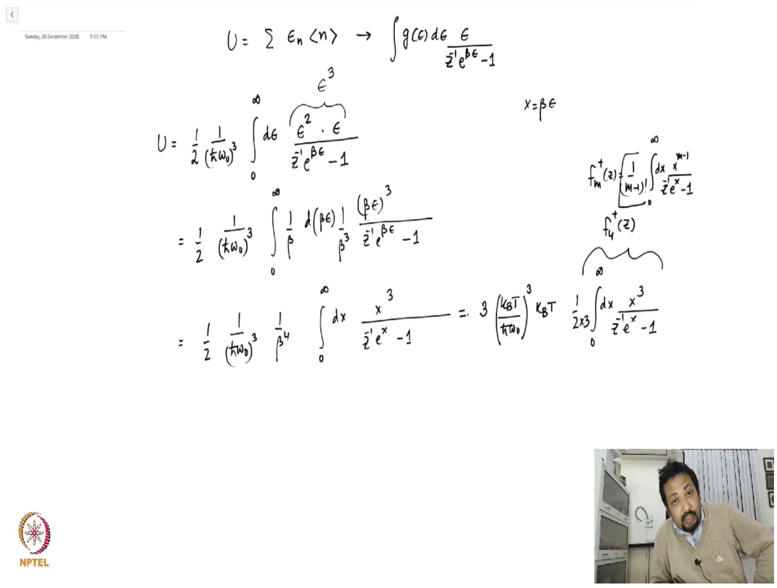

And the energy is as usual the energy is sum over epsa n average of n which is the occupation number and this becomes a continuous using the continuous density of state this goes to g epsa d epsa times epsa divided by Z inverse e to the power beta epsa minus 1. So, that we write down this as g epsa is half 1 over h bar omega naught whole cube integral 0 to infinity d epsilon epsa cube epsa square times epsa divided by Z inverse e to the power beta epsa minus 1.

Now, as usual we will say will use the transformation of variable x equal to beta epsilon. So, that I have 0 to infinity 1 over beta d of beta epsilon and then I have an epsa cube over here. So, that makes it beta epsa whole cube and I have 1 over beta cube Z inverse e to the power beta epsa minus 1, sorry there is a cube outside and over h bar omega naught.

So, this becomes half the beta cube and beta gives you beta to the power 4. So, I have h bar omega naught whole cube 1 over beta to the power 4, 0 to infinity dx x cube Z inverse e to the power x minus 1 which we rewrite as k B T over h bar omega naught whole cube times another factor of k B T and then I am left out with half 0 to infinity dx x cube Z inverse e to the power x minus 1.

Clearly, this integral is related to my function f m plus of Z which was 1 over m minus 1 factorial 0 to infinity dx x to the power m minus 1 Z inverse e to the power x minus 1. Now, you see that whatever I have within the integrant tells me that this is f 4 plus of z, but the factorial does not match. This is the 3 factorial for f 4 I have a half that is easily remedied by bringing a 3 factor in the denominator as well as in the numerator.

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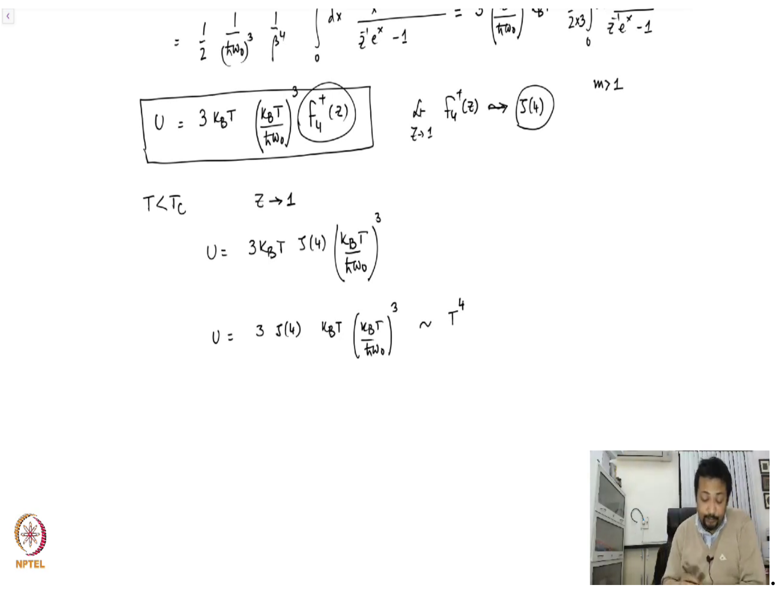

So, I have 3 k B T k B T over h bar omega naught whole cube and I have f 4 plus of Z and this is my energy I am looking at. Now, the idea is for T less than T c, I know Z goes to 1. Therefore, my f 4 plus of Z since m is greater than 1 as Z tends to 1, this function value go sorry goes to zeta of 4.

So, I will replace this by zeta of 4 where I write down this as 3 k B T zeta 4 and then I have k B T over h bar omega naught whole cube. So, that U is equal to let us say 3 zeta 4 k B T and k B T over h bar omega naught whole cube which implies the energy the internal energy scales with T to the power 4.

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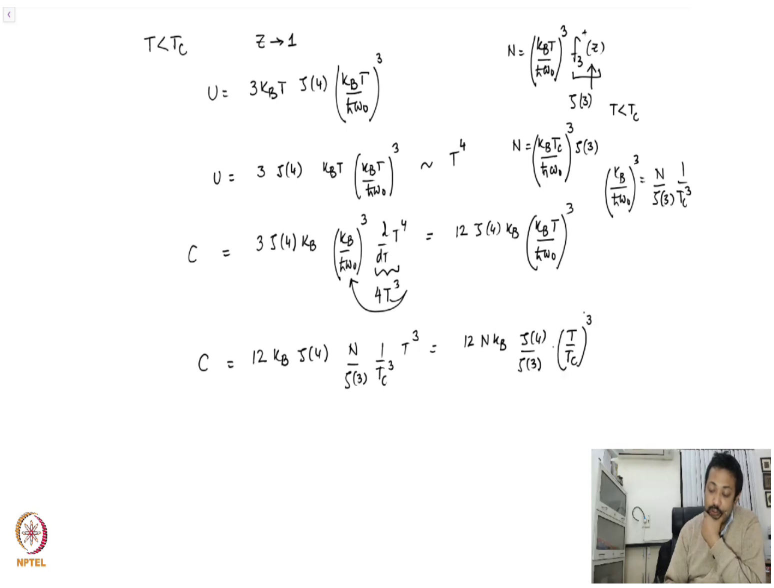

The specific heat at constant volume is given by 3 zeta 4 is del U del t. So, that I have k B, let us keep k B over here and then I have k B over h bar omega naught whole cube d dT of T to the power 4 and that is 12 zeta 4 k B k B T over h bar omega naught whole cube. So, this derivative brings in a factor 4 times T cube and this T cube I reobserve in the bracket to recast my equation for specific heat in this way.

Now, if you recall your N particle number was k B T over h bar omega naught whole cube f 3 plus Z. For T greater than T c from this is the equation from which you have to determine the fugacity, but for T less than T c my life is little bit simpler because this function now becomes zeta of 3.

So, that N is equal to k B T over h bar at T c we will substitute for at T c whole cube zeta of 3. Therefore, I can immediately see the expression and I can see that k B over h bar omega naught whole cube is nothing, but N over zeta 3 and 1 over T c cube. So, my specific heat becomes 12 k B zeta 4 and then I have N over zeta 3 1 over T c whole cube times T cube which takes a very simple form, which gives me 12 N k B zeta 4 over zeta 3 times T over T c whole cube.

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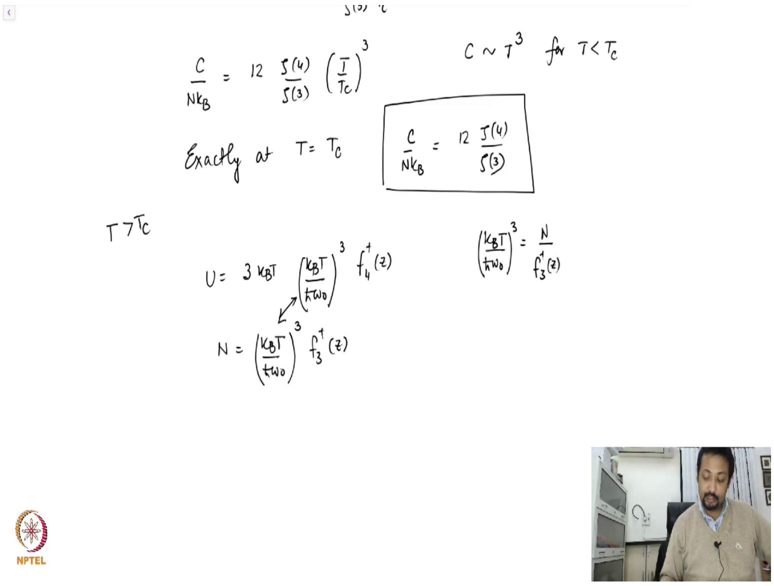

So, that C over N k B is going to be 12 zeta 4 over zeta 3 times T over T c whole cube. And exactly at T is equal to T c, the specific heat is given by 12 zeta 4 over zeta 3. The specific heat for T less than T c would scale as T cube for T less than T c. What about the high temperature case? Well, for that we know that U was given by 3 k B T k B T over h bar omega naught whole cube f plus 4 of Z.

And N this is the case where we are looking at T greater than T c and N is given by k B T over h bar omega naught whole cube f 3 plus Z. So, I can now eliminate this k B T h bar omega naught which is over here. So, I can write down k B T over h bar omega naught whole cube as N over f plus 3 of Z.

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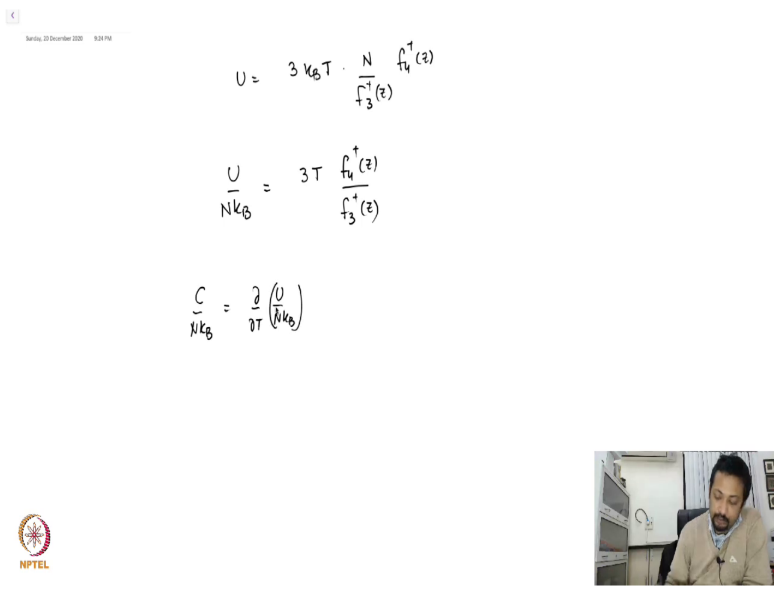

And I have the expression U is equal to 3 k B T times N over f plus 3 of Z times f 4 plus of Z. So, therefore, U over N k B is going to be 3 T times f 4 plus of Z and f 3 plus of Z. So, we want to do the again repeat the elaborate calculation and its instructive to do it. The specific heat C over N k B is going to be del del T of U over N k B.

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$$\frac{C}{Nk_B} = \frac{\partial}{\partial T}\left(\frac{U}{Nk_B}\right)$$

$$= \frac{\partial}{\partial T}\left[3T\,\frac{f_4^+(z)}{f_3^+(z)}\right] = 3\,\frac{f_4^+(z)}{f_3^+(z)} + 3T\left[\frac{f_3^+(z)\frac{df_4^+}{dT} - f_4^+(z)\frac{df_3^+}{dT}}{f_3^+(z)^2}\right]$$

$$\frac{C}{Nk_B} = 3\,\frac{f_4^+(z)}{f_3^+(z)} + 3T\,\frac{f_3^+(z)\frac{df_4^+}{dT} - f_4^+(z)\frac{df_3^+(z)}{dT}}{\left[f_3^+(z)\right]^2}$$

$$\frac{df_m^+}{dT} = \frac{1}{z}\,f_{m-1}^+(z)$$

This means this is going to be del del T of 3 times T times f 4 plus of Z f 3 plus of Z, which simplifies to 3 times f 4 plus of Z divided by f 3 plus of Z. First the derivative with respect to temperature and then you have plus 3 T divided by f 3 plus of Z whole square times f 3 plus of Z df 4 plus dZ sorry dT minus f 4 plus of Z df 3 plus dT.

Let us be careful with this to see whether everything is consistent now. So, that C over N k B is going to be 3 f 4 plus of Z f 3 plus of Z plus 3 times T f 3 plus of Z df 4 plus of dT minus f 4 plus of Z df 3 plus of Z dT divided by f 3 plus of Z square. Now, I know that d dT of f m plus is 1 by Z f m minus 1 plus of Z. This we did in the previous lectures times.

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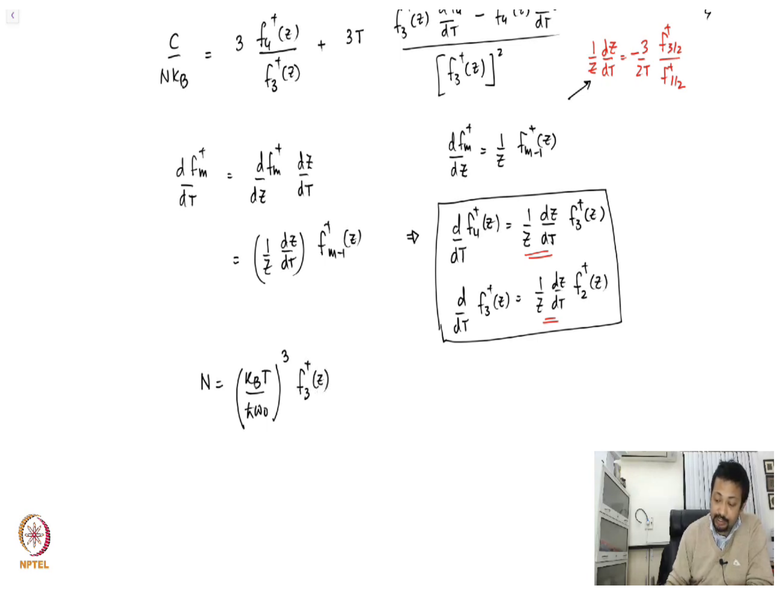

So, now, I know that d dT of f m plus is d dZ of f m plus and then dZ of Dt. And df m plus dZ is going to be 1 by Z f m minus 1 plus as Z. So, that this becomes 1 by Z dZ dT times f plus m minus 1 of Z. And this implies that f plus 5 sorry f plus. So, I have a derivative of d dT of f plus 4 of Z which is 1 over Z dZ dT times f 3 plus of Z and d dT of f 3 plus of Z is going to be 1 over Z dZ dT of f 2 plus of Z.

So, we are left to determine these derivatives. Now, in the classical in the ideal Bose gas we have seen that this derivative dZ dT is minus 3 over 2 T f plus 3 by 2 divided by f plus half. So, word of caution is do not use this here because you have a different density of states. Please be careful in blindly using results. Instead we start from the same starting from point from where using which we derived such a relation that is to look at the particle number.

Particle number is given by for a harmonically trapped Bose gas is given by k B T over divided by h bar omega naught whole cube times f plus 3 of Z.

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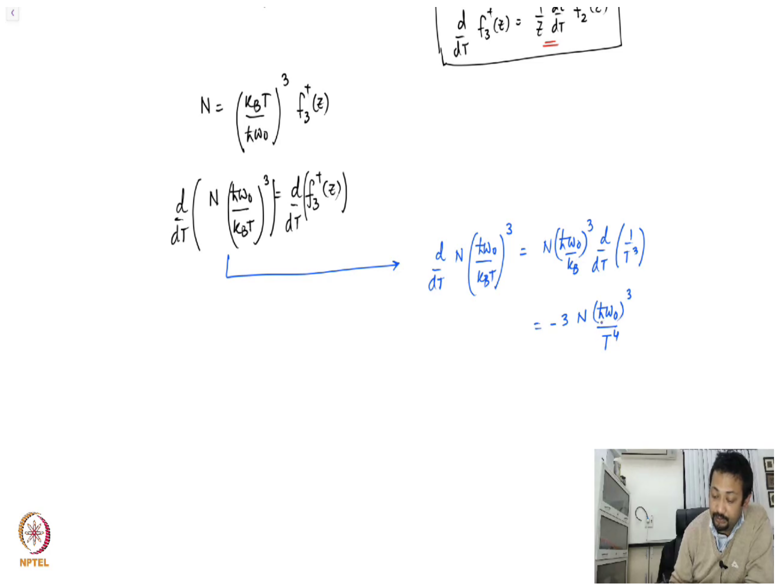

So, that I have N times h bar omega naught k B T whole cube is going to be f of f 3 of Z. Now, let us take a derivative with respect to temperature both in the left hand side as well as in the right hand side and let us look at the left hand side.

The left hand side is d dT of N h bar omega naught over k B T whole cube which is going to be N h bar omega naught whole cube d dT of 1 over T cube and that is a straightforward derivative. You are going to have minus 3 N h bar omega naught whole cube divided by T to the power 4 and I have missed a k B over here.

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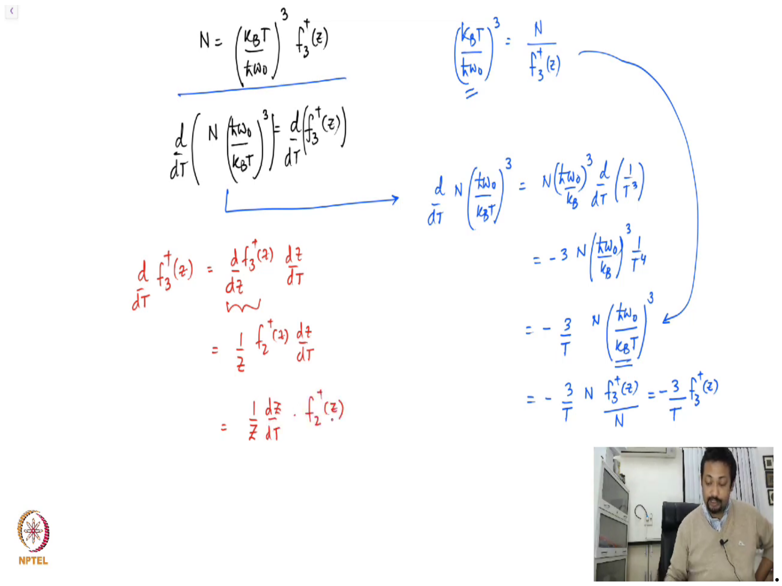

So, that this k B 1 over T to the power 4. So, that I have minus 3 over T times N h bar omega naught divided by k B T whole cube. But, if you look at this expression then k B T h bar omega naught whole cube is going to be N divided by f plus 3 of Z. And therefore, you have the answer minus 3 over T times N and I am going to use this expression over here noting that what I have here is just 1 over this value the left hand side.

So, that this gives me f plus 3 over Z divided by N and this answer is minus 3 over T f 3 of Z. The right hand side is d dT of f 3 of Z and that is going to be dZ d dZ of f 3 of Z times dZ dT and this derivative is 1 over Z f 2 plus Z times dZ dT. So, that the left hand side is 1 over Z dZ of dT times f 2 plus Z.

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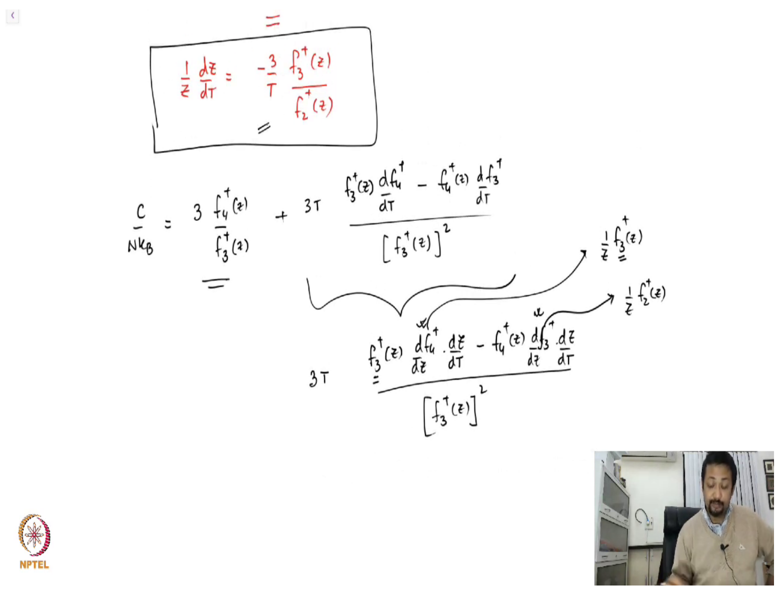

I can take this take this combine this to give me 1 over Z dZ dT is equal to minus 3 over T f 3 plus of Z divided by f 2 plus of Z. So, once I have this I take the expression of the specific heat which is going to be 3 f 4 plus divided by f 3 plus both functions of Z plus 3 T. Let us just write it down once again. It is a little bit laborious, but it is ok. So, I have f 3 plus of Z df 4 plus dT minus f 4 plus of Z df 3 plus dT divided by f 3 plus of Z whole square.

Let us take this expression and carry forward the calculation I know that I have the left hand I have this part we will add it at the right time. So, this is 3 T times f 3 plus of Z and this is d dZ of f 4 plus times dZ dT minus f 4 plus of Z d dZ of f 3 plus times dZ dT divided by f 3 plus of Z whole square.

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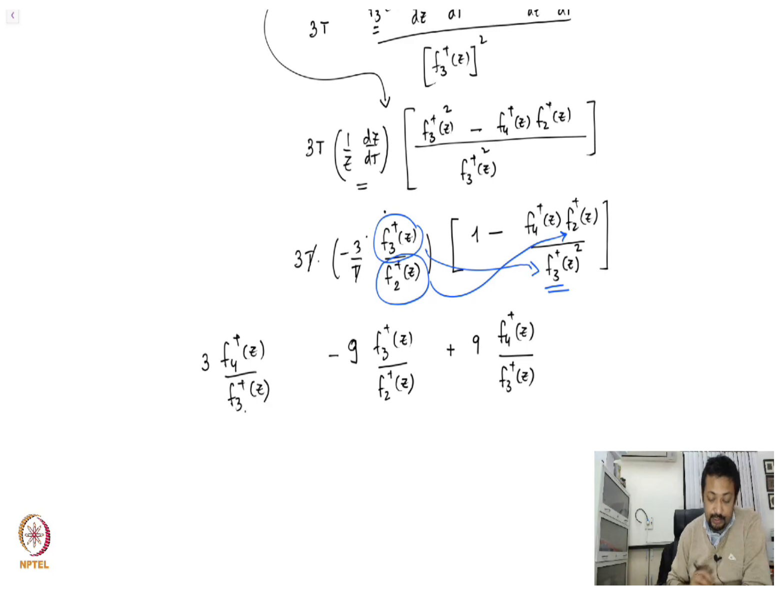

So, that this is 3. This I already know this derivatives. This derivative is 1 by Z f 3 plus of Z and this derivative is 1 by Z f 2 plus Z. Therefore, I have 1 over Z dZ dT that comes outside as a common factor and then I have f 3 multiplying f 3 giving me f 3 plus Z whole square minus f 4 plus of Z f 2 plus of Z divided by f 3 plus of Z whole square.

This also I know I just calculated one by Z dZ dT and the answer is this. So, I take this and substitute over here to give me 3 times T minus 3 by T f 3 plus of Z divided by I have f 2 plus of Z and this is 1 minus f 4 plus of Z f 2 plus of Z divided by f 3 plus of Z whole square.

So, that this expression T T gets cancelled out and the first term you see is minus 9 f 3 plus of Z divided by f 2 plus of Z. The second expression is plus 9 f 4 plus of Z and if you see that this factor of f 3 plus now cancels one factor of f 3 plus from the denominator and the f 2 plus completely cancels the f 2 plus over here.

So, that you have f 3 plus of Z. Now, look at it carefully and see let us add the first term. The first term is 3 f 4 plus of Z divided by f 3 plus of Z. So, that this and this are the same terms, well same forms

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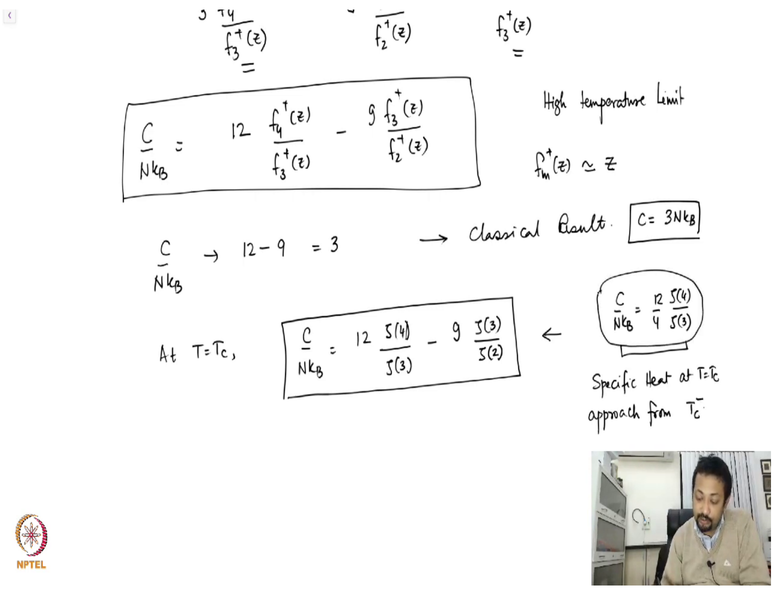

Therefore, the specific heat now becomes 9 plus 3 is 12 f 4 plus of Z divided by f 3 plus of Z minus 9 f 3 plus of Z divided by f 2 plus of Z. Question is this looks very nice. Now, we go to check whether we have arrived at the right and expression. So, for that the first thing to look is the high temperature expansion sorry the high temperature limit.

And in the high temperature limit Z is very small and I know that all of this f m plus Z is approximately Z. So, that the specific heat C over N k B which is 12 minus 9 which is equal to 3 and this is the classical result which is C is going to be 3 N k B. Now, is the interesting thing is what happens at T equal to T c? At T equal to T c one has C over N k B as 12 times zeta 4 divided by zeta 3 minus 9 times zeta 3 divided by zeta 2 the Riemann zeta function.

And here it is interesting because I clearly see that there is a difference right. There is a difference and this does not match the specific heat which we obtained at T equal to T c which was C over N k B was 12 by 4 zeta 4 over zeta 3. This is the result of the specific heat at T c approaching from T c minus. So, you are approaching that critical temperature from below the critical temperature and once you do that this is the result that you get.

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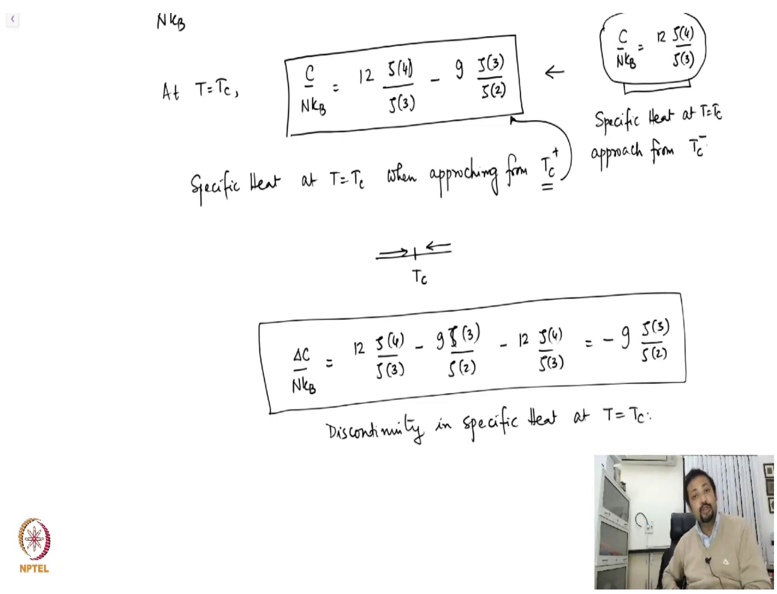

In contrast specific heat at T is equal to T c when approaching from T c plus. So, this is your T c. T C plus is approaching from this, T c minus is this and when you approach from T c plus you get this expression. So, there is a difference. And that difference delta C over N k B is 12 zeta 4 divided by zeta 3 minus 9 of zeta 3 divided by zeta 2 minus 12; sorry this is just 12, not my mistake minus 12 zeta 4 over zeta 3 which is minus 9 zeta 3 over zeta 2.

So, there is a discontinuity in the specific heat in specific heat at the critical temperature T equal to T c.