Statistical Mechanics
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Lecture - 59
Bose - Einstein Condensation in a Harmonically Trapped Bose Gas

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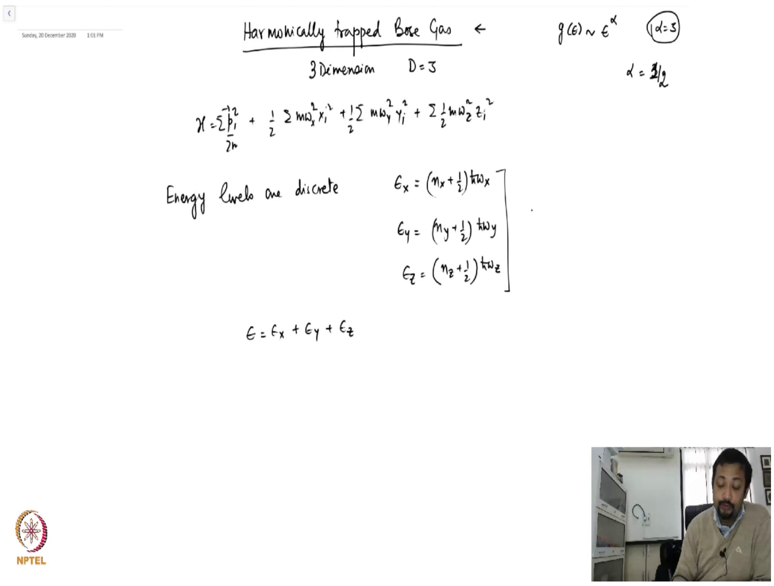

So, we want to now look at a Harmonically Trapped Bose Gas and our motivation is that this is typically what is realized in experiments, but the second point is also that we will see that g epsilon is different, even though that this system is in 3 dimension. So, that D is equal to 3 your g epsa is has a different behavior as epsa to the power alpha where alpha is equal to 3 for a free bose gas we knew that alpha was 3 by sorry, alpha was 1 by 2, but here we want to look at it.

Now, the Hamiltonian is p square over twice m plus half m omega square x i square plus half. So, one can have an anisotropic harmonic potential. So, that you have omega y square y i square plus half m omega z square z i square the half factor should we not be missed, but the energy levels are discrete.

So, that we have epsa x is n x plus half h bar omega x, epsa y is n y plus half h bar omega y and epsa z is n z plus half h bar omega z. So, the total energy is now epsa x plus epsa y plus epsa z.

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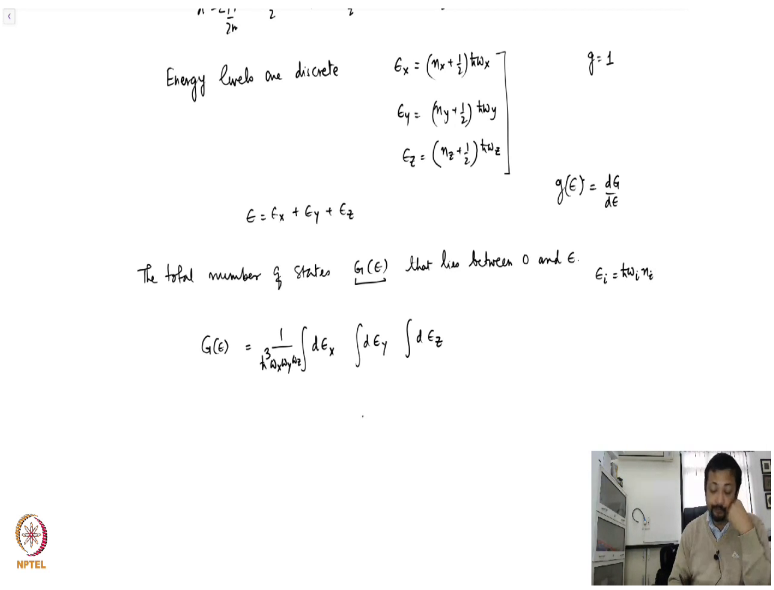

We will ignore the ground state energy and try to figure out the total number of state. The total number of states G epsa that lies between 0 and epsilon. The idea is that once I know this number then g epsa the density of state is given by d G d epsa. Further we also consider particles which do not have internal degrees of freedom. So, that we will set the factor g is equal to 1, if you do have particles which has internal degrees of freedom, then you need to multiply that factor.

Now, g epsa capital G of epsilon is then given by integral d epsa x d epsa y and d epsa z divided by h cube omega x not much of a space here. So, 1 over h bar whole cube omega x omega y and omega z is equal to capital G of epsa.

Here, we have essentially is defined the variables or defined a coordinate system, if the variable h bar omega i n i and this gives me the total density of total number of states between the energies 0 and epsilon, but one has to be careful about the integration limit.

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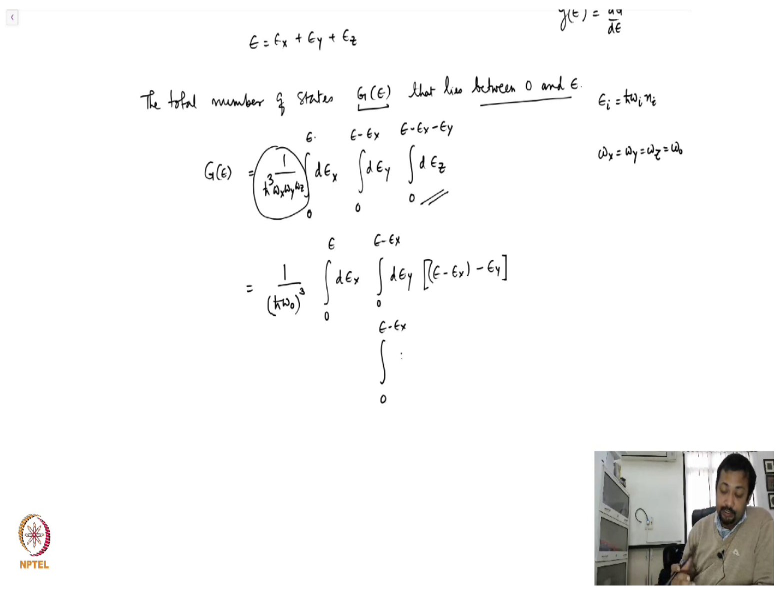

Clearly, epsa z can go from epsa minus epsa x minus epsa y and epsa y can be from 0 to epsa minus epsa x and epsilon x will be the from 0 to epsilon. Well, we want to carry forward this cumbersome factor that sits outside, but here we make the assumption that omega x equal to omega y is equal to omega z is equal to omega naught.

So, that I have h bar omega whole cube and then the integral is 0 to epsilon d epsa x, 0 to epsa minus epsa x d epsa y, if you carry out the integral over epsa z you will be left out with epsa minus epsilon x minus epsilon y 0 to epsa minus epsa x.

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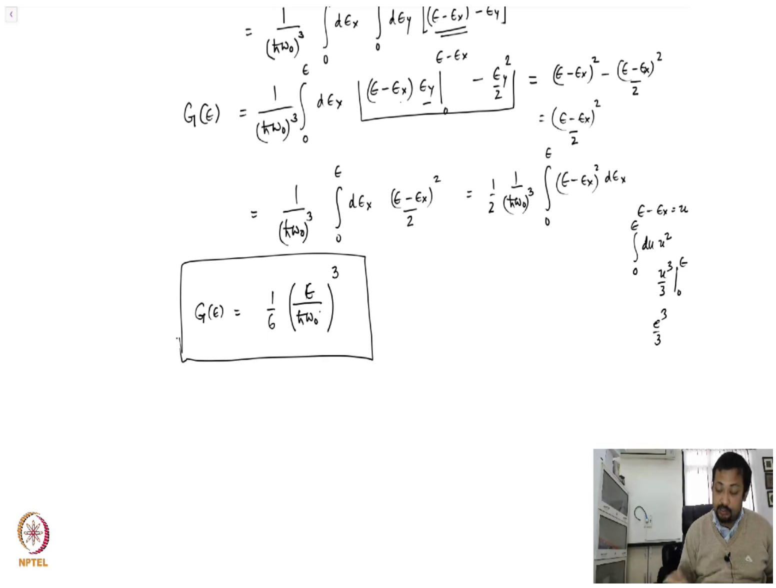

The first integral, let us now carry out the integration over epsa y.

So, at the first term is going to be epsa minus epsa x times epsa y 0 to epsilon minus epsilon x minus epsa y square by 2 and then I have an integral over epsilon x 0 to epsa outside I have a factor h naught omega whole cube and this is your total number of energy states that is available between the energies 0 and epsa.

So, 1 over h naught omega naught whole cube 0 to epsilon d epsilon x substitute this, this part can simplify to epsa minus epsa x whole square minus epsa minus epsa x whole square divided by 2 so, that I have epsa minus epsa x whole square by 2. So, epsilon minus epsilon x whole square divided by 2, which is going to be half 1 over h naught omega 1 over h bar omega naught whole cube, 0 to epsa, epsa minus epsa x whole square d epsa x.

This integration is very easy to do, if you substitute epsa minus epsa x as u then essentially you have 0 to epsilon u square d u which is going to be u cube over 3 0 to epsilon and that gives you epsa cube over 3. So, that the total number of energy states lying between 0 and epsilon is going to be 1 by 6 1 by h naught omega epsilon by h naught omega whole cube.

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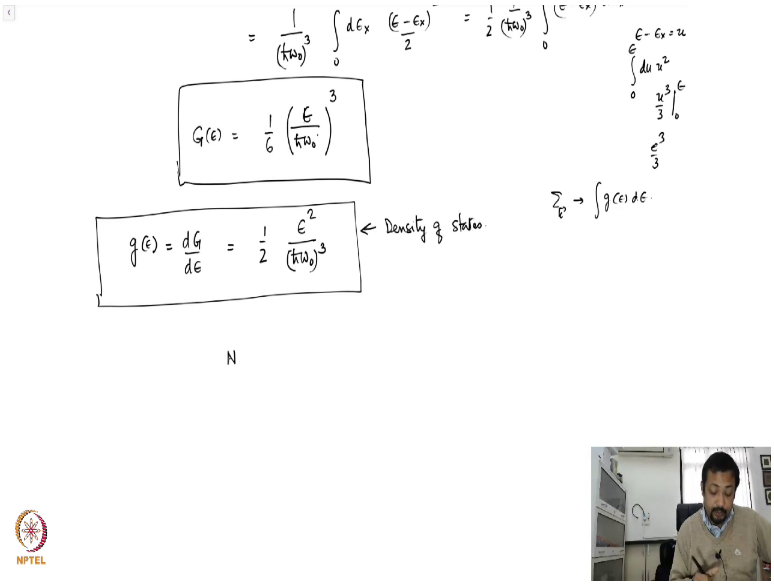

So, this has a very nice x expression. The density of states is g epsa small g of epsa and that is which is of primary importance to us is d G d epsa, which is going to be half epsa square divided by h bar omega naught whole cube. We are going to work with this expression for the density of states why is it so, important to us? Because, I know that all the discrete sums over k I can replace by integration g epsilon d epsilon and that is precisely what we plan to do.

The starting point then is going to be the particle number; well one can also look at the canonical grand.

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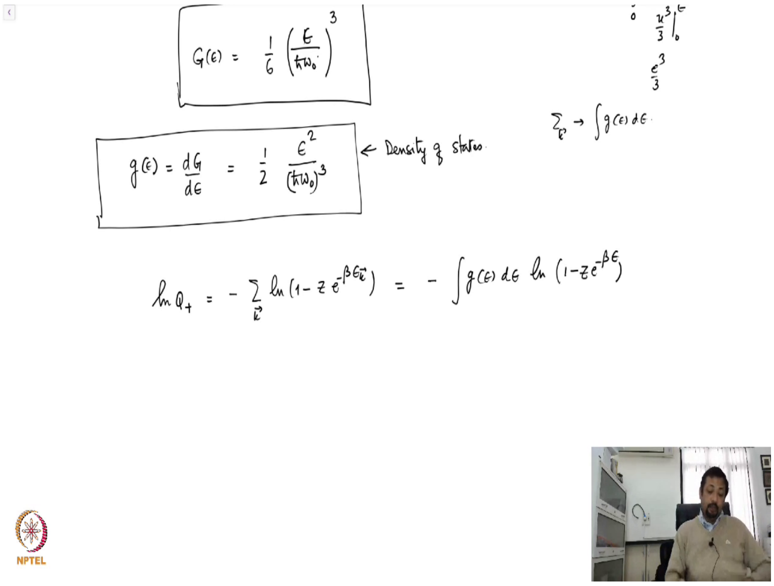

Let us look at the grand canonical partition function and the grand canonical partition function is given by minus sum of a k l n 1 minus z e to the power minus beta epsilon k which is going to be minus integral g epsa d epsa l n of 1 minus z e to the power minus beta epsa. Now, remember we also have to separate out the ground state that is the half h bar omega.

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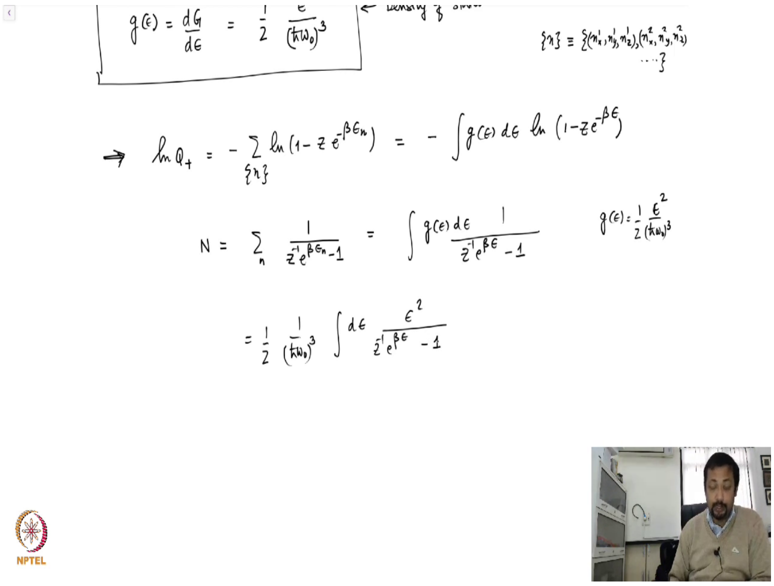

So, here one has to be careful because, the sum over k is now going to be replaced by the sum over n, where this set is essentially equivalent to n 1 x n 1 y n 1 z then I have n 2 x n 2 y n 2 z so on and so forth right. But we will simply write down this and not separate out the ground state right now we will not worry about that. In fact, we in the calculation that we want to do later on, we do not need this grand canonical partition function we want to calculate the critical temperature first.

The critical temperature for that we need for the critical temperature for the Bose Einstein condenser to appear in such a system.

Now, for that I need the total particle number and the total particle number is given by sum over n 1 over z inverse e to the power beta epsa minus 1 and this is going to be integral g epsilon d epsilon 1 over z inverse e to the power beta epsa minus 1, that we plugin the expression for g epsa which was half 1 over h bar omega naught whole cube times epsilon square.

So, I have a half factor, 1 over h bar omega naught whole cube and then I have integral of d epsa epsa square divided by z inverse e to the power beta epsa minus 1.

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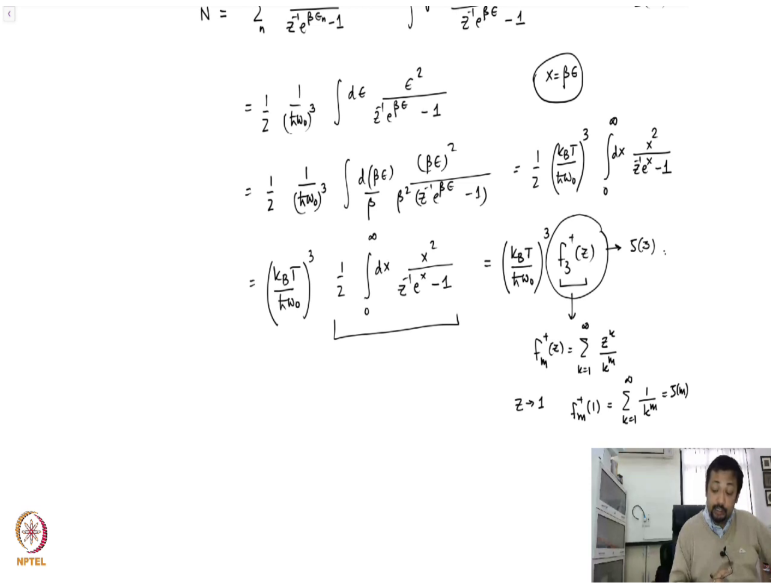

This integral, now I can substitute x is equal to beta epsa and if x is equal to beta epsa you see, this means that, I have 1 over h naught h bar omega naught whole cube integration d of beta epsa. I have divided by beta and I have beta epsa whole square over z inverse e to the power beta epsa minus 1 divided by beta square,

So that the expression takes a nice form omega naught whole cube integral 0 to infinity d x x cube sorry, x square z inverse e to the power x minus 1. And this is k B T over h naught omega whole cube half of 0 to infinity d x x square z inverse e to the power x minus 1, but this form of the integral is very very familiar to me.

So, I know that this is going to be k B T h naught h bar omega naught whole cube f of 3 plus z right. Now, clearly as you decrease the temperature your z also changes. So, you hit upon the limit where this function will have a limiting value of if you recall that f m plus of z is sum over k equal to 1 to infinity, I have z to the power k divided by k to the power m and when I hit z to 1, I have f m plus 1 going to be sum over k equal to 1 to infinity, 1 over k to the power m which is zeta m the Riemann zeta function.

So, this value of this function, now goes and hits the upper bound which is zeta of 3.

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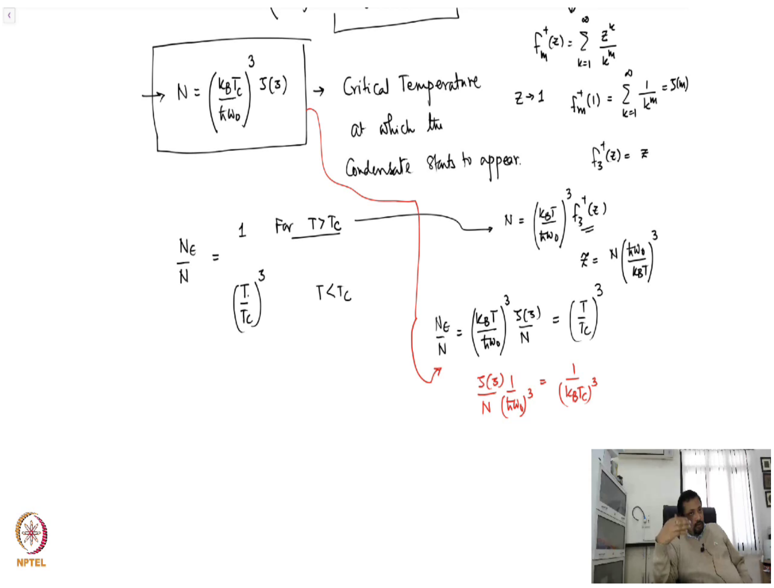

So, your temporal the condition that the condensate starts to appear is given by N is going to be k B T h bar naught h bar omega naught whole cube zeta times 3 and we put a subscript over here that denotes that this equation essentially gives me the critical temperature, at which the condensate starts to appear.

So, essentially it is at this temperature, you have the particles starting to populate the k equal to 0 momentum state. The ground sorry, but the k equal to 0, but the ground state of the system because it is no longer a free particle system one has to keep that in mind. Therefore, using our analysis that we did for the free particle gas, we know that in the excited states if I just look at this fraction N epsilon divided by N the total number of particle this is going to be 1 for T greater than T c.

So, once again then when you have high temperatures you will have k B T c sorry, k B T over h naught h bar omega naught whole cube f 3 plus z, in this case you will have this equation that is valid and you have to determine the fugacity from this relation and that effectively tells you that z is to the lowest order, I know that f 3 plus z is going to be z to the lowest order. So, that z is going to be N times h bar omega k B T whole cube, this is going to be to the to the lowest order.

For T less than T c, I have f 3 plus z being substituted by zeta of 3. So, that you have n epsilon for T less than T c, one can do the very simple calculation, we will do it on the side and plug it in there N epsilon over N is going to be I have k B T over h bar omega loss whole cube zeta of 3 divided by N. And if you see that this quantity is going to be using this relation I have zeta 3 if I use this relation over here, I write down essentially zeta 3 divided by N 1 by h bar omega naught whole cube is going to be 1 over k B T c whole cube.

So, that this expression becomes T over T c whole cube. Remarkably, you see this factor has now changed, for an ideal Bose gas this factor was T by T c raised to the power 3 half, but this is now T by T c raised to the power 3 and this factor solely depends on the density of states. When we basically intuitively said that how why does a specific heat of an ideal Bose gas near 0 temperature scales as T to the power 3 by 2, we said that look I can excite particles for finite temperatures,

I can excite them and they are at most going to get energy levels go to energy levels with the momentum values which is k m determined by h bar square k bar square k m square over twice m is equal to k B T. However, it should all and then we said that the total energy was volume times k m raised to the power d by 2 sorry k m raised to the power d times k B T, because each of these particles carry an thermal energy k B T, but this k m raised to the power d follows from the density of states.

So, even though you can still be in 3 dimension, but as you are seeing in this example my value of d the exponent that comes in is clearly different it is no longer 3 by 2, but it is 3 and that is solely the property of density of states that you should keep in mind.

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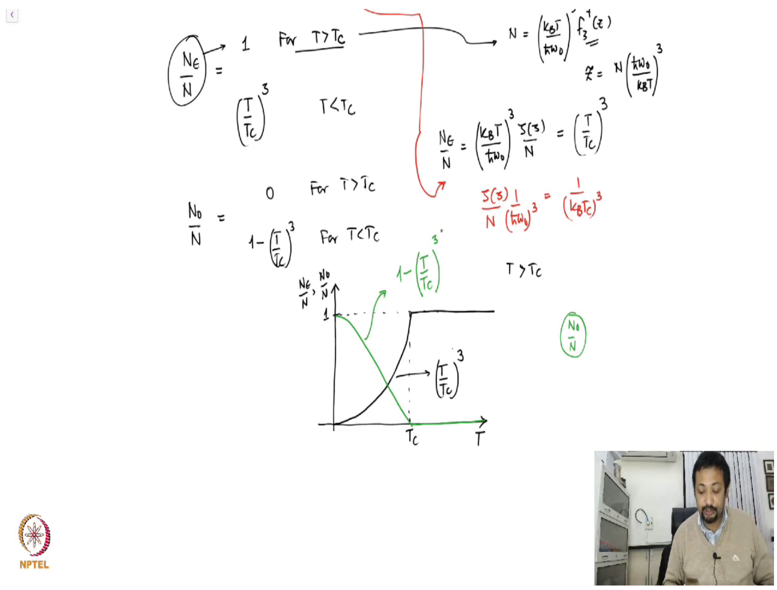

For temperature for the ground state N 0 over N is going to be 0 for T greater than T c and it follows from this expression is going to be 1 minus T over T c whole cube for T less than T c.

So, one can of course, plot this now. So, as we did for the free particle case we can plot this fraction of N epsa over N and N 0 over N as a function of temperature. So, this is my unity and this is my T c. So, I know that for T greater than T c my ratio of N epsa over N is unity. So, this remains like this and as I cross the critical temperature this expression follows T over T c, let us write it here T over T c whole cube.

In contrast, the fraction N 0 over N the number of particles populating the ground state is 0 for all temperatures below T c, because you can accommodate this particles in the excited states, and as you keep on reducing the temperature closer to below T c, then essentially you start populating the ground state more and more, so that at t equal to 0 this ratio is 1 and therefore, you have a nice plot that goes.

So, this is let us do it a little bit more carefully; that goes like this and this is 1 minus T over T c raised to the power 3.