Statistical Mechanics
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Lecture - 58
Specific Heat of an Ideal Bose Gas - Part 02

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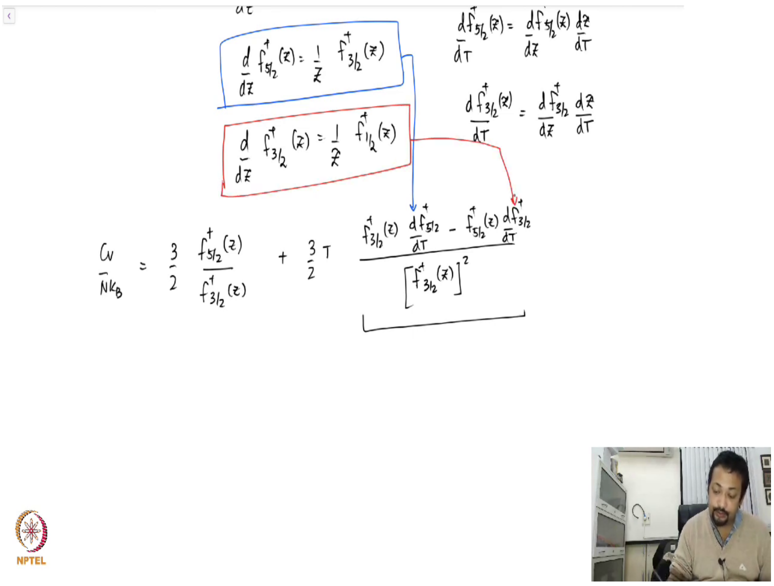

So, let us go back to the original calculation that we started off with. So, the specific heat C v over NK B is going to be 3 by 2 f plus 5 by 2 of z divided by f plus 3 by 2 of z plus 3 by 2 T f plus 3 by 2 of z whole square f plus 3 by 2 z df plus 5 by 2 dT minus f plus 5 by 2 of z d of f plus 3 by 2 dT. This expression now, what we will do here is we will replace.

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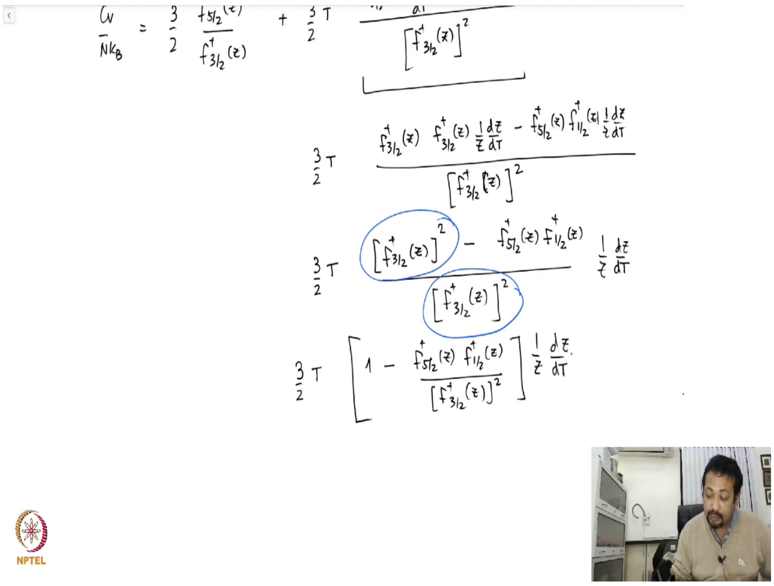

We will reuse this over here and this over here, which means this gives us 3 by 2 T f plus 3 by 2 of z f plus 3 by 2 of z times 1 over z dz dT minus f plus 5 by 2 over z times f plus half over z times 1 over z dz dT divided by f plus 3 over 2 of z whole square. And therefore, this becomes 3 by 2 f plus 3 by 2 over z whole square minus f plus 5 over 2 z times f plus half of z divided by f of f plus of 3 by 2 z whole square 1 over z dz dT.

So, we can simplify this further and we will have 3 by 2 T times this gives you 1, if you take this divided by this; this gives you 1 and you have 1 minus f plus 5 by 2 z f plus half of z divided by f plus 3 by 2 of z whole square the whole thing in a bracket and I have 1 by z dz dT.

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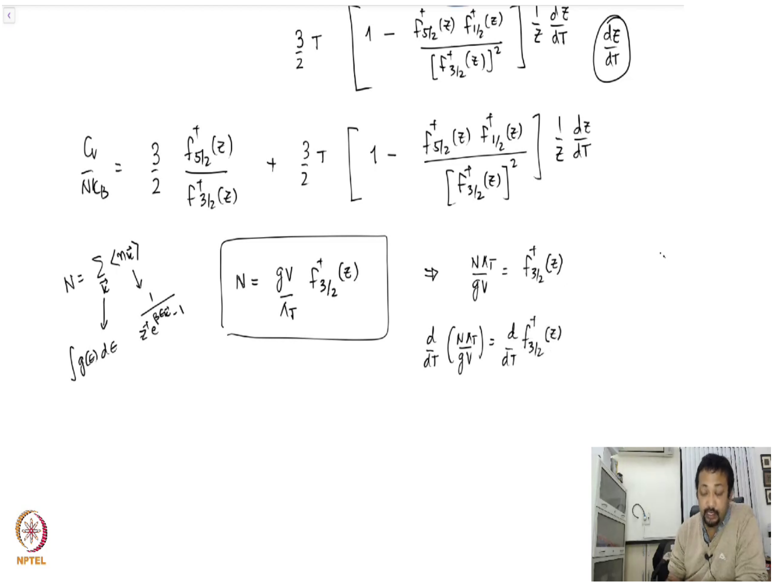

So, that the specific heat over NK B, do not forget the first term; I have three-half f plus 5 by 2 of z f plus 3 by 2 of z plus 3 by 2 T 1 minus f plus 5 by 2 of z f plus half of z f plus 3 by 2 z whole square 1 over z dz dT. Now, it remains for us to determine this derivative dz of dT and I note that this is not very simple task.

Why is that? Because, it is not that z is equal that z is equal to e to the power beta mu; but one can be tempted to just take the derivative with of beta with respect to T and we done away with. But one has to also note that the chemical potential needs to be determined as a function of T. So, this is a very complicated non trivial task, but you can simplify this.

For that, we start off with the expression for the particle number which was N was gV over lambda T f plus three-half z. And if you do not recall how we got this, here we wrote down the total particle number as the sum over the average occupation number in the energy levels and in the kth in the k energy levels. And this we knew that has the expression z inverse e to the power beta epsa k minus 1 and this sum into an integral in terms of the density of states g epsa d epsa;

Where, epsa goes as epsa to the power half; sorry g epsa goes as epsa to the power half and then, we got this result like this. So, this means that N lambda T over gV is going to be f plus 3 by 2 of z. Good. Now, I want to take a derivative with respect to temperature in the left hand side as well as in the right hand side d dT of f plus 3 by 2 of z.

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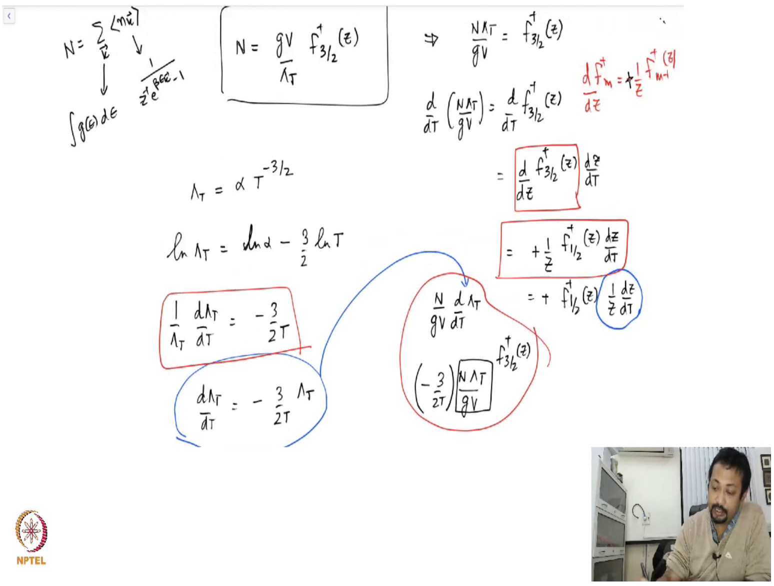

The right hand side is very easy to evaluate. We will do that in a straightforward way by writing d dz. So, the derivative with respect to temperature, I will convert it into a derivative with respect to z, z and then I will have dz of dT.

But this one, I know because f plus m d of m, I derived this as minus 1 over z f plus m minus 1 of z. So, it follows that this derivative is minus 1 over z f plus half three-half minus 1 is half z dz dT. So, that I have minus f plus half of z 1 over z dz dT.

You see in the right hand side, I precisely have the term which I am looking for which is already there in the specific heat. I now, have to evaluate the left hand side. The left hand side is n lam sorry N gV d d T of lambda T. Lambda T, I know I will write down this alpha T to the power minus three-half. Alpha is a constant which I am not very interested in.

Right now, in my expression, so that ln lambda T is ln alpha minus three-half ln T and therefore, 1 over lambda T d lambda T dT is going to be minus 3 by 2 T which means d lambda T dT is going to be minus 3 by 2 T times lambda T. And if I replace this expression over here, I come up with N lambda T over gV minus 3 by 2 T.

But surprisingly, this quantity I have already started off with. So, which is f plus 3 by 2 of z. So, if I use all this information now, this part, this part and then this going in the left hand side, I have my mistake, there cannot be a minus. Here this is going to be plus. So, this is plus, the equation we derived was plus here.

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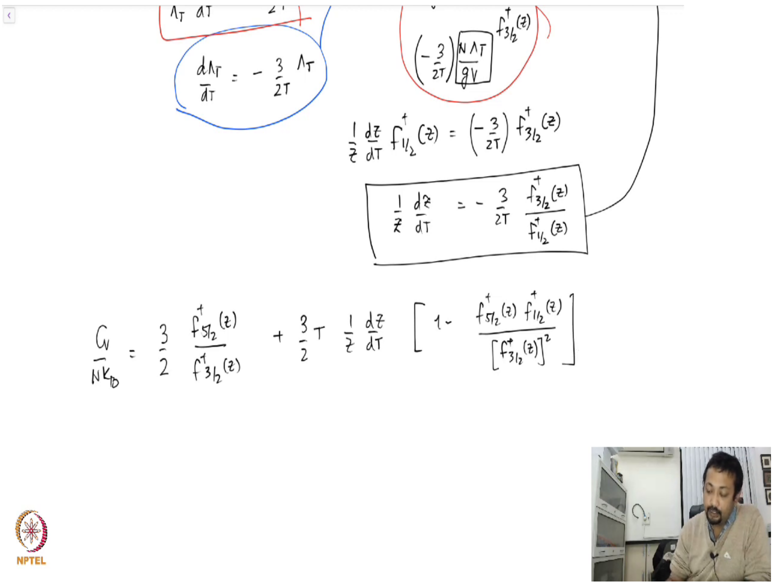

I have f plus half of z times 1 over z dz dT is going to be minus 3 by 2 T f plus 3 by 2 of z. So, that 1 over z dz dT is going to be minus 3 over 2 T f plus 3 by 2 of z divided by f plus half of z and this result. Now I take from here and substitute it over here in the expression for my specific heat. So, that I come up with C v over NK B was 3 by 2 f plus 5 by 2 z f plus 3 by 2 z plus 3 by 2 T 1 by z dz dT 1 minus f plus 5 by 2 z f plus half z f plus 3 by 2 of z whole square.

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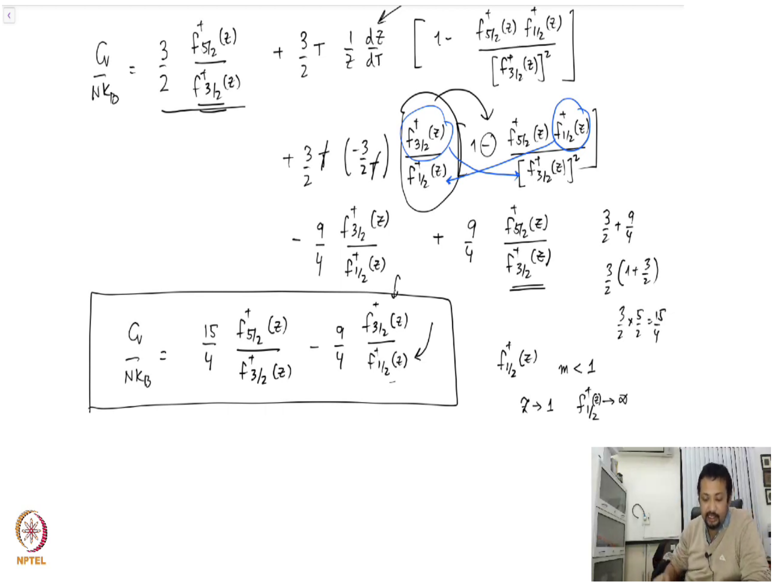

And this is going to be plus 3 by 2 T. So, this over here, I have minus 3 by 2 and 1 by times 1 by T and I have f plus I have f plus 3 by 2 z divided by f plus half of z 1 minus f plus 5 by 2 of z f plus half of z divided by f plus 3 by 2 of z whole square complete the bracket. Take this inside. If you take this inside, then you see first of all, this T and this T gets cancelled out and you have the product of the minus come, minus gives you this gives.

So, product of minus and plus gives you a minus. So, that you have minus 9 by 4 and then the first term is f plus 3 by 2 of z divided by f plus half of z. Now, you notice that one factor of f plus 3 by 2 of z cancels with one factor here. And the f plus half in the numerator is completely cancelled by the f plus half in the denominator. So, that you will have f plus 5 by 2 of z divided by f plus 3 by 2 of z and then you have a minus of 9 by 4 and a minus over here that gives you 9 by 4.

If you follow this carefully you see that this term has the same form as this term, the pre factors are different, we simply add up which is 3 by 2 plus 9 by 4 which gives you 3 by 2 1 plus 3 by 2 that is 5 by. So, 3 by 2 into 5 by 2 which is 15 over 4 and you have C v over NK B is equal to 15 over 4 f plus 5 by 2 of z divided by f plus 3 by 2 of z minus 9 by 4 f plus 3 by 2 of z. And divided by f plus half of z and this is the expression that we are originally after.

But we have to run checks and the first check that we have to run is does it match at T c with the expression. We had earlier when we derived the specific heat for temperatures less than T c. Now, you would see the expression and if you look at focus on the second term in the expression, you see the in the denominator. I have f plus half of z and here for this function m is less than 1.

So, that as z tends to 1, this f plus half tends to of z tends to infinity. So, as you keep on decreasing the temperature, the denominator blows up. So, that the second term starts contributing less and less and less until at T c, this vanishes completely.

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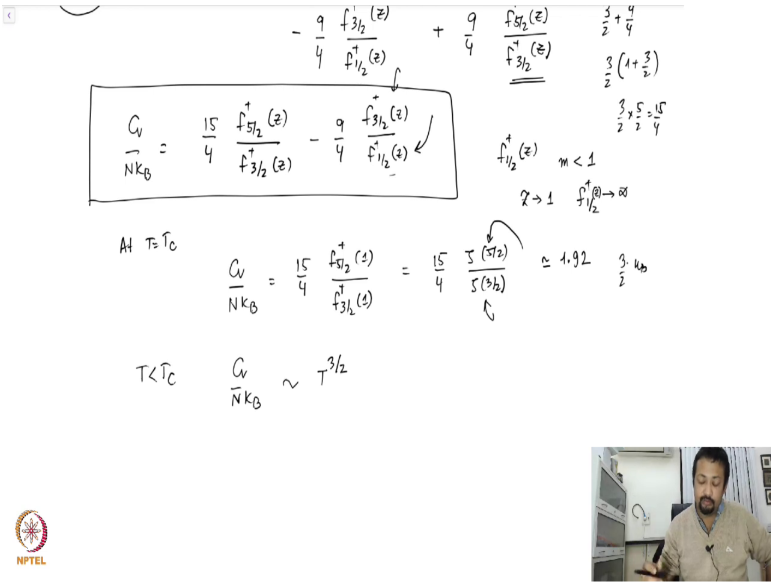

So, at T is equal to T c, you simply have C v over NK B is equal to 15 over 4 f 5 by 2 plus 1 divided by f plus 3 by 2 of 1 which is 15 by 4 zeta 5 by 2 and zeta three-half. Exactly matches the specific heat that we did when we looked at the case T less than T c.

For specific heat T less than T c, I know that C v over NK B goes as T to the power three-half right interestingly. Now, if you put in these numbers for the Riemann zeta function, you will see at T c this number is approximately 1.92.

It is more than what Dulong-Petit law tells you. That is, essentially 3 by 2 k B right and k B. So, well, in this case, it is going to be 3 by 2. If you now just go back for a second, go back to the specific heat, this expression and look at very high temperatures. Do I get back my classical result that is my second check?

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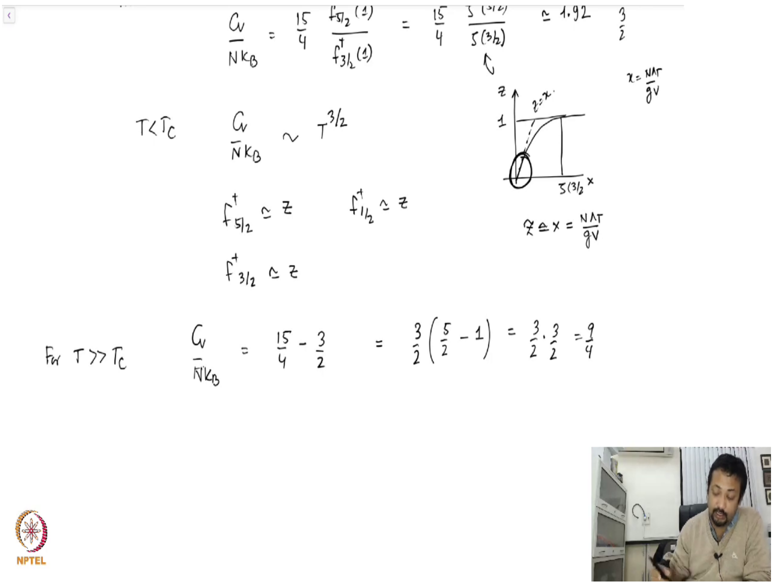

So, I know that for high temperatures, if I plot z as a function of x, where x was n lambda T over gV. This is the behavior and this is one beyond zeta three-half and for small enough values. This is going to be z is equal to x and we are interested in this region, for small than of values of x, so that I have z is approximately equal to x equal to N lambda T over gV.

If that is the case, then f plus 5 by 2 is approximately equal to z; f plus 3 by 2 is approximately equal to z and f plus half is also approximately equal to z. That is a leading order correction that is a leading order term.

So, that I have C v over NK B for temperatures much much larger than the critical temperature. I have 15 by 4 minus 3 by 2. Then, this answer is going to be if I take 3 by 2 common, I have 5 by 2 minus 1 which is going to be 3 by 2 into 3 by 2 that is 9 by 4 and there is something wrong here.

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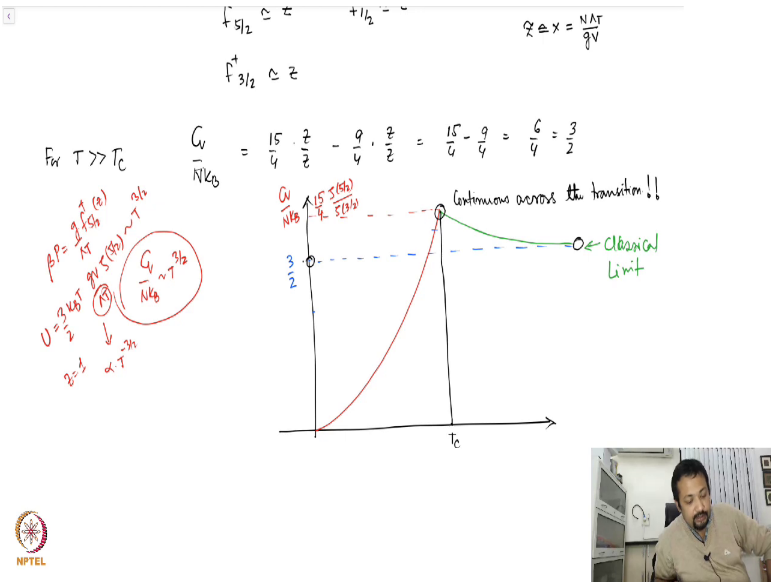

So, I know where the mistake is. For very high temperatures, for T much much larger than T c; therefore, C v over NK B becomes 15 by 4, 15 by 4 times z divided by z minus 9 by 4 times z divided by z which is 15 by 4 minus 9 by 4 which is equal to 6 by 4 and that is exactly equal to 3 by 2.

So, far away from the critical temperature, you get back the classical result of Dulong-Petit law. So, if I now want to plot the specific heat for T less than T c, C v over NK B goes as T to the power 3-half. So, this is C v over NK B right.

If you are still if you have forgotten how do we got this result, then I know that beta P was g over lambda T f plus 5 by 2 times z. And from this, we calculated U was 3 by 2 k B T gV over lambda T f plus 5 by 2 of time z and we substituted z equal to 1 for T less than T c which gave us zeta 5 by 2. And since therefore, I have a 1 by lambda T in the this goes as T to the power three-half. Since this quantity goes as alpha T to the power minus 3-half.

So, therefore, I plot now this value is going to be 15 by 4 zeta of 5 by 2 divided by zeta of 3 by 2 right and that is roughly 1.92 and yet, we have another value which is 3 by 2 and that is the Dulong-Petit law and then, well, it looks a little ugly. So, I think one has to be careful now. Let us draw it little bit to scale. So, this is roughly 2 and therefore, this distance is 2 means this distance is 1 and therefore, I have 1.5 is this.

So, this is going to be 3 by 2 approximately and therefore, one expects that the high temperature curve should look something like this right. So, this is the classical limit and this is where your critical temperature T c is, it is continuous across the transition. The specific, it is continuous across the transition; not only that, exactly at the critical point, its value is larger than the classical limit of three-half, that is essentially of three-half that is the Dulong-Petit law.

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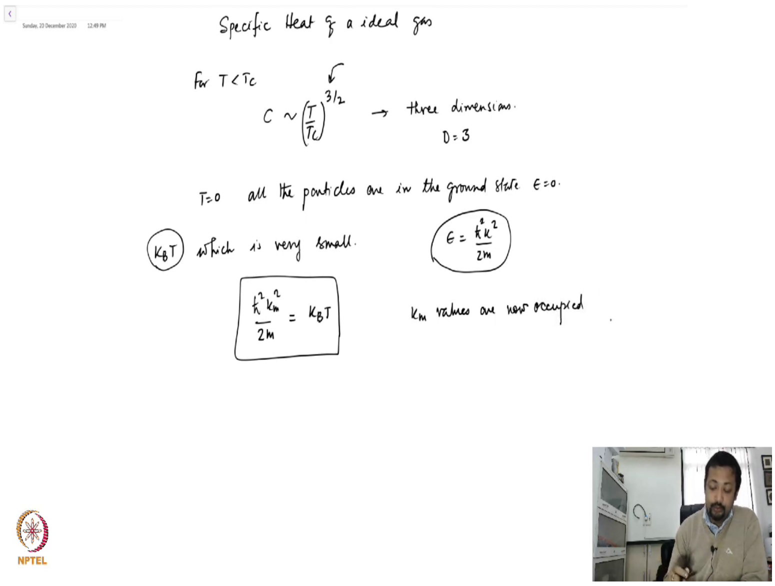

The specific heat of an ideal Bose gas for T less than T c, the specific heat goes as T over T c raised to the power three-half. Now, this result is for three-dimensions, which means the dimensionality of space is 3; D is equal to 3. Now, we want to argue for any arbitrary dimension and in fact, we want to provide a physical intuition into this kind of a behavior for a Bose gas. Strictly at T is equal to 0, all the particles are in the ground state with epsilon equal to 0 and as you see therefore, the specific heat vanishes at T is equal to 0.

Now, suppose we give the system a little temperature k B T which is very small. Now, once you give this temperature to the system. Then essentially, you are exciting some of the particles and the some of these particles can go to the higher excited states. So, you have, but you have a limit right.

Your free particle energy states have the energies h square epsa sorry, h square k square over twice m and when you give energy of the order of k B T to this particles, you can at most fill up certain number of your case values of your k values. And that k values is given by h square k m square over twice m is equal to k B T. So, this k m values are now occupied.

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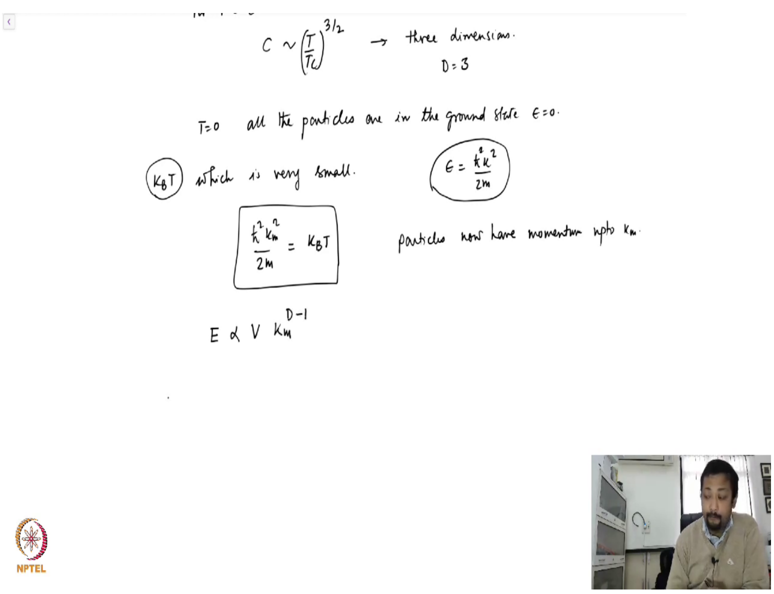

So, we will say. So, particles now have momentum up to k m, they do not go above it, because they do not have that sufficient thermal energy to go beyond that right. Now, the energy, it is a very simple understanding. The energy is proportional to the volume and to the density of state which goes as k m D minus 1; sorry. Now, each of these particles which are now excited will carry an energy k B T.

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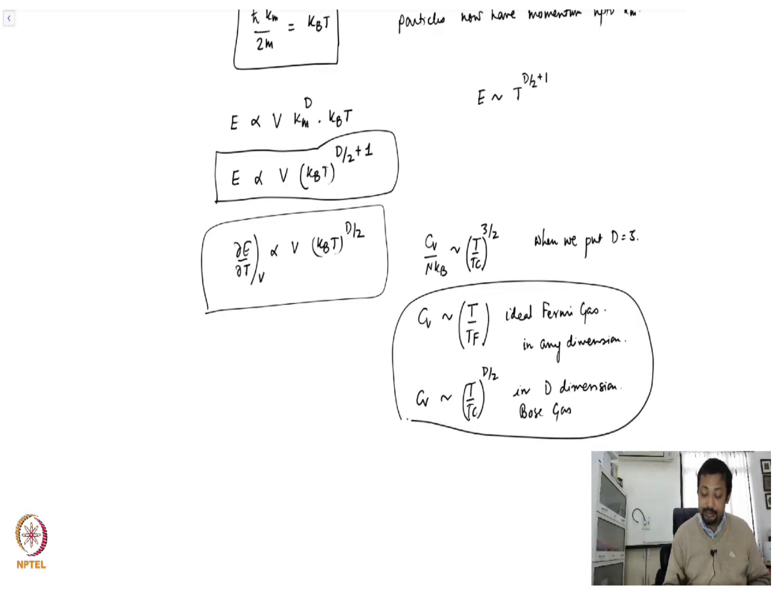

So, that the total energy of the system now goes to proportional to V k m raised to the power D times k B T right, which becomes V k B T raised to the power D by 2 plus 1. I am not worried about the factors over here. There is a proportionality constant. I can lump everything together in the proportionality constant. Once I have see, if you look at this structure of the energy.

So, for any arbitrary dimension, therefore, the energy must go as T to the power D by 2 plus 1. Therefore, the specific heat which is a derivative of this energy with respect to temperature will be proportional to volume health constant will be proportional to k B T raised to the power D by 2.

And in then, three-dimensions, we have C v over NK B going as T to the power 3 by 2, when we put D is equal to 3. So, this is a more general result and unlike a fermis gas, where your specific heat was proportional to linear. So, one can write down this as T v over T c and here, we had T over T F, the this was for an ideal fermi gas in any dimension.

So, for any arbitrary dimension D, your specific heat for an ideal Fermi gas was always linear. In contrast, for a the specific heat, for a ideal Bose gas will scale as T over T c raised to the power D by 2 in any arbitrary dimension. In any in, not in any; but we will say in D dimensions. So, there is a remarkable difference for a Bose specific heat, for a Bose gas and specific heat for a fermi gas.