Statistical Mechanics

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Lecture - 57
Specific Heat of an Ideal Bose Gas - Part 01

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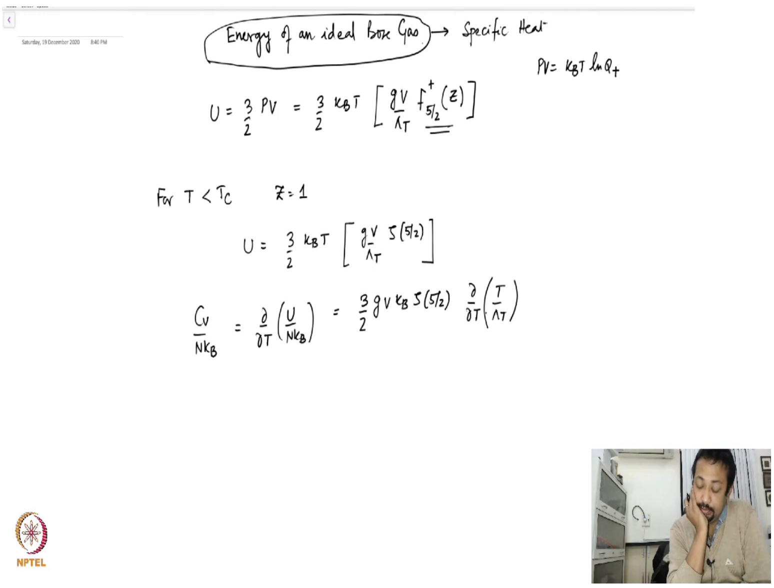

Next interest that we want to do very carefully is the energy of an Ideal Bose gas and therefore, the specific heat. Our interest essentially lies in evaluating the energy of the Ideal Bose gas because we have been until it route to access the specific heat, which is actually the measurable quantity in the experiment.

Now, you I know that for a non-relativistic gas, two-third of the energy density is equal to the pressure. So, therefore, U is going to be three-half times PV and PV, we just now saw was K

B T ln Q plus. So, that I have three-half of times K B T times gV over lambda T f plus 5 by 2 z.

Again, we ignore the term, the second term in ln Q plus because we are working in a thermodynamic limit and just as in the pressure, we saw that this is going to be; I can ignore that. We will also ignore it in the internal energy right. Good. For T less than T c, this quantity z is equal to 1.

So, that U becomes three-half K B T gV over lambda T times zeta of 5 by 2; correct? Good. So, I want to calculate the specific heat now. So, we will do it in this reduced form. We will write down C v over NK B which is going to be d dT of U over NK B del del T of U over NK B.

And that is g three-half gV times K B times zeta 5 by 2 times del del T of T over lambda T. Correct? Ok. So, this is let us see how can we do it very easily.

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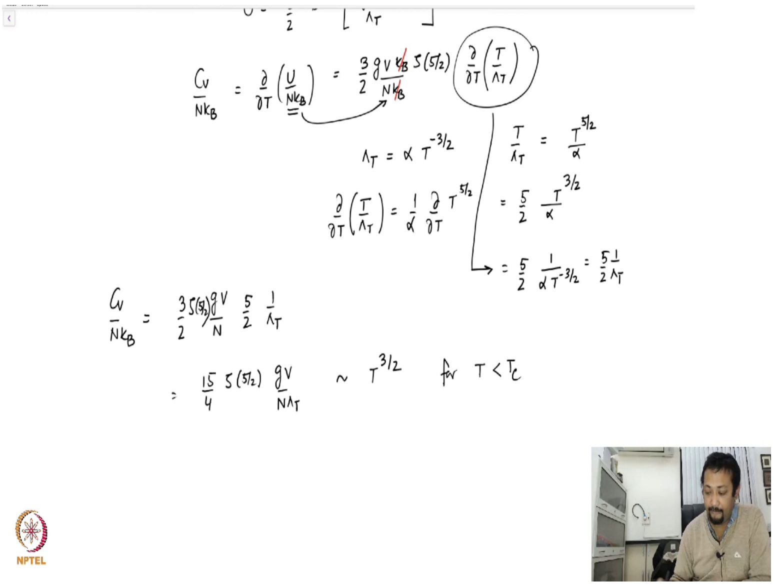

Now, lambda T is alpha T to the power minus three-half right. So, that T over lambda T is going to be alpha times T to the power 5 by 2 and therefore, del del T of T over lambda T is going to be alpha del del T of T to the power 5 by 2, which is 5 by 2 times alpha T to the power three-half.

Can we express it in terms of this? I am sorry. This is my mistake. So, this has to be T to the power by alpha. So, that this is 1 over alpha and this is by alpha which is 5 by 2 alpha T to the power minus three-half and you have 5 by 2 lambda 1 by lambda T.

So, that C v over NK B is 3 by 2. My mistake, there has to be an NK B over here which follows from this and which I have missed. So, be careful with the calculations. So, I have gV over N, the K B and the K B cancels out and this I should have been more attentive to and then, I have zeta 5 by 2 and this derivative is essentially this.

So, that I have 5 by 2 1 over lambda T. So, that I have 15 by 4 zeta 5 by 2 gV over N lambda T and this therefore, goes as T to the power three-half, for T less than T c.

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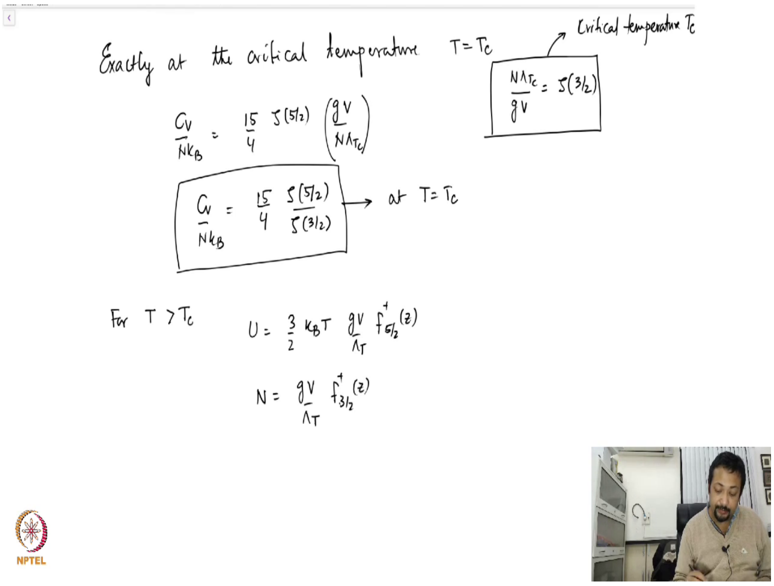

Now, exactly at the critical temperature T is equal to T c, I have C v over NK B is going to be 15 by 4 zeta 5 by 2 gV over N lambda T c. But I know that N lambda T c over gV is equal to zeta of three-half because this is the criteria for obtaining the critical temperature T of c.

So, I will replace this, then as 15 by 4 zeta 5 by 2 divided by zeta of 3 by 2 as C v over NK B and this is the value that you get at T is equal to T c. Of course, what is left now is to look at what happens to the specific heat for T greater than T c. The second case to study is for T greater than T c and we start off with the usual expression that U is going to be 3 by 2 K B T times PV.

And therefore, that is gV over lambda T f plus 5 by 2 z. Now, here again, I have to determine the fugacity from the total particle number which I had gV over N lambda T f plus 3 by 2 of z.

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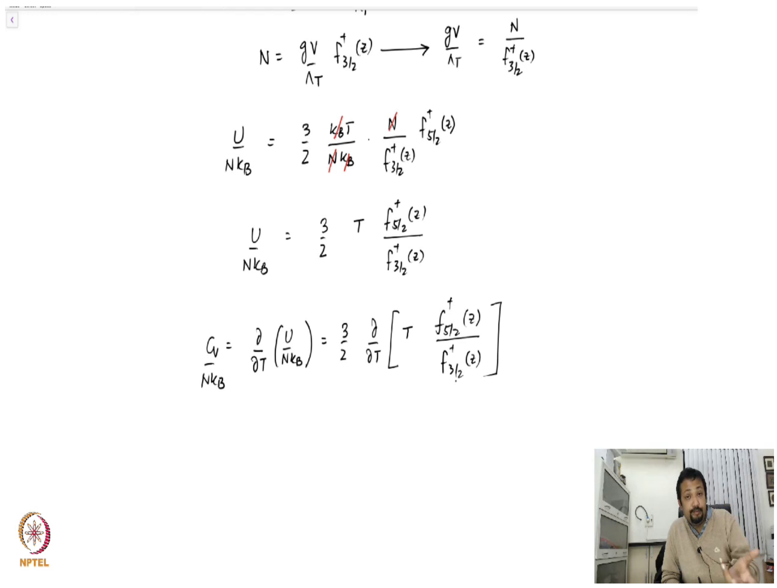

So, that U over NK B is going to be 3 by 2 K B T divided by NK B times gV over lambda T is N gV over lambda T. I use this relation to write down gV over lambda T as N over f plus 3 by 2 z. I can of course, determine the fugacity again and then, replace it over here; but it is not necessary to do.

In this analysis, one can do it more elegantly. So, f plus 3 by 2 z times f plus 5 by 2 z. Lot of things cancel out; K B, K B cancels out; N, N cancels out and I have U over NK B is going to be 3 by 2 T f plus 5 by 2 z divided by f plus 3 by 2 z. Therefore, the specific heat now becomes C v over NK B is del del T of U over NK B which is del del T of T over f plus 5 by 2 z divided by f plus 3 by 2 z.

Now again, one has to be careful here. Please just do not take a derivative of temperature and (Refer Time: 09:41) because that is just only one contribution. You should also note that z is e to the power beta mu and therefore, that is a function of temperature. At, but these functions f plus that is written over here are complicated functions.

There are no representation of these functions. These are integrals essentially and you cannot represent them in terms of simple functions. So, carrying the derivative is little bit more subtle.

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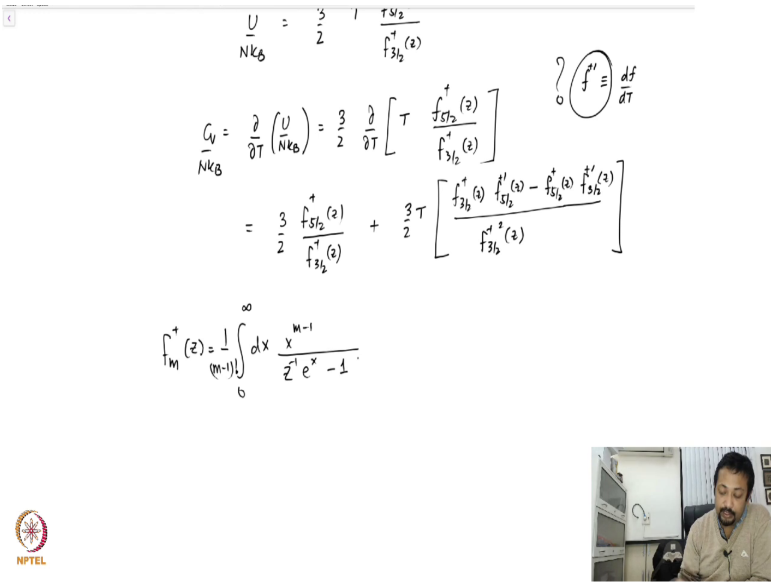

But our purpose is to do that. Let us just evaluate the trivial term. The trivial term is the derivative of the first term that is del del T of T and you are left out with f plus 5 by 2 z divided by f plus 3 by 2 of z and then, you have 3 by 2 T, the derivative gives you f plus 3 by 2 whole square of z f plus 3 by 2 z f dash of 5 by 2 z minus f of 5 by 2. So, there is a plus, there is a plus z f dash of 3 by 2 z. This is the complication.

So, where f dash plus is identical to d f d T. Now, the first thing to evaluate these derivatives; how can I do this, how do I evaluate such quantities? For that, we note f m plus of z is integral 0 to infinity 1 over m minus 1. So, we go back to the original expression for these function, d x x to the power m minus 1 z inverse e to the power x minus 1.

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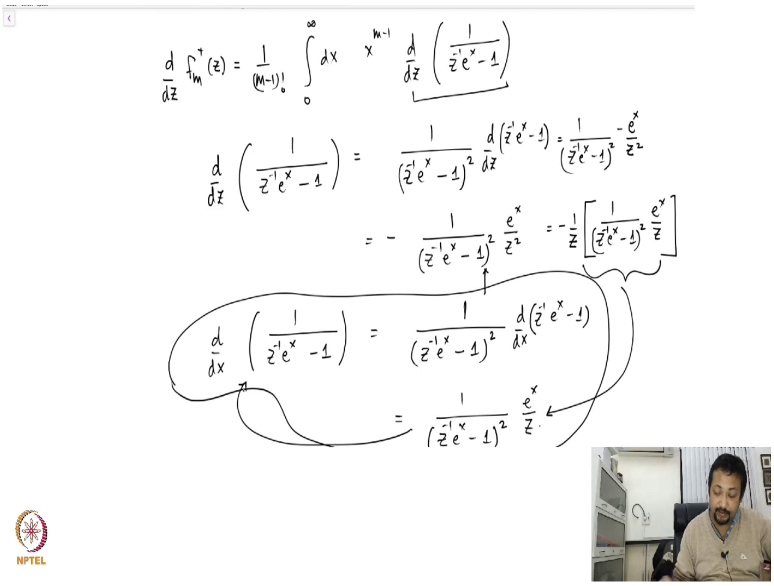

So, that now, if I take a derivative with respect to z this means f m plus z is 1 over m minus 1 factorial 0 to infinity d x x to the power m minus 1 d d z of 1 over z inverse e to the power x minus 1. Let us do this. I mean there is a simple way of doing it, but I want to illustrate this. So, d d z of 1 over z inverse e to the power x minus 1 is going to be 1 over z inverse e to the power x minus 1 whole square.

And then, I have d d z of z inverse e to the power x. So, d d z of z inverse e to the power x minus 1 which is 1 over z inverse e to the power x minus 1 whole square. I have e to the power x d d z of 1 by z which gives me minus z square. So, that this derivative, I have minus 1 over z inverse e to the power x minus 1 e to the power x over z square ok.

What we do next? Well, we want to convert this; the derivative with respect to z to a derivative with respect to x. For that, we also note that if I just do d d x of 1 over z inverse e to the power x minus 1, I will come up with not some sorry this is a whole square over here; something which is different, but something which is something which is very similar.

This and then, I have d d x of z inverse e to the power x minus 1 which gives me 1 over z inverse e to the power x minus 1 whole square times e to the power x over z. So, by looking at this that we have evaluated and then, I look at this expression, I can see that I can write down this as this.

And then, I can write down the rest of it as z inverse e to the power x minus 1 whole square e to the power x by z. But surprisingly, this quantity is this which in turn is this.

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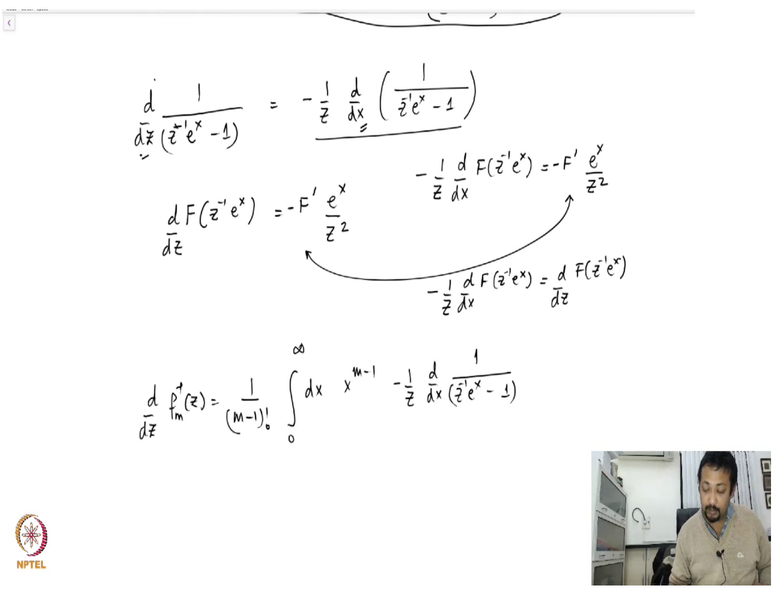

So, therefore, I have d d z of 1 over z inverse e to the power x minus 1 is minus 1 by z of d d x of 1 over z inverse e to the power x minus 1. Now, this you could have easily guessed if you considered that if I have a function let us say a function capital F of z inverse e to the power x and if I want to take this.

Then this is going to be F dash e to the power x over z square minus. And if you want to take d d x of F of z inverse e to the power x, then this is just going to be F dash right this is just a chain rule of differentiation. You are going to have e to the power x over z and if you compare this and this.

Then you realize d d x of minus 1 over z minus 1 over z d d x of z inverse e to the power x is actually minus z square which is d d z of e to the F z inverse e to the power x; good. So, now that we are here and the reason is see, I want to convert the derivative with respect to z to a derivative with respect to x and there is an ulterior motive for that.

Because I know here the integral is with respect to x and if I replace this derivative with respect to z up to a derivative with respect to x, I can use integration by parts. I can make my life a little bit more simpler. So, I have d d z of f plus m of z as 1 over m minus 1 factorial 0 to infinity d d x x to the power m minus 1 and then, I had a d d z which I can see that is going to be minus 1 over z minus 1 over z d d x of z inverse e to the power x minus 1.

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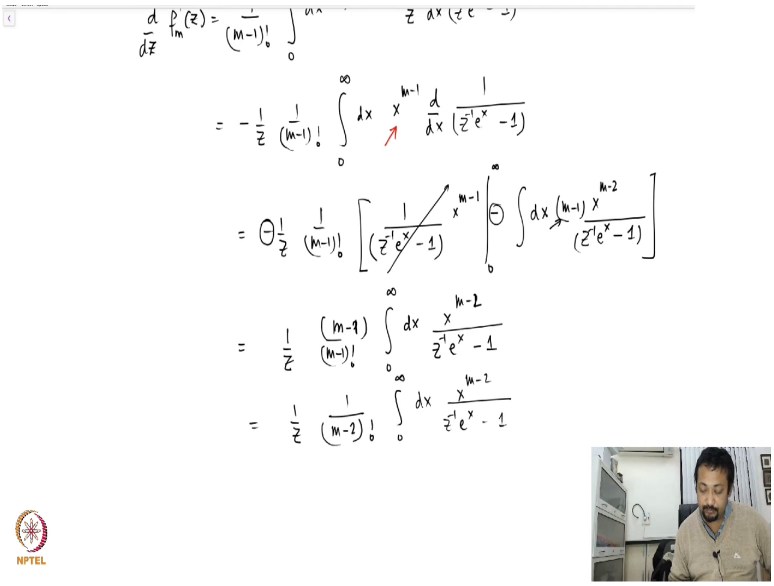

So, that this is minus 1 over z 1 over m minus 1 factorial 0 to infinity d x x to the power m minus 1 d d x of z inverse e to the power x minus 1. Now, let us integrate by parts. When I integrate by parts, I have minus 1 by z m minus 1 factorial and now in this case, I take this 1 as the first function so that I have d x of this.

So, that gives me z inverse e to the power minus of this times x to the power m minus 1 minus integral d x m minus 1 x to the power m minus 2 over z inverse e to the power x minus 1. The limits are 0 to infinity. This one vanishes in both the limits. So, that I have 1 by z this minus and this minus makes it a plus m minus 1 factorial d 0 to infinity, I have m minus 1 in the numerator.

That comes from this, d x x to the power m minus 2 divided by z inverse e to the power x minus 1 and this is 1 over m minus 2 factorial 0 to infinity d x x to the power m minus 2 z inverse e to the power x minus 1 which surprisingly is f plus m minus 1 of z.

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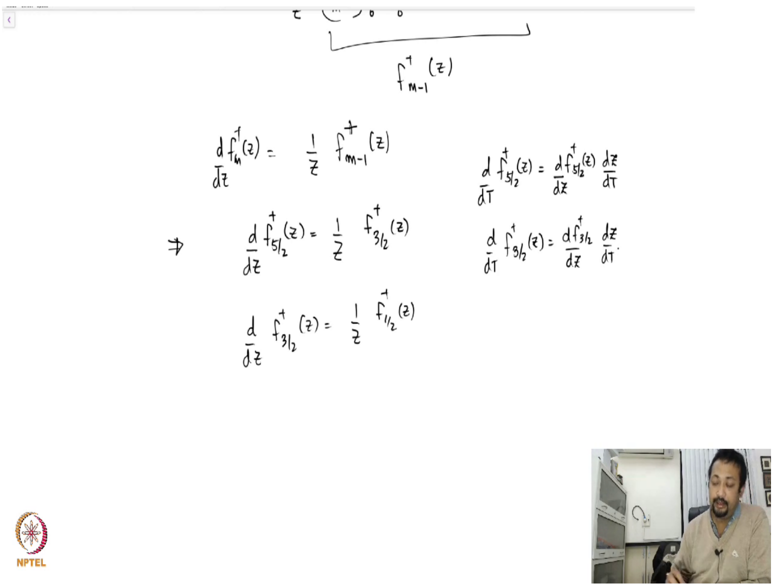

So, that I have z of f plus m minus 1. I have f plus m minus 1 of z; d d z of f plus m of z. Therefore, this implies that d d z of f plus 5 by 2 z is going to be 1 by z f plus 3 by 2 z and d d z of f plus three-half of z is going to be 1 by z f plus half of z correct.

Now, d d T of f plus five-half of z is going to be d d z of f plus five-half of z times d z d T and similarly, d d T of f plus three-half of z is going to be d f plus three-half of z d z d z d T right.