Statistical Mechanics
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Lecture - 55
Bose-Einstein Condensation

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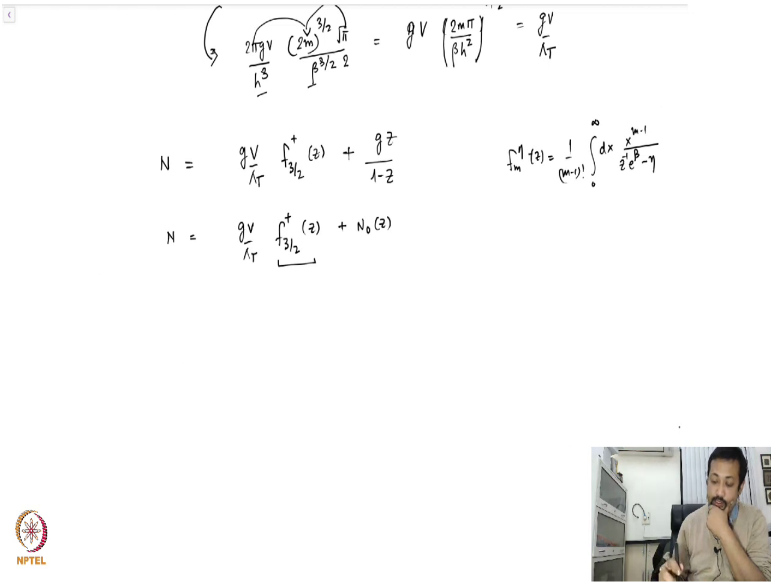

So, now we have this expression for N which is essentially g V over lambda T f of 3 half plus Z, the plus denoting that we are looking at a bosonic system N 0 Z. So, as usual the fugacity has to be determined from this relation, but if we look at this as this function f of m eta of Z was 1 over m minus 1 factorial 0 to infinity d x x to the power m minus 1 Z inverse e to the power beta minus eta.

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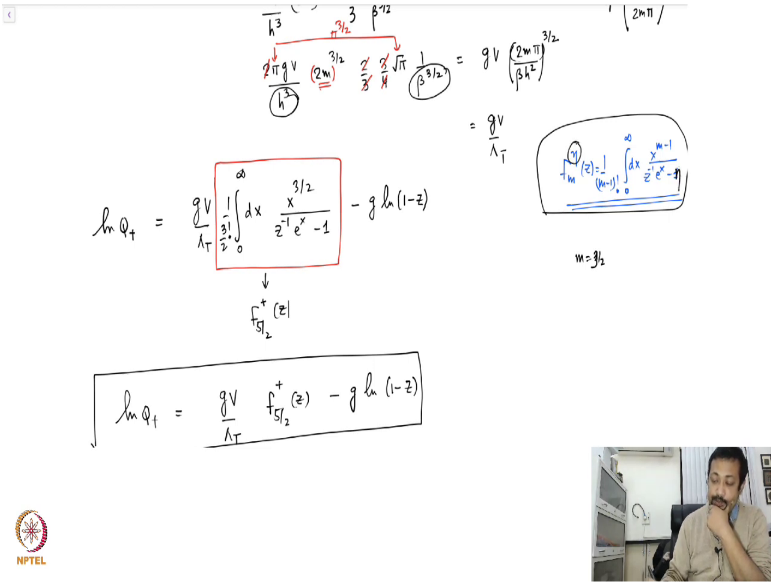

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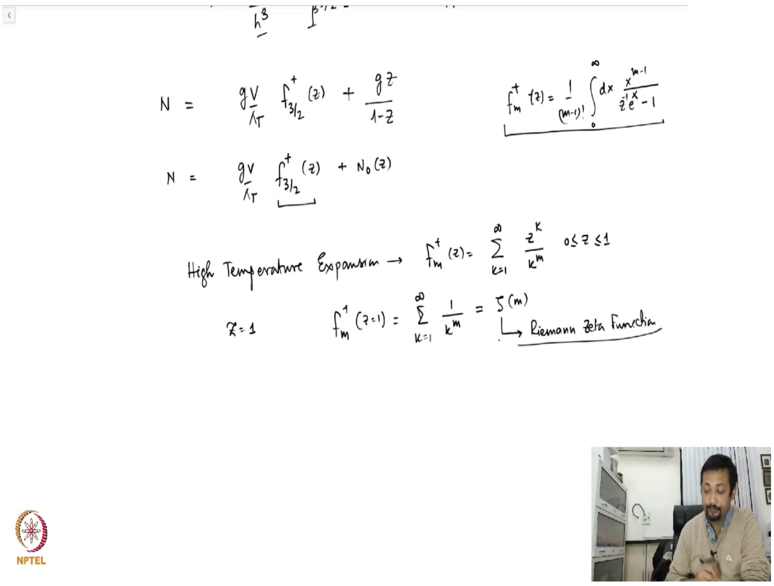

I think we did make an error while we sorry this has to be minus of eta for a specific eta. If it was plus then I know that this is just going to be f m plus offset is just going to be minus of 1 beta epsilon. Terribly sorry for the mistake, this has to be e to the power x minus 1.

Now, this is a very complicated function and usually the it is representation in terms of simple functions are not possible, but we do know the high temperature expansion of this ah. The high temperature expansion of f m; so, now, high temperature expansion gives us f m plus of Z was 1 over m minus 1 factorial this was already taken care of.

It was simply sum over k equal to 1 to infinity Z to the power k k to the power m for Z greater than equal to 0 less than equal to 1. One should note that if I set Z equal to 1, so, f m plus of Z equal to 1 becomes sum over k equal to 1 to infinity 1 over k to the power m and this is zeta of m when the zeta function is the Riemann zeta function.

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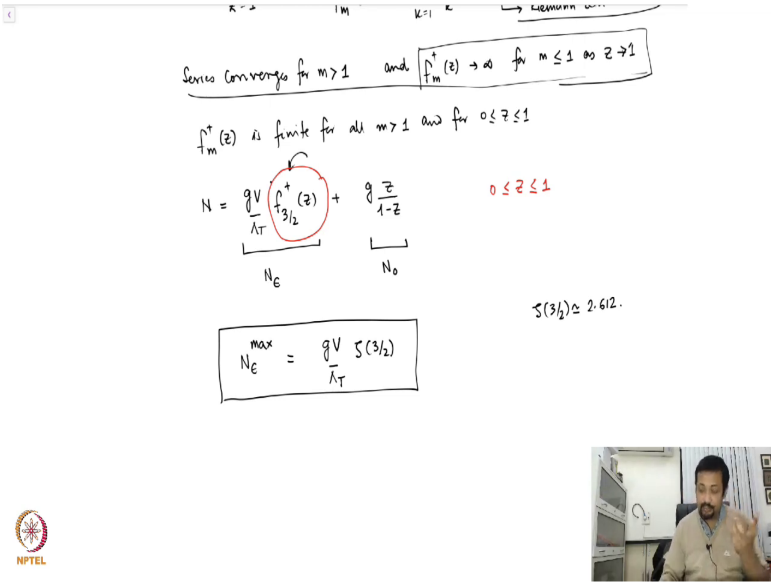

The series converges for m greater than 1 and f plus m of Z tends to infinity, for m less than 1 as Z tends to 1. So, series converges for m greater than 1 and f plus m goes to infinity, if m is less than equal to 1 as Z tends to 1. On the other hand, f plus m of Z is finite for all m greater than 1 and for Z greater than equal to 0 less than equal to 1. So, coming back to this then I have coming back to this expression for the particle number is f of 3 half plus of Z plus g Z 1 minus Z.

The first term that you see over here is the term is the number of particles which is contained in the excited states and this is the number of particles which are contained in the ground state

N equal to 0. So, this is the term, the first term is the term which is contained which essentially represents the particles contained in the states other than the ground state.

But now, we have clearly seen that this term is limited by this function right and for a Bose gas I know that the maximum value of Z can happen ah. There is a limit on values of Z that is Z can be greater than equal to 0 or must be less than equal to 1. So, clearly when Z approaches 1, then this part the first term we will write down this as N epsilon maximum there is a maximum number of particles which you can accommodate in these excited states.

And that number is given by, g V over lambda T zeta 3 half, where zeta 3 half is approximately 2.612. Now, this is a very important inference. You see from the expression for the total number of particles in your system that you can accommodate only a certain number of particles in the excited states. There is a limit upper bound on that.

So, if you keep on adding particles to the system you are going to hit this limit and then if you keep on adding more and more particles where do they go? They go to the ground state and that is when your condensation starts to appear. Sorry not the condensation starts to happen and the condensate starts to appear.

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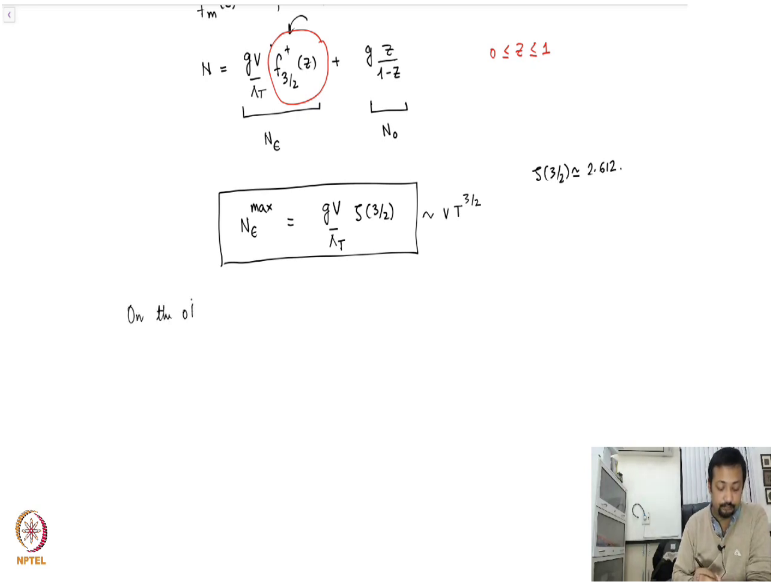

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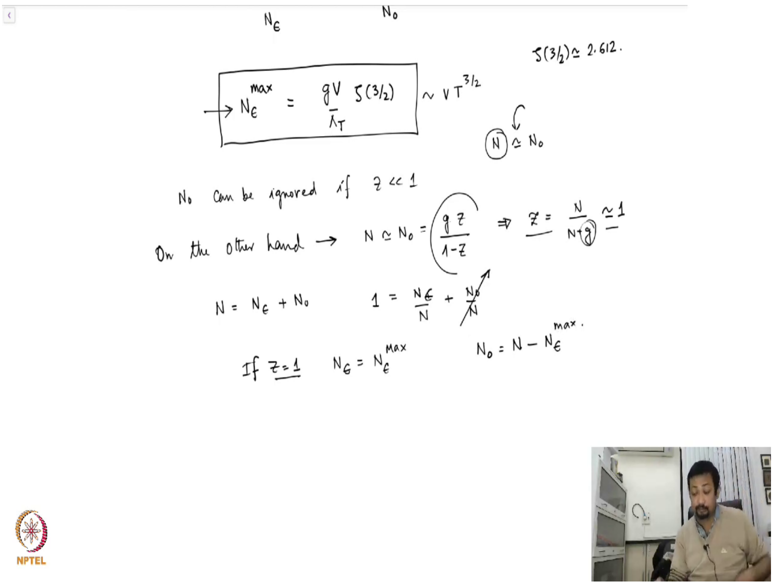

This goes as V T to the power 3 half. On the other hand not well let us. So, the second term, this is the term N 0 can be ignored if Z is very very less than 1. So, if you are not hitting the limit of Z to 1. On the other hand, if N the total particle number is dominated by the number of particles in the ground state. So, that you have N 0 is approximately this, you see what happens to Z.

This implies that Z is approximately N over N plus not approximately, but exactly N by N plus g, g is a tiny number compared to N which is 10 to the power 23 typically and therefore, you see this value is 1. So, whenever this condition is satisfied that you actually see that the number of particles in the ground states start dominating the total number of particles in the system, so most of the particles have gone into the ground state then you have actually hit the limit Z equal to 1.

So, in the thermodynamic limit, we rewrite this equation which we had over here and a little up. So, we had the expression N was equal to N epsilon plus N 0. We rewrite this equation as 1 is equal to N epsilon over N plus N 0 over N. Clearly, if N 0 is not sufficiently large, so that you have not hit the Z to 1 limit, then in the thermodynamic limit you can ignore this term and most of the particles are accommodated in the excited states.

In contrast whenever you have populated all the excited levels there is a limit this epsilon max you have reached you start populating the ground state and that is when; if your Z is equal to 1 and N epsilon is equal to N epsilon max then you have populated all your excited energy levels and then essentially N 0 is given by you have hit the limit of Z to one as Z is equal to 1. So, you see the N 0 diverges. So, essentially the number of particles in the ground state is given by total number of particles minus N epsa max right.

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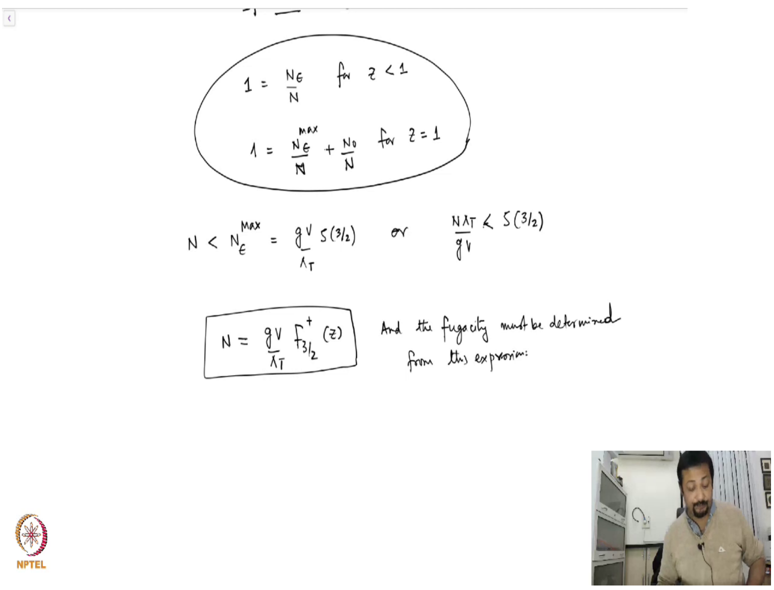

So, we conclude therefore, that N epsa over N is equal to 1 for Z less than 1 and 1 is equal to N epsilon by over N oops plus N 0 over N for Z is equal to 1. Therefore, we can see that when the total particle number is less than N epsa max, which is g V over lambda T times zeta 3 half.

Or we can rewrite this condition as N lambda T over g V is less than zeta of 3 half that means you can still accommodate the particles in the excited levels. So, that you have this expression for N. And the fugacity must be determined from this expression as we have done it for a fermion system or an ideal fermi gas.

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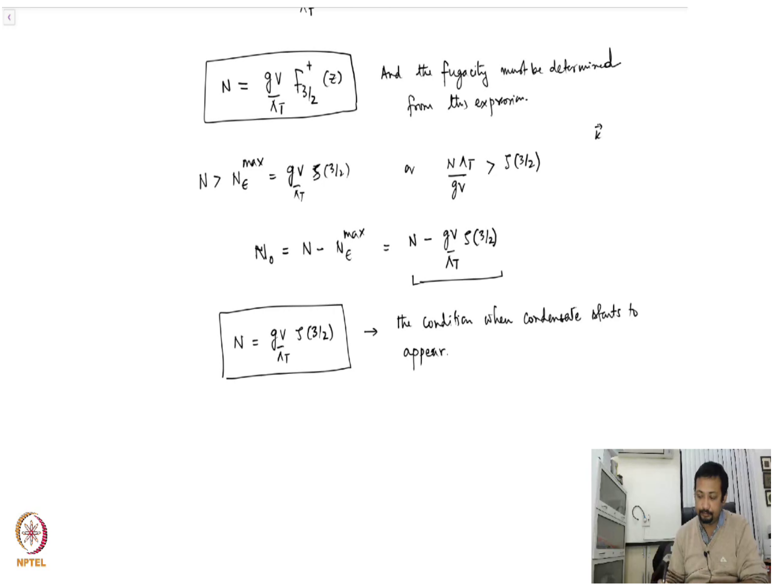

Now, in contrast when N is greater than N epsilon max which is lets g V over lambda T zeta 3 half or N lambda T over g V is greater than zeta of 3 half then you have hit the limit in putting particles in the excited states. Therefore, in that case your N 0 is given by N minus N epsilon max which is N minus g V over lambda T zeta 3 half.

This means that there is a condition at which point the condensate can appear. I can figure out a temperature. So, I can clearly see that based on this the condensate starts to appear when N is g V over lambda T zeta to the power 3 half. That means, when your number of particles in the system has hit the upper bound upper limit of N of the number of particles you can put in excited levels.

The condition when condensate starts to appear. Again at this point I would like to iterate that look even though we are talking about the condensation and whenever one talks about the condensation one talks about visualize the condensation in a real space, it is not really a condensation in real space. It is actually a condensation in a momentum space, all particles.

So, whatever we have been talking about is labeled by this momentum vector k, the single particle levels are leveled by the momentum vector k. So, what is actually happening is you are actually populating the state with the zero momentum. So, it is a if you can visualize the momentum space then you are going to see a peak which is starting to appear in the zero momentum state and that is exactly what people have seen in experiments.

So, this effectively also means, this essentially means, that for large enough temperatures you have sufficiently large amount of thermal energy to excite the particles in the excited states. So, that they are in the excited states. But as you keep on decreasing the temperature as the temperature comes down sorry, the density becomes high or either you could have the temperature coming down becomes smaller and smaller.

Or the density is raised which means you keep on adding particles and particles until you hit this criteria. What essentially happens is the positive correlation between the bosons ensures that they go and on because they are attractive. So, they go and sit at the ground state energy epsilon equal to 0.

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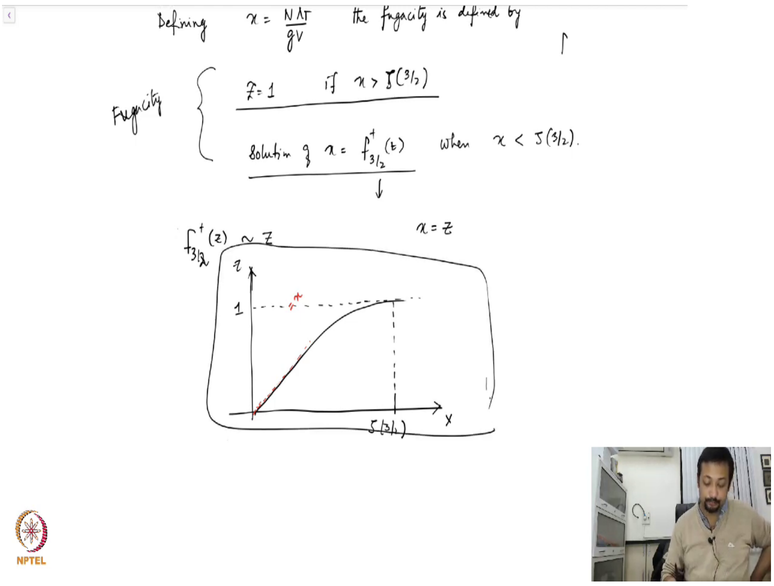

So, defining x equal to N lambda T over g V, then the fugacity is defined by Z is equal to 1 if x is greater than zeta of 3 half and solution of x is equal to f plus 3 half Z. So, fugacity, something is wrong here. Fugacity is equal to 1 if x is greater than zeta 3 half and or is the solution of this when x is less than zeta to the power 3 half.

Recall the first leading order correction to this function f 3 half of Z. f 3 half goes as Z for small enough Z. So, that you have the solution x equal to Z. If you want to plot the fugacity, so, this hits a value 1 whenever your x is greater than zeta 3 half, but for small enough you will have Z is equal to x ok not really so, the not drawn to schematics. So, let us just quickly draw it first. For small enough I am going to have and you expect that the curve becomes something like this way.

So, let us shift it a little bit to the right, so that we can say that this is going to be your zeta 3 half and not this. So, this is how your curve should look.

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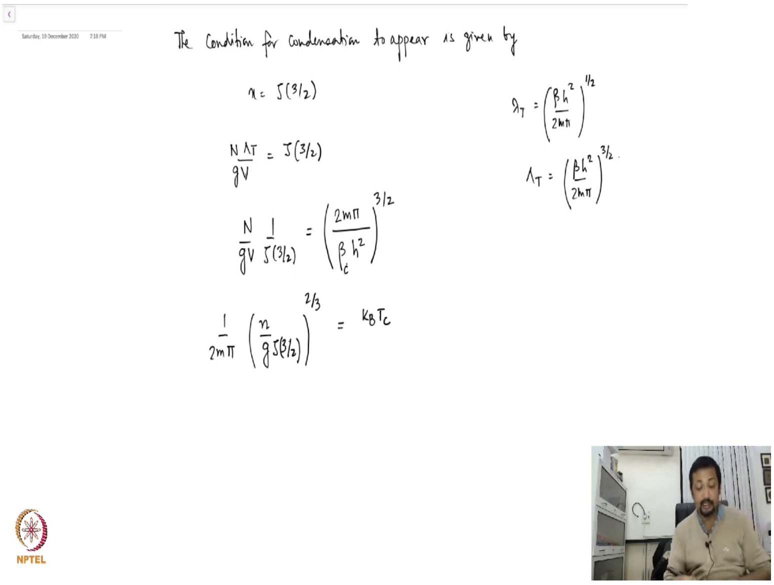

The condition for condensation to appear is given by x equal to zeta 3 half right which is N lambda T over g V is going to be zeta of 3 half. So, now, lambda T is beta h square over 2 m pi raised to the power half. So, that the thermal de Broglie volume is beta h square over 2 m pi raised to the power 3 half. So, then N over g V 1 over zeta 3 half is equal to 1 by beta h square twice m pi raised to the power 3 half.

So, capital N by V is a small n the number density g zeta 3 half raised to the power two-third. I have 1 over twice m pi which is going to be your k B T c. So, we will replace the beta here by beta c indicating that this gives me the temperature.

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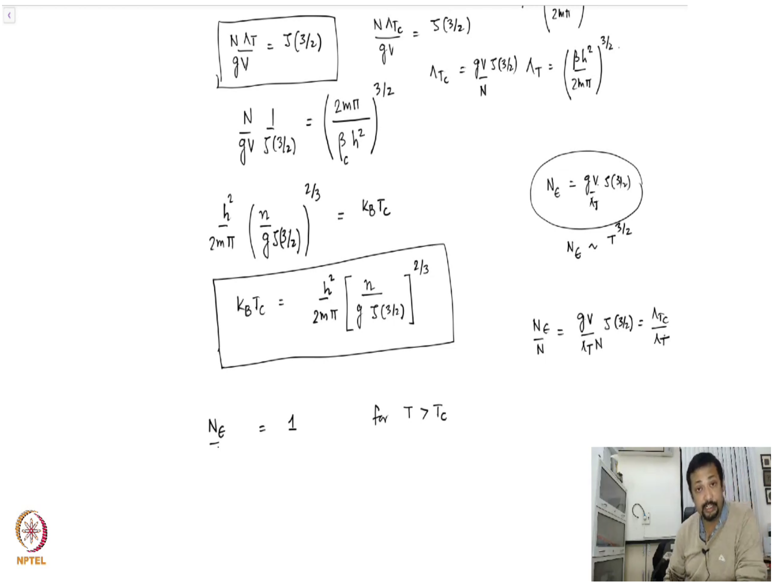

So, you have a critical temperature k B of T c which is given by 1 over 2 m pi n over g times zeta of 3 half raised to the power 2 by 3. I am missing out a factor of h square. So, that this has to be h square in the denominator. So, this is the critical temperature that you are looking at. At this temperature your condensation starts to happen and you starts to see a condenser that appears at the k equal to 0 momentum state.

Now, the particle number in the excited state N epsilon over N is equal to 1 for T greater than T c because all the particles that you put in essentially goes into the excited states if you are away from this critical temperature, but the moment you hit the critical temperature you essentially hit a limit of g V over lambda T zeta 3 half. So, that this N epsilon goes as T to the power V T to the power 3 half.

And you clearly see that from this condition that we had that this gives me the temperature at which the condensate starts to appear. So, I can sort of write down this as lambda T c g V is equal to zeta 3 half. So, that lambda T c is equal to g V over N zeta 3 half right. Or essentially I can write down g V over N and if I look over here then g V over zeta times zeta 3 half divided by N which is N epsilon over N is g V lambda T N zeta of 3 half which is essentially lambda T c over lambda T.

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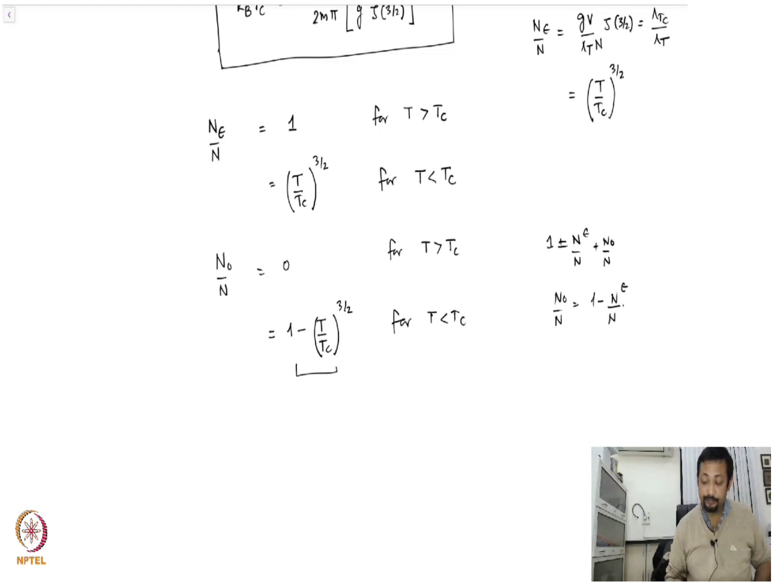

So, this is therefore, T over T c raised to the power 3 half. So, this is T over T c raised to the power 3 half for T less than T c. So, as you start hitting the critical temperature you get a you start to get this. Now, the number of particles in the ground state is 0 for T greater than T c.

However, as you cross the critical temperature then this becomes one minus T over T c raised to the power 3 by 2 and this expression follows from the fact that 1 plus 1 is equal to N epsilon by N plus N 0 over n. So, that N 0 over N is 1 minus N epsilon over N. If you carefully study these two expressions one can plot them all. So, you see that as you approach 0 and 0 temperature the number of this thing vanishes.

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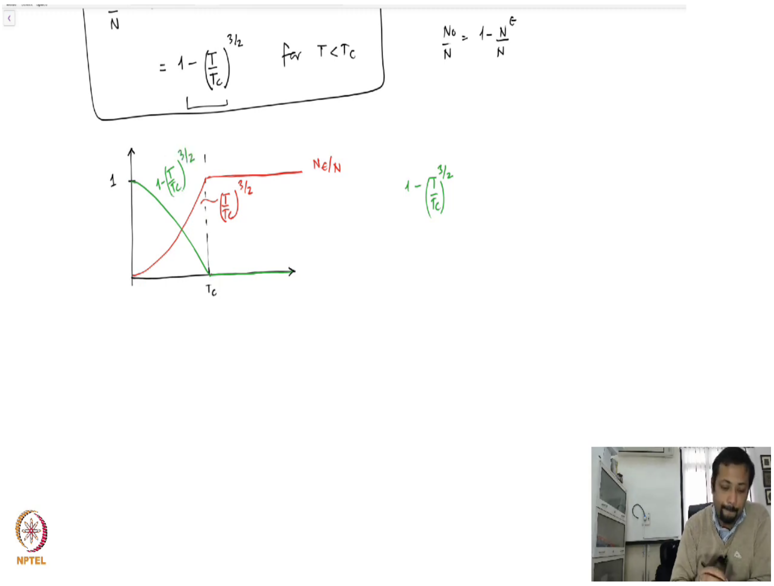

So, that I have this is my T c let us say; we will draw a vertical line and let us say this is one then this is going to be N epsilon over N number of particles in the excited states for T greater than T c and this goes as this T over T c raised to the power 3 half. For temperatures higher than T c the occupation in the ground state energy level is 0.

And as you start going below the critical temperature the occupation in ground state starts to increase going as T to the power 3 by 2. So, that sorry, 1 minus T over T c raised to the power 3 by 2 and so that you have the plot which looks something like this. So, that this is 1 minus T over T c raised to the power 3 half.