Statistical Mechanics
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Lecture - 50
High Temperature Expansion

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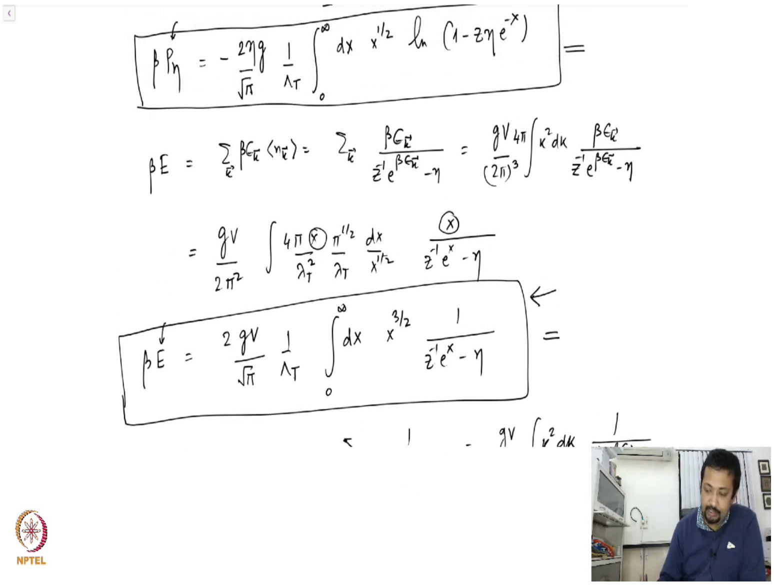

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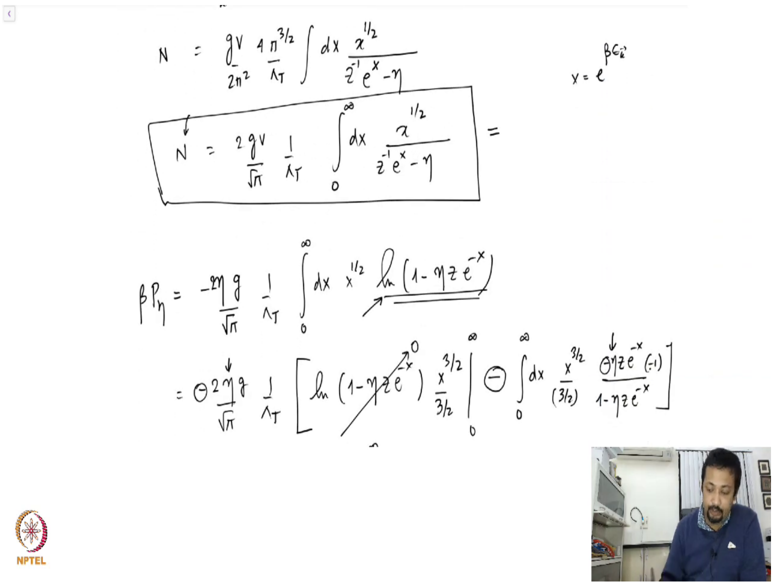

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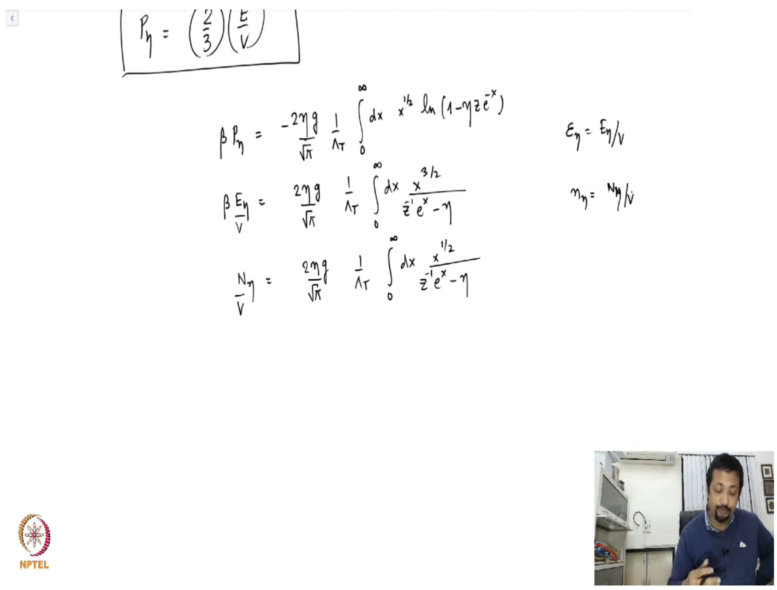

Now, if you look at these expressions; then you see that P eta E and N; I can recast them as beta; P eta is equal to minus m sorry; minus eta g; 2 eta g over square root pi 1 over lambda t; integral 0 to infinity dx x to the power half, ln 1 minus eta; z e to the power minus x.

Beta E eta is equal to twice eta g v over square root pi; 1 over lambda t, 0 to infinity; dx; x to the power 3 half z inverse, e to the power x minus eta and N eta is going to be twice eta g; square root pi; V; 1 over lambda T, 0 to infinity; dx x to the power half divided by z inverse, e to the power x minus eta right.

So, I can simplify a little bit further; in the sense I can take the volume to the right hand side over here and over here. And define the energy density epsa eta as E eta over V and the number density n eta as capital N eta over V.

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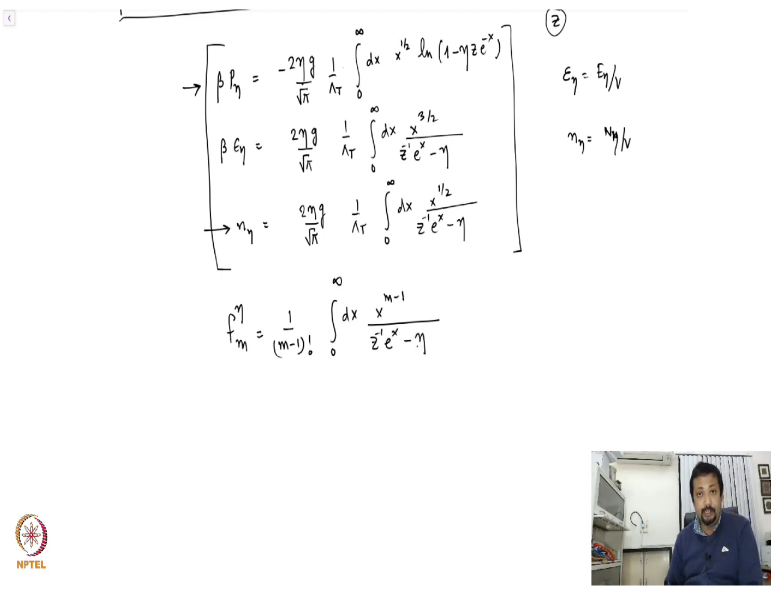

So, this means I have epsa eta and I have n eta right, but if you look at this expressions carefully, this describes the thermodynamics of an ideal quantum gas, but everything is in terms of the fugacity z.

And if I want to obtain the equation of state, I have to determine z from this equation number density and replace this over here in this expression. For this, I define a function f eta of m; as 1 over m minus 1 factorial, integration 0 to infinity; dx; x to the power m minus 1 divided by z inverse; e to the power x minus eta right.

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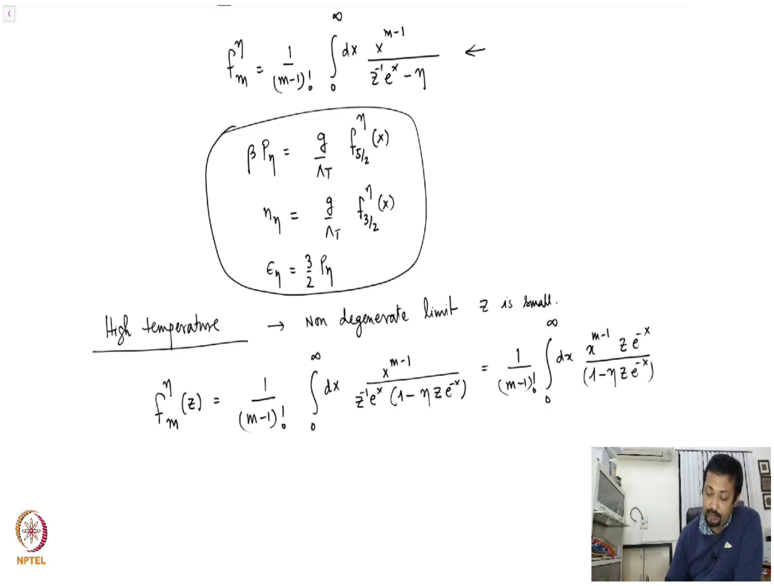

In terms of f eta; beta P eta the pressure is 1. So, in terms of this integral the pressure becomes g over lambda T; f 5 by 2 eta of x and this follows from this expression that we have over here; this expression. n eta is going to be g over lambda T; f of 3 half eta of x and epsa eta which is the energy density is going to be 3 half beta; sorry 3 half of p eta.

So, these are the expressions that we are going to work with. Now, first what I want to do is I want to look at the high temperature limit. The reason I want to look at the high temperature limit first is here I can take care of both the ideal fermionic gas and ideal bosonic gas, using the same formalism; I do not have to do anything special. It is a low temperature which is has to be treated separately right.

Now, in the high temperature limit which is also called the non degenerate limit; z is small so that I have f eta of m as a function of z is equal to 1 by m minus 1; factorial integration; 0 to infinity dx; x to the power m minus 1 divided by; I can take z inverse, e to the power x outside; this becomes eta z; e to the power minus x which is 1 over m minus 1 factorial; 0 to infinity, d of x; x to the power m minus 1, this is z e to the power minus x divided by 1 minus eta; z e to the power minus x.

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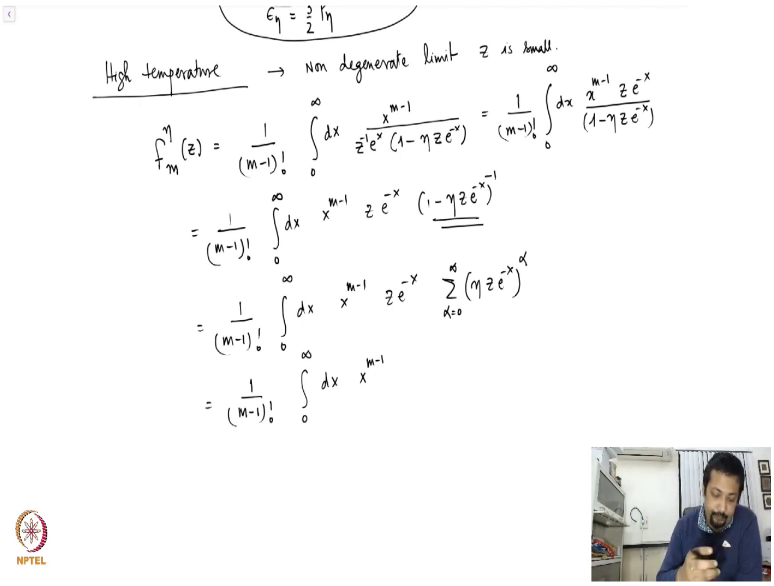

And this is 1 minus m minus 1 factorial; 0 to infinity, dx; x to the power m minus 1; z e to the power minus x; 1 minus eta z; e to the power minus x raised to the power minus 1. So, now I can do a binomial expansion of this term that gives me m minus 1 factorial; 0 to infinity, d of x; x to the power m minus 1; z e to the power minus x; sum over alpha; eta z; e to the power minus x raised to the power alpha. So, that I have alpha is equal to 0 to infinity; m minus 1 factorial, 0 to infinity; dx x to the power m minus 1, I have eta to the power alpha; sum over alpha is equal to 0 ok sorry I cannot take it out.

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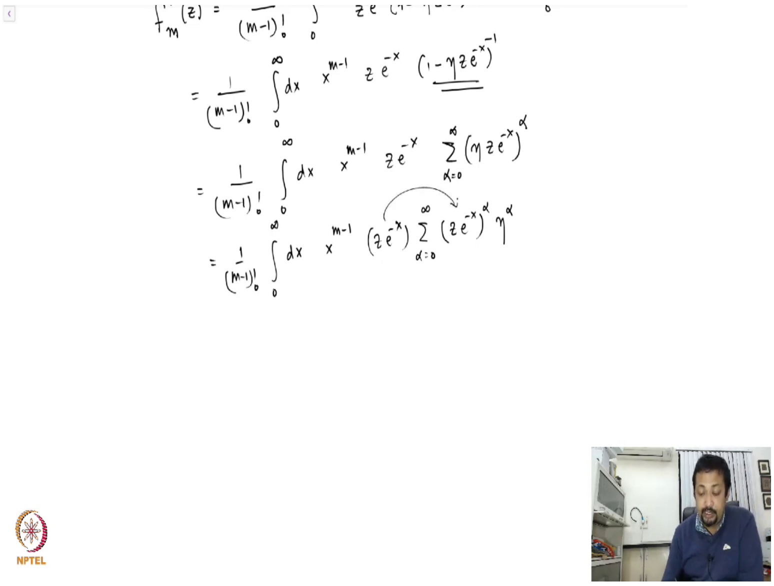

So, that I have 1 over m minus 1 factorial; 0 to infinity, d of x; x to the power m minus 1; z e to the power minus x; sum over alpha is equal to 0 to infinity; z e to the power minus x raised to the power alpha, eta to the power alpha.

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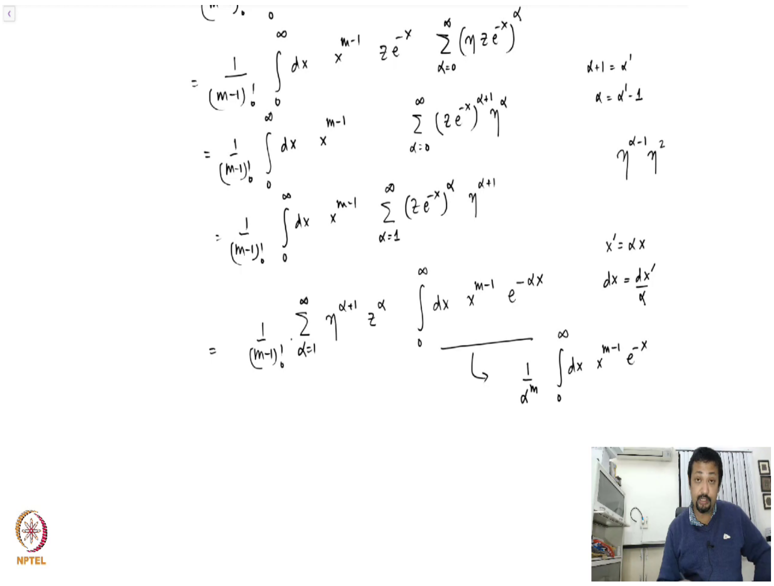

I can bring it inside over here so that this becomes alpha plus 1 and then I can change the indices m minus 1; 0 to infinity, d of x; x to the power m minus 1. So, that I can write alpha plus 1 is equal to let us say alpha prime. So, that alpha is alpha prime minus 1 and when alpha is equal to 0, alpha prime runs from.

So that I can change the indices by writing alpha, so that I can change the indices by writing alpha plus 1 is equal to alpha prime; so, that alpha is equal to alpha prime minus 1. And therefore, I have the new sum should run from 1 to infinity; I am going to have alpha equal to 1 to infinity; z e to the power minus x raised to the power alpha and then I am going to have eta to the power alpha minus 1.

But, I also note that eta square is plus 1; so I am just going to multiply eta to the power alpha minus 1 with eta square because eta square is 1; so it does not matter to me and to get me alpha plus 1, there is a sum over alpha. So, then I have 1 minus m minus 1 factorial; sum over alpha equal to 1 to infinity, I have eta alpha plus 1, I have z to the power alpha and then I have 0 to infinity; dx; x to the power m minus 1, e to the power minus alpha x.

And this quantity is well known in the sense that I know how this thing is going to go. And this in this integral; I can replace x prime as alpha x; so that dx becomes x prime; dx prime over alpha.

And you see that this integral, I can recast as 1 over alpha to the power m; 0 to infinity dx; x to the power m minus 1, e to the power minus x, which is nothing, but the 1 minus m minus 1 factorial will cancel with this. And therefore you are going to be left out with alpha equal to 1 to infinity; alpha plus 1, z to the power alpha divided by alpha to the power m.

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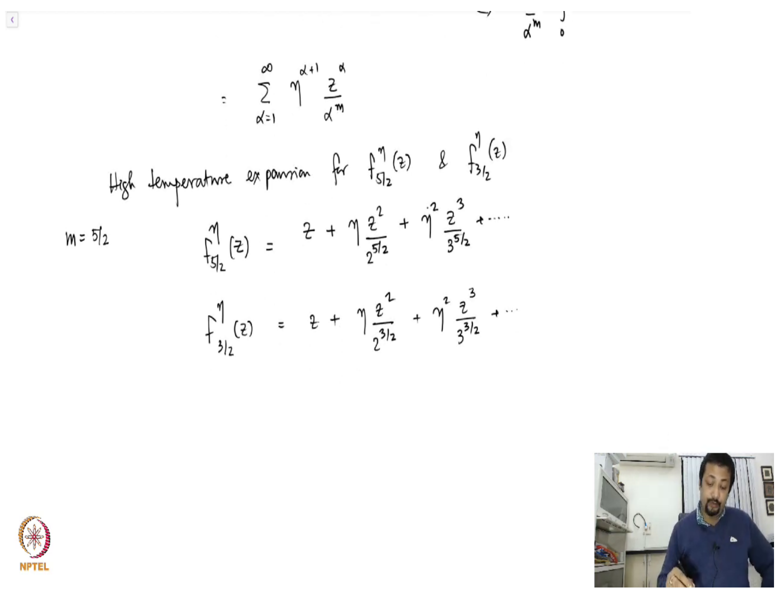

So, which means the high temperature expansion which we just worked out for f 5 half eta of z and f 3 half; eta of z is the following are the following. So, f of 5 half; eta of z means m is equal to 5 half is going to be; if I put alpha equal to 1, I have eta square z plus eta z square divided by alpha is 2; 2 to the power 5 by 2 plus eta square.

z cube, I have 3 to the power 5 by 2 plus higher order terms. And I am going to have 3 half of z is equal to z plus eta; z square; 2 to the power 3 by 2 plus eta square; z cube, 3 to the power 3 by 2 so on and so forth. Since eta square is equal to 1; I can as well remove the eta square from here.

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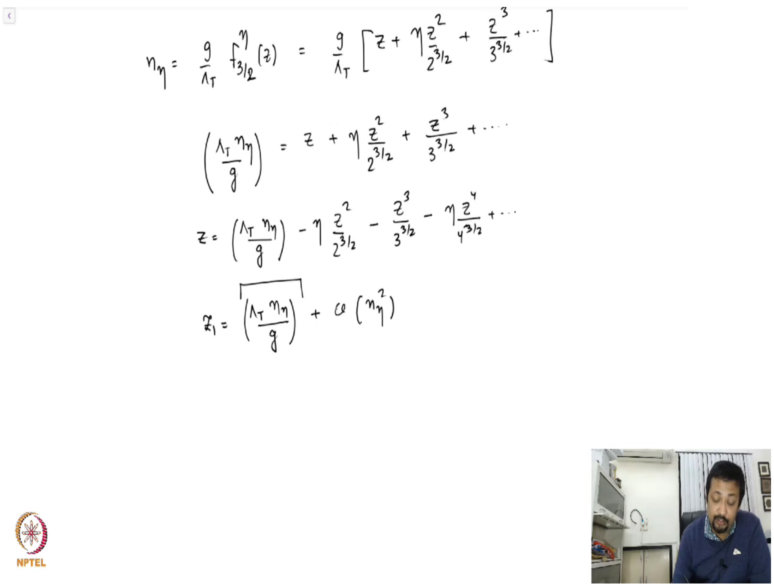

We are going to use; we are going to use these two high temperature expansion of this int function f. Now, I know that n eta was g over lambda T; f 3 by 2 eta of z and the idea is to determine z; the fugacity from this equation. So, if I write it down, then this becomes g over lambda T; z plus eta z square 2 to the power 3 half plus z cube; 3 to the power 3 half plus higher order terms.

So, that I have lambda T and eta over g is going to be z plus eta z square over 2 to the power 3 by 2 plus z cube; 3 to the power 3 by 2 plus so on and so forth right. Now, I want to determine z from this equation perturbatively; how to do the perturbative determination?

So, I can see that z is going to be lambda T and eta divide by g minus eta z square 2 to the power 3 by 2 minus z cube; 3 to the power 3 by 2. And the; if you want to take the next term, it is going to be again eta; z 4; 4 to the power 3 by 2 plus higher order terms good.

So, I immediately say that to the lowest order; my z 1 is going to be lambda T; n eta over g plus the next order is going to be of order n eta square because the next term is of the order of z square. So, if you plug in this value just this value of z over here then you see that the next term is of the order of n eta square.

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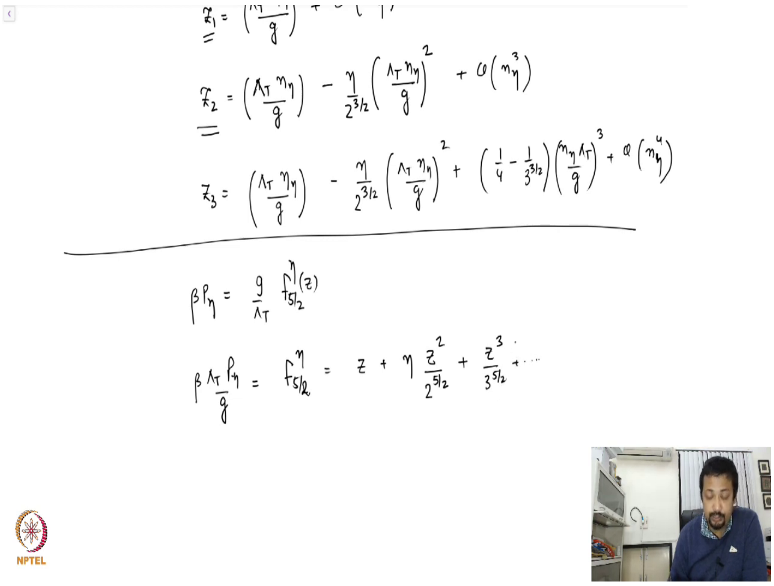

What about z 2? z 2 is going to be n lambda T; n eta over g minus eta, I am going to substitute z 1 into this expression that gives me lambda T; n eta over g whole square; 2 to the power 3 by 2. And then I have order of n eta whole cube which comes from this. I can determine z 3 by substituting z 2 in this expression right.

So, and you are going to get the answer as lambda T; n eta over g minus eta raised to the power 2 to the power 3 by 2 lambda T; n eta over g whole square plus 1 by 4 minus 3 to the power 3 half; n eta, lambda T over g raised to the power whole cube plus order; n eta raised to the power 4.

So, this is the perturbative way of evaluating the fugacity z. Now, the idea is to use this and to look at beta P eta, which was g over lambda T; f of 5 by 2 of z times eta. So, let us see this is going to be beta; lambda T, P eta over g is going to be f of 5 half; eta over z. And this answer is z; if you go back to this x approximation, it is going to be z plus eta z square; 2 to the power 5 by 2 plus z cube; 3 to the power 5 by 2 plus higher order terms.

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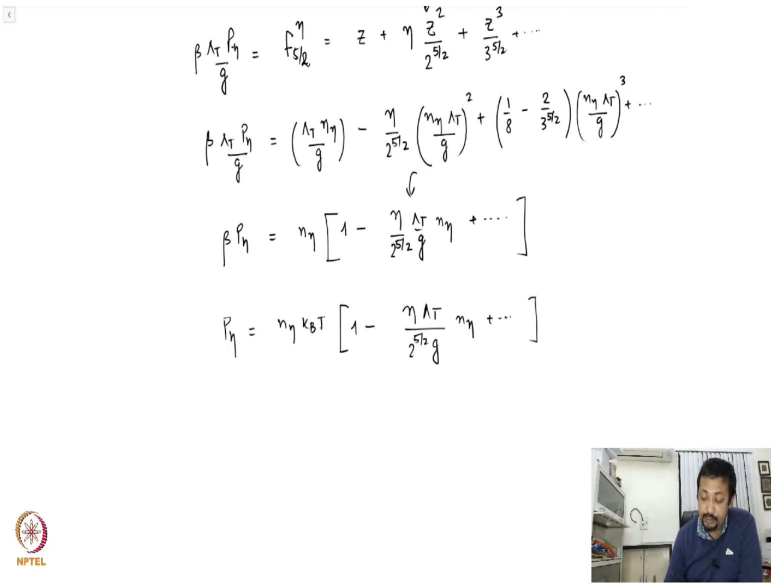

Now, if I substitute z 3 in the value of here; in the expression for beta p eta, then I see that this is going to be lambda T eta; sorry this is going to be lambda T; n eta over g minus 2 sorry.

See, if I now substitute this value of z 3 in this expression; I will come up with the expression that this is going to be lambda T; n eta over g minus eta; 2 to the power 5 by 2; n eta, lambda T over g whole square plus 1 by 8 minus 2; 3 to the power 5 by 2 and eta; lambda T over g whole cube plus higher order terms.

So, this is going to be beta lambda T; P eta over g. So, you immediately see that the lambda T over g cancels out from the left hand side. So, that I have n eta; I can take this common as 1 minus eta; 2 to the power 5 by 2, 1 factor of lambda T by g is going to cancel out from the left hand side. So, I am going to be left out with lambda T over g times n eta plus higher order terms.

In fact, just to determine; this correction it is sufficient to keep up to the first order in z 1 so that the thermodynamic pressure of an ideal quantum gas using the canonical formalism comes out to be P eta, as n eta k B T; 1 minus beta lambda T; 2 to the power 5 by 2 g n eta plus higher order terms. And you immediately see that this is the second variant coefficient; it is a correction over the ideal gas equation P is equal to n k B T.

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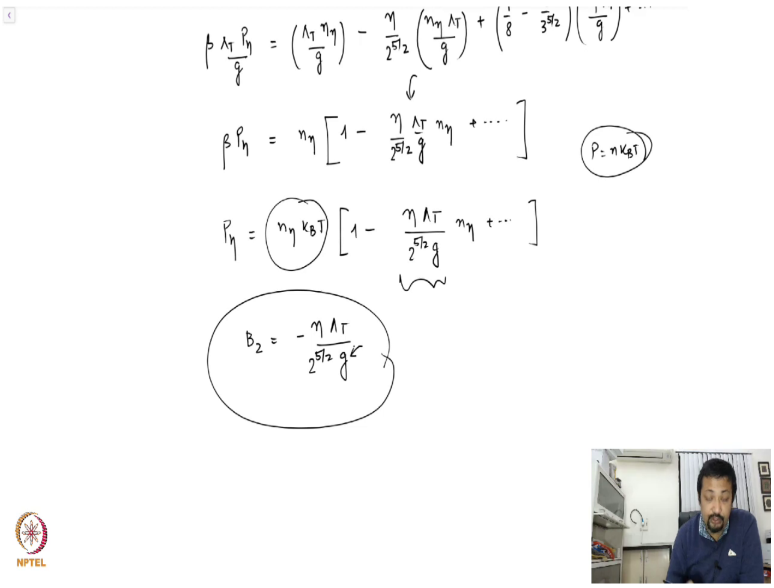

And that correction B 2 is minus eta; lambda T over 2 to the power 5 by 2 g, it is an identical correction that we obtained in the canonical and symbol; when we did this in the using the canonical and symbol, except that there is an additional factor of g which we could have also taken care there, but we did not now we have done it explicitly in this case.