Statistical Mechanics
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Lecture - 27
Microcanonical Ultrarelativistic Gas and Quantum Solid

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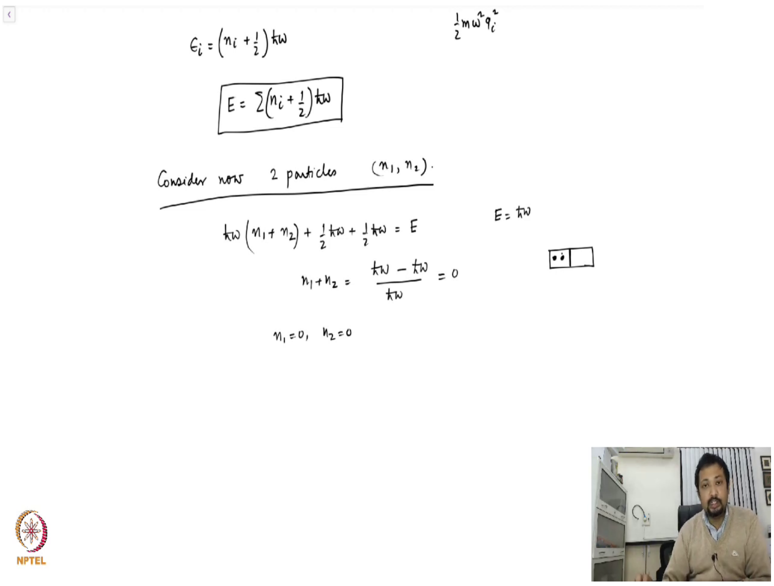

Think about it that I have two boxes and I am placing both of them in one of the box, nothing else is allowed; no other freedom.

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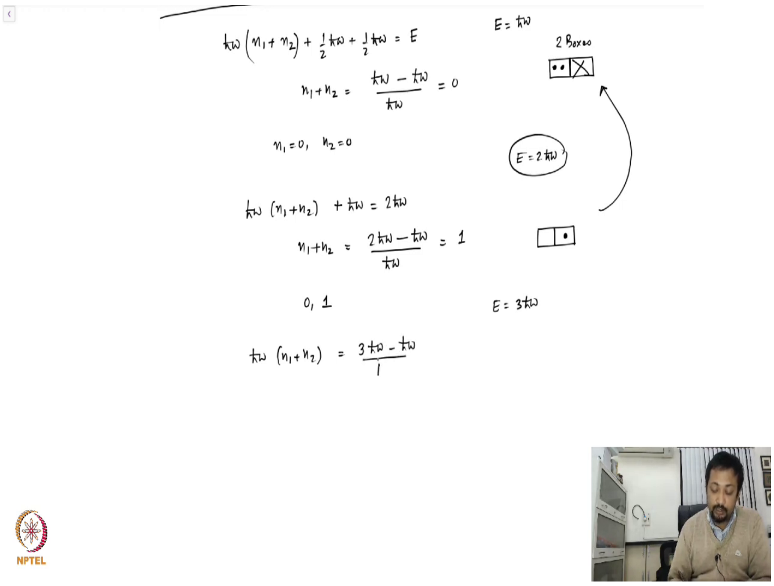

If you have n 1 plus n 2 plus h bar omega; if you set E is equal to 2 h bar omega, then this essentially means that you have twice h bar omega and n 1 plus n 2 is 2 h bar omega minus h bar omega divided by h bar omega that is 1. So, effectively now you have to figure out the combination of n 1 and n 2 which will give you one; this total energy 2 h bar omega. So, one possibility is 0 and 1 right.

So, therefore, this is like having 2 boxes in which you place one of the particles in this box and no particles in this box. In this particular case, this is the direct contrast because in this particular case what you had was you were not allowed to put anything in this. So, effectively as if even though you have two boxes even though you have two boxes, you have 0 particles to play with right.

What about the case when E is equal to 3 h bar omega? So, you have n 1 plus n 2 is going to be 3 h bar omega minus h bar omega and we can bring this h bar omega here and you will have 2.

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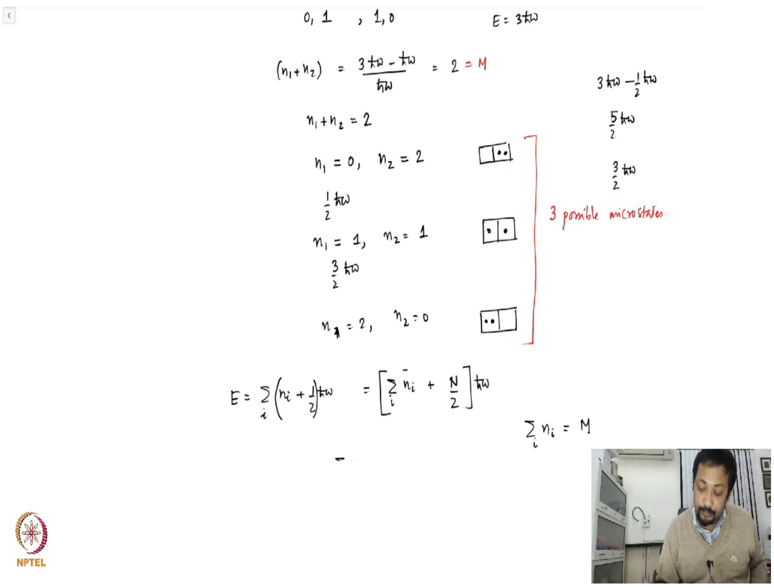

So, n 1 plus n 2 is going to be 2; that means, I have n 1 is equal to 0 n 2 is going to be n 1 equal to 0 gives me half h bar omega and I have to take it to 3 h bar omega minus half h bar omega. So, that is going to be 5 by 2 h bar omega. So, I have to place n 2 in a level where I have the whose energy is going to be 5 by 2 h bar omega and that happens when n 2 is equal to 2.

What happens when n 1 is equal to 1? This gives me 3 half h bar omega and I need another 3 half h bar omega right. So, that n 2 can also be 1 which is obvious because n 1 plus n 2 has to be 1 and then finally, I have n 1 is equal to 2 and I have n 2 is equal to 0. So, as if I have now

2 boxes, I am placing 0 particles in this box and both of them are over here and here the configuration is this and here the configuration is this.

So, I hope the general idea is clear to you now. So, going back to the general case where I had E is equal to sum over i n i plus half h bar times omega which gives me sum over i n i plus n by 2 times h bar of omega.

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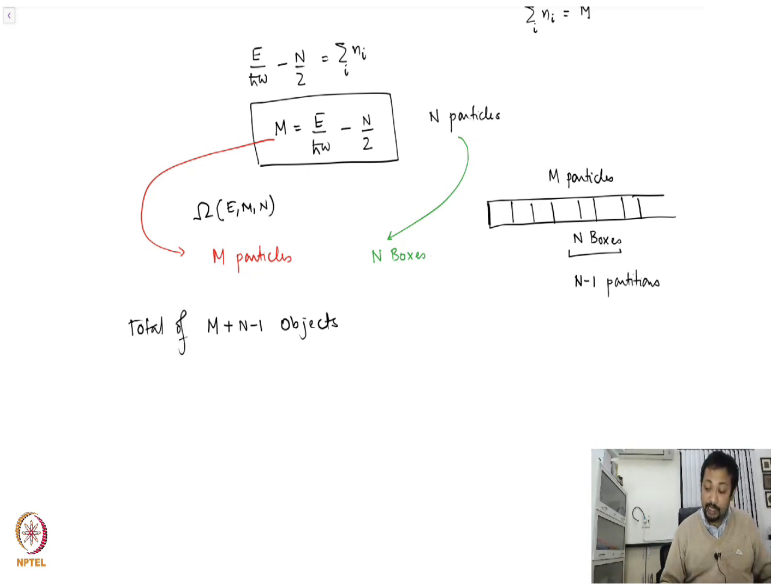

So, that this h bar omega minus n by 2 is going to be sum over i n i and clearly this sum over i which is the sum over the occupation numbers is going to be M an integer. So, I have M is equal to E over h bar omega minus N by 2. Now, the idea is with N particles. So, the entropy for the total number of microstates is omega is going to be a function of E, M and N.

Idea is now the interpretation is as we have seen in the earlier examples if you look over here, then 2 boxes means 2 particles sorry the other way around 2 particles n 1 and n 2; two boxes.

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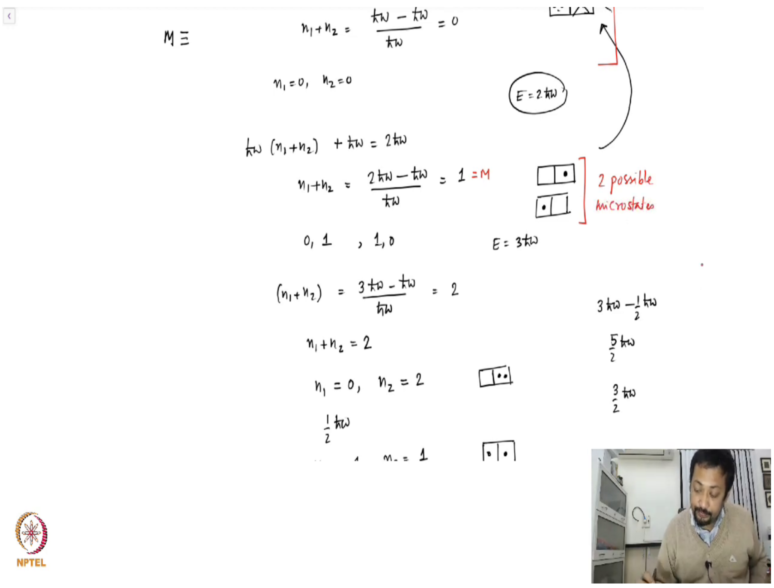

This quantity which is identical to M is 0 means no particle, you have no particles to play with. They have to be there in that box when you have this case. In this case M is equal to 1. This is equal to your M and you have seen that n 1 n 2 either has a combination 0 and 1 or 1 comma 0 that is also a valid thing in which you place one of them here, the other one here it is like 2 boxes and you are just putting one particle either in this box; in this box.

So, there are two possible microstates two possible microstates here and in this case, as we have seen in detail where N M the value of M was 2; there are three possible microstates. So, the general idea is that this is now interpreted as M particles and this is now interpreted as N boxes.

Therefore, its like you have N boxes and you have M particles. Well you are thinking about particles, you can think about M marbles also which you want to put them in the boxes and this effectively means that N boxes means the end of these things are fixed. So, you have N minus 1 partitions.

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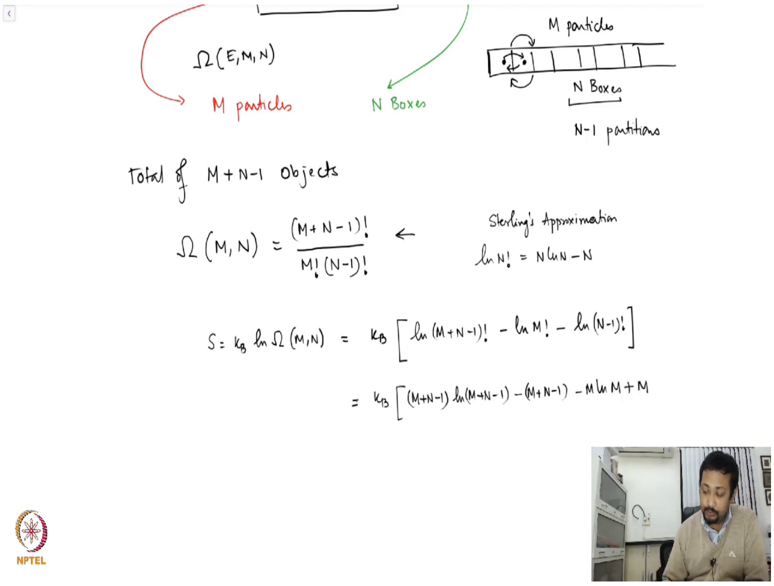

So, total of M plus N minus 1 objects and for this all possible arrangements that are possible are M plus N minus 1 factorial, but then you see if you interchange this partition with this one, you are doing nothing nothing changes. Similarly if you have placed a particle over here and if you have placed a particle in this box; if you interchange these particles, you do not get a new microstate.

So, clearly this is an over counting and the over counting is exactly given by M factorial because you can interchange between the particles and that is not going to give you anything else new and you can interchange between the partitions that is also not going to give you anything new. Therefore the total number of microstates M comma N is going to be this.

So, now, when we have this expression for the total number of microstates, we want to write down the entropy of the system as K B ln of omega M comma N which essentially means that this is K B ln of M plus N minus 1 factorial minus ln of N factorial minus ln of N minus 1 factorial.

So, if I now use a Sterling's approximation if you recall the Sterling approximation, essentially approximates log of a factorial N factorial for large enough N as N ln N minus N. We write down this as K B M plus N minus 1 ln M plus N minus 1 minus M plus N minus 1 minus M ln M plus M do not make the mistakes with the signs.

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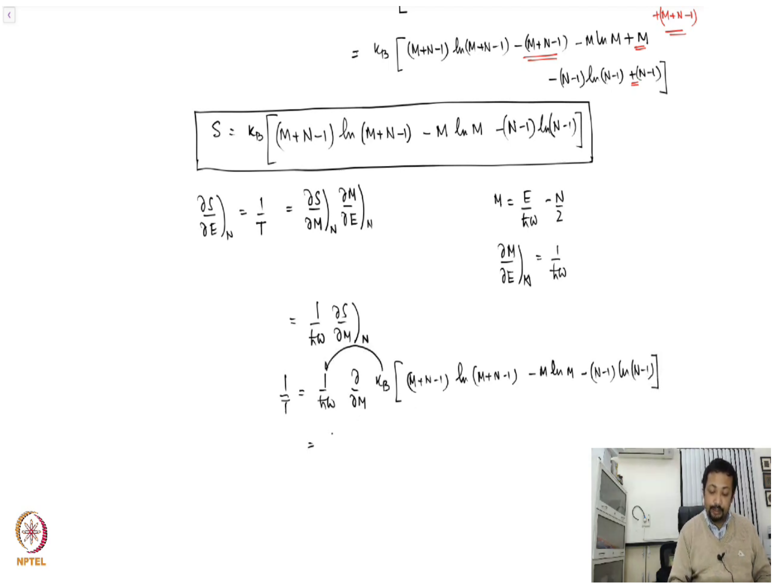

And then you have minus N minus 1 ln of N minus 1 plus N minus 1. Look at this expression carefully and you see that this plus and this plus gives you a plus M plus N minus 1. So, this factor now cancels with this factor and you have S is equal to K B M plus N minus 1 ln of M plus N minus 1 minus M ln M minus N minus 1 ln N minus 1.

This is the entropy of the system to proceed further we note the thermodynamic relation del S del E and held fixed is 1 by T. Now this although it looks very complicated, this expression for the entropy looks complicated and taking derivative might be very cumbersome. One can simplify this just writing del S del M N held constant times del M del E N held constant recall that M is going to be E over h bar minus N over 2.

So, that del M del E gives you N held constant gives you just 1 over h bar omega and you have del S del M N held constant 1 over h bar omega. So, that you have 1 over h bar 1 by T is equal to 1 over h bar omega del del M of K B M plus N minus 1 ln of M plus N minus 1 minus M ln M minus N minus 1 ln of N minus 1.

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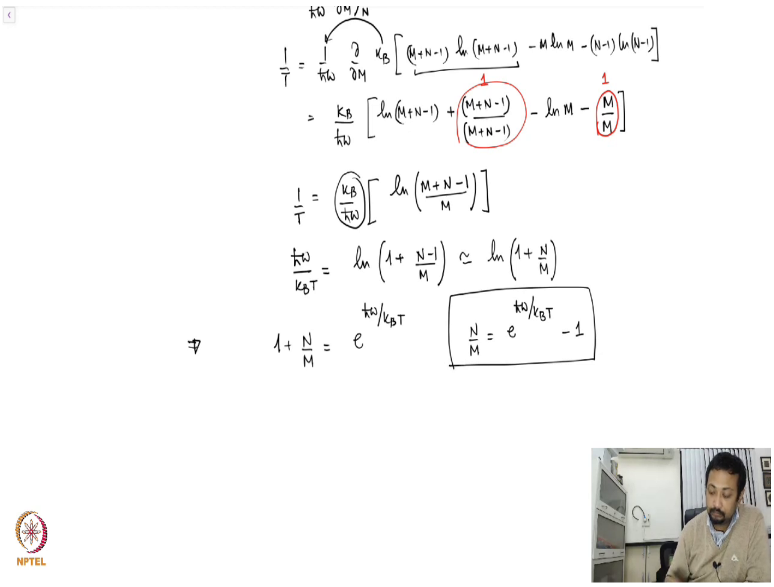

K B comes outside and sits on top of h bar omega K B divided by h bar omega. And if you take the derivative of the first term, you are going to get ln M plus N minus 1 plus M plus N minus 1 derivative of the log is going to give you 1 by M plus N minus 1 and similarly you are going to get ln of M minus M divided by M.

So, that you see this expression this term is 1, this term is 1 which cancels with each other because both of them have opposite signs, you have ln M plus N minus 1 divided by M. This part I am going to recast by writing h bar omega over K B T is going to be ln of 1 plus N minus 1 over M.

Now, since we are in the thermodynamic limit such that N is very large, this is going to be approximated as ln 1 plus N over M and this implies that ln sorry 1 plus N over M is going to be e to the power h bar omega over K B T. So, that N over M is going to be e to the power h bar omega over K B T minus 1.

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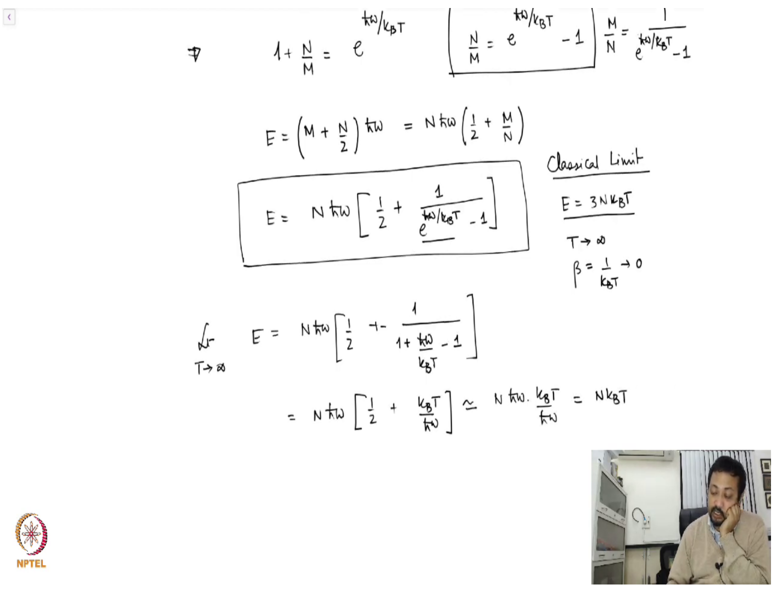

Now, E where did we write that? We take this expression, E is going to be M plus N by 2 times h bar of omega. If we take N constant common and outside, this becomes half plus M by N. So, that the energy is N h bar omega half plus M by N follows from this equation is 1 over e to the power h bar omega over K B T minus 1. So, that I have 1 over e to the power h bar omega over K B T minus 1 right.

Now, the question is I want to cross check and validate my result. So, the best way to check it first thing that you check is the classical limit. So, in the classical limit, we have seen that this system is going to have an energy 3 N K B T. We have already done this. Hence so, to see whether we go to this limit, what we need to do is to substitute for T to infinity. So, that beta which is equal to 1 over K B T is small. It goes to 0 therefore, you see I can expand the exponential.

So, I have N h bar omega half 1 plus 1 plus h bar omega over K B T minus 1 E limit of T to infinity which is going to be N h bar omega half plus K B T over h bar omega. And since my K B T is very large compared to this p factor of half; this is going to be N h bar omega times K B T over h bar omega which gives me the result in K B T which is precisely the result for this because we are looking at a only 1 dimensional system.

We said in the very beginning that 3 dimensions can be independent can be treated independent of each other. So, if you just do it for 1 dimension you can just adding the results. So, for a 3 dimension, you are going to have two more factors of N K B T coming in and you will have this.

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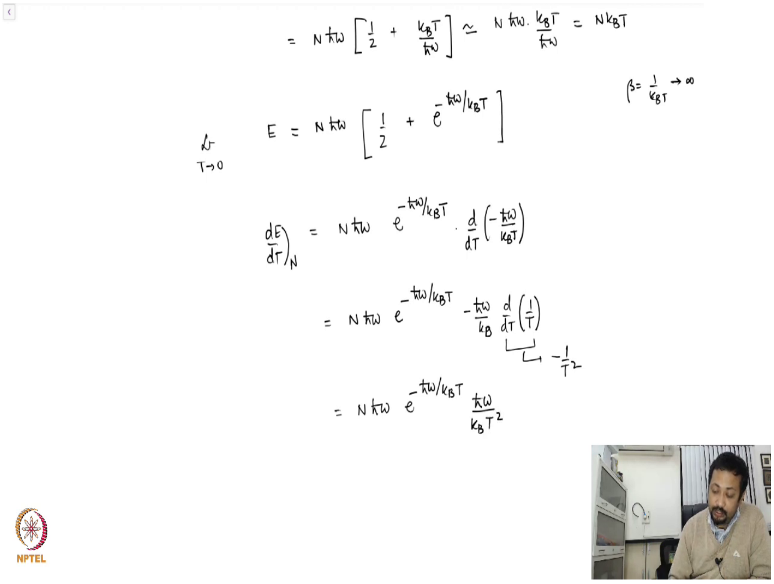

Limit of T to 0 is when this factor the exponential factor dominates because 1 over K B T beta goes to infinity and you have N h bar omega half plus e to the power minus h bar omega over K B T.

Once again the specific heat now becomes dE dT at constant N which is N h bar omega e to the power minus h bar omega over K B T times d dT of minus h bar omega over K B T which is going to be N h bar omega e to the power minus h bar omega over K B T. I have a minus h bar omega over K B and the derivative d dT of 1 over T which is going to give me minus 1 over T square. So, this is minus 1 over T square.

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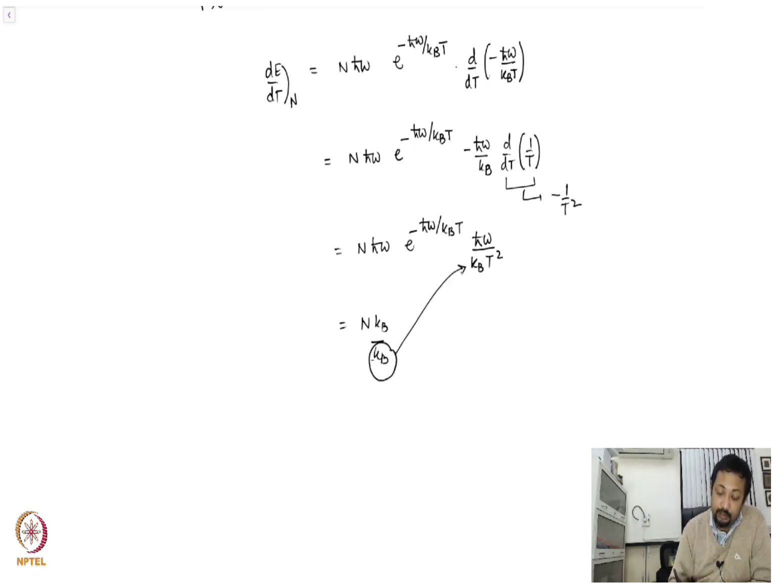

So, I have N h bar omega minus h bar omega over K B T. I am going to have h bar omega over K B T square. Let us recast this a little bit rearrange multiply by K B and divide by K B. So, that this K B, now I plug it over here to give me h bar omega divided by K B T whole square.

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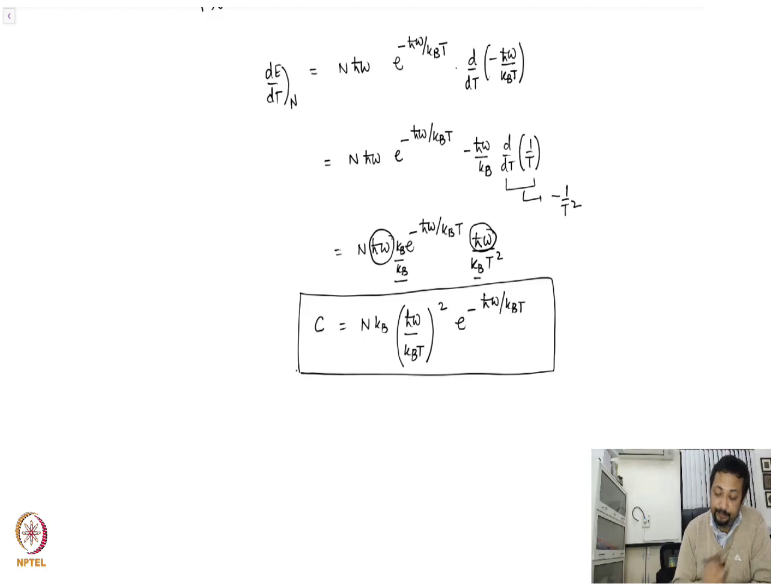

So, there is one factor of h bar omega one factor of h bar omega that gives me this and I have multiplied with K B divided by K B. So, I have multiplied by K B and divided by K B and this and this gives me K B T square e to the power minus h bar omega over K B T. So, this is going to be your specific heat.