Fluid Dynamics for Astrophysics
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Lecture - 09
Potential flows

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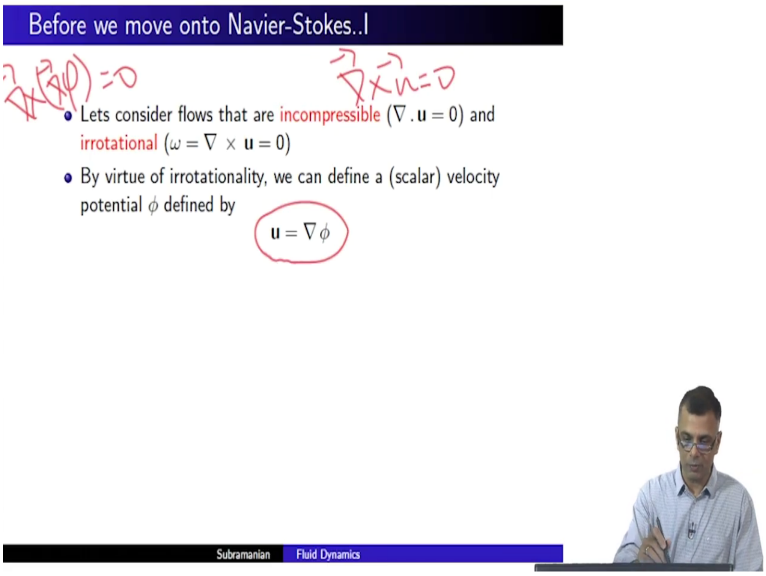

Hi, so we are back. As promised we will move on to the Navier-stokes equation in its traditional form, but before doing that I figured you know, we can make use of whatever we have learned so far especially with regard to stream functions and vorticity and everything to investigate a few interesting properties before moving on to the Navier-stokes equation. Which is in some sense one of the cornerstones of fluid dynamics right.

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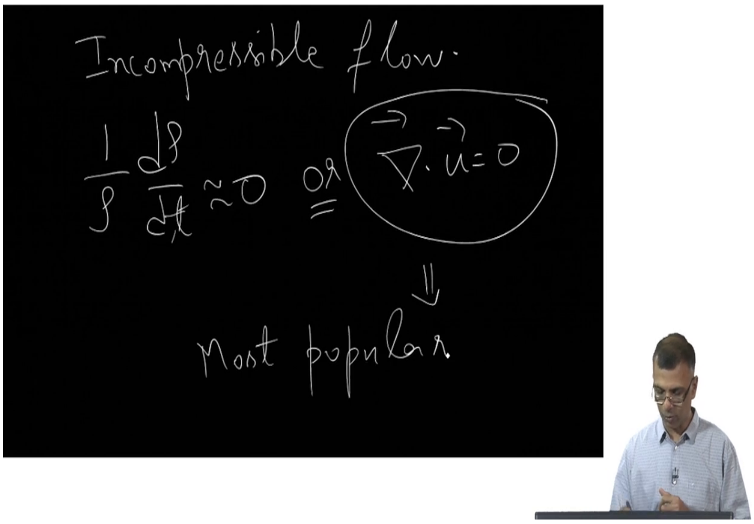

So, let us first consider flows that are incompressible, in other words characterized by you know divergence of u equals 0. Remember that the definition of incompressibility and incompressible flow is 1 for which a Lagrangian observer; an observer sitting on top of a fluid parcel does not discern any changes in density. This is very important to keep this in mind an incompressible flow, is one where; is one where the straight this is equal to 0.

In other words the changes in density as discerned by a Lagrangian observer are 0 or equivalently and this is usually what is normally referred to this is the most popular definition for incompressible flows and that is what we have got here ok. So, that is what we have got here.

And let us also consider flows that are irrotational; in other words the vorticity which is defined by the curl of u is 0; in other words and I will say that an irrotational flow is one which is inviscid, why? Let us look at it.

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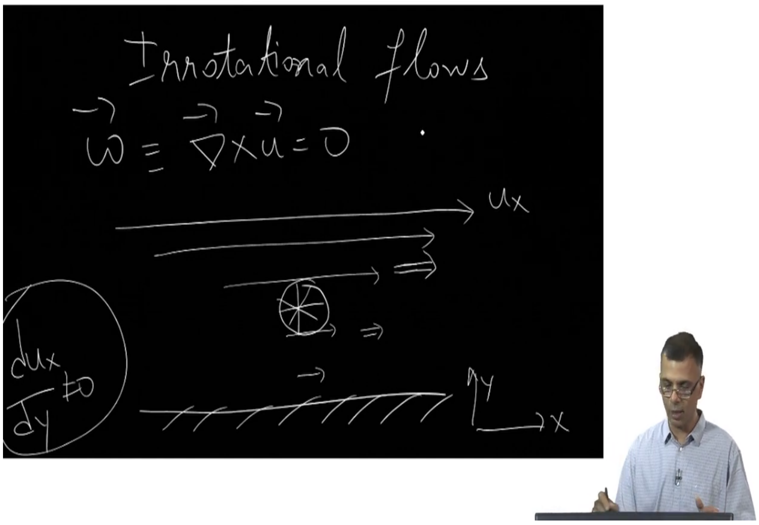

I said irrotational flows where the vorticity this is equal to 0. Now let us we have already discussed this, but its useful to recap it.

Let us consider a situation where you have a viscous flow. The same diagram that we have sketched many many times before here is the sort of bottom layer and here is the free surface of the fluid. Consider flow of a viscous fluid such as honey again you have x and y so this would be u of x.

So, a situation where you have a gradient of the u of x as you proceed in y in other, words a flow of a viscous liquid such as honey where you know the top layer is moving fast the next bottom layer is moving a little slower next bottom layer is moving a little slower so on so forth ah..

This would be one where if you placed a little paddle wheel; in here it would rotate because the force here on top is slightly larger than the force here. So, it would rotate and this follows so in.

So, this is a situation where you know d u x so this is a situation where curl of u is not 0 ok, you can arrive at this also by you know by looking at the definition of curl in. You know you remember how a curl is written like this and u x u y u z. This how so, only when you have these cross terms d over d x of u y or d over d y of u x like so.

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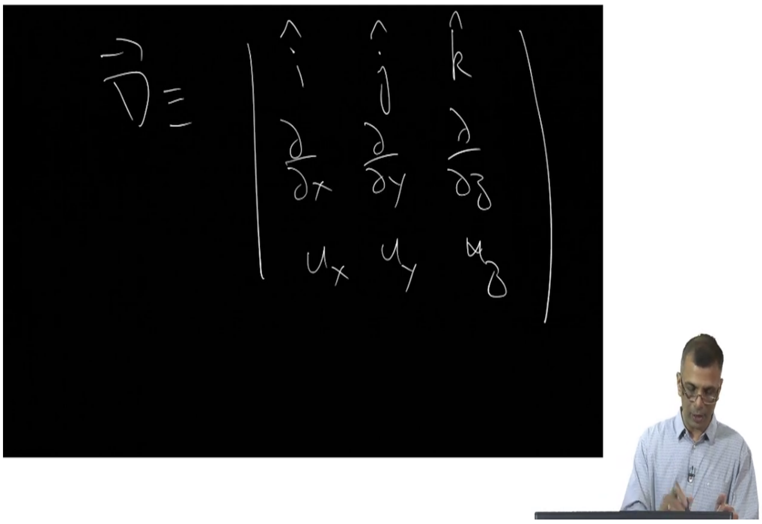

Only when these cross terms are non zero you have a you have a curl that is non zero. If that is not so then the then the flow is irrotational and this paddle wheel will not rotate and what is the situation where the paddle wheel will not rotate? Its one; well where the flow is irrotational or there is really no gradient there is really no sticking between the layers and the flow is effectively inviscid.

So, let us consider situations where both of these are true; a flow that is incompressible and also irrotational. This is not just for you know mathematical convenience well part of it is really; there are many elegant solutions that can be you know derived using these two conditions, but it is also something that is of great practical importance in many situations ok. So, let us consider a situation where both of these are satisfied.

Now, by virtue of irrotationality by virtue of the fact that you know the curl of u is 0 you can define a scalar potential like so, why? Because the curl of the gradient of a scalar function is always 0 right, the curl of u is 0. We already said that; we already said that in other words the curl of the gradient of phi is equal to 0. In other words, by virtue of irrotationality we can define a velocity potential right.

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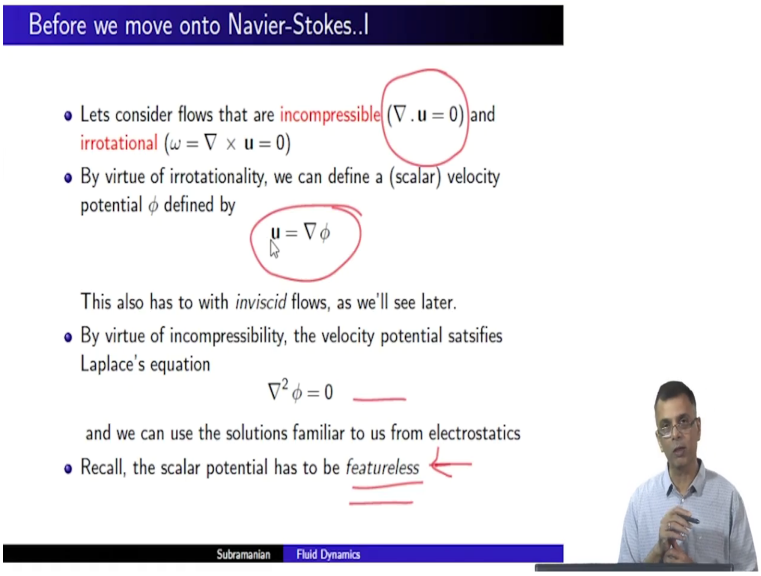

So, this has to do with inviscid flows as we will see later; I just explained this to you. By virtue of incompressibility right; so we already we have already said that u is defined as a gradient of a scalar potential. On top of it we are saying that the flow is in incompressible. In other words the divergence of the gradient which boils down to this Laplacian here is equal to 0 right.

So, we are looking at solutions of Laplace's equation for the velocity potential ok. And this is something that is well known in from many fields say from electrostatics so, we can take advantage of many of these results in dealing with flows that are effectively incompressible and irrotational ok.

So, remember that this is a consequence of both these; only if both of these are satisfied you can you can have you know you can have a potential you can solve Laplace's equation to get the potential and from the potential you can get the velocity field. And we can like we said we can use solutions that are familiar to us from electrostatics.

Recall the scalar potential has to be featureless, I simply say recall and this is from let me see what I have got in my yeah. So, before saying that let us explain this one; this statement a little bit.

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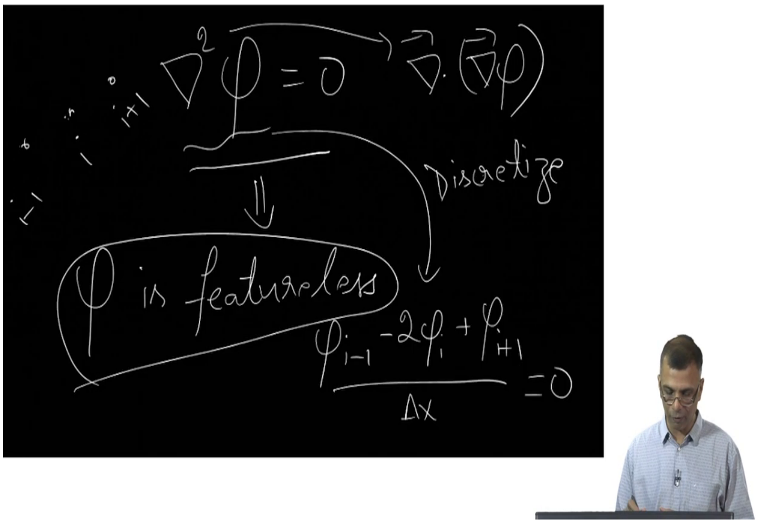

So, what are we talking about? We are talking about right. Or well this came from the fact this simply came from the fact that that is what this is.

Now, I made the statement that this implies that phi is featureless, by that I mean this is the phi does not have hills or valleys or anything it is a nice flat surface. You either know this from electrostatics or you remember it or even if not we can. So, this is essentially Laplace's equation. So, any solution to Laplace's equation is featureless one way of understanding this is by discretizing this right.

How? Say in just one dimension, you discretize this second derivative right. How does it look like? It looks like something like phi i minus 1 minus 2 plus over either 2 delta x or delta x. That is what this one is the discretized version of that ok. So, this is a phi i plus 1, what this is really saying; and you can you can verify this you discretize a double derivative just in one dimension and that is what this is.

What this is really saying is that the middle point phi i. So, that you are discretizing on a grid like this yeah this is i this is i minus 1 and this is i plus 1 you can discretize in the other direction also, but its not.

So, important for the for the purposes. So, what you are really saying is that the middle point is really the average of the value of phi at point i is really the average yeah of and this if this is equal to 0; that means, the numerator has to be equal to 0 right.

So, what you are saying is that phi i is really the average of this and that. So, you are really averaging if there are hills or valleys yeah, you are averaging really removes features, if you think about it, if there is a; if there is a peak here and a valley here if there is a peak here and a valley here what this process is really doing is averaging over the peak in the valley and getting an average value at the middle, and you are you keep doing this ok.

So, in doing so you ensure that the solution to this equation is featureless in doing so, you ensure that this is feature this is one way of understanding this there are other ways of understanding this also ok. So, I figured out, point this out before moving forward because this is such an important point.

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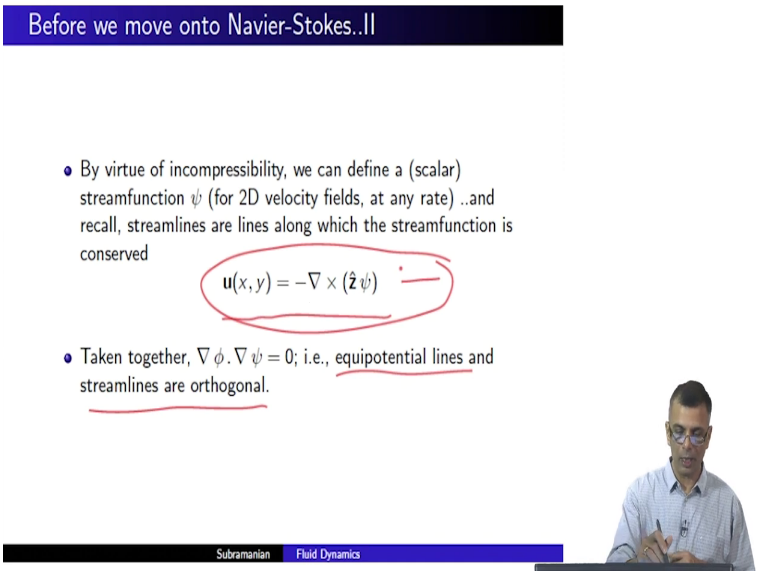

The other thing to remember is that by virtue of incompressibility by virtue of this condition of this condition yeah, you can define a scalar stream function psi, well it is for 2D velocity fields at any rate and recall streamlines are lines along which you know the stream function this is conserved right.

You can write down by virtue of incompressibility by virtue of the fact that the divergence of u has is always equal to 0 that is the assumption we are making. So, the divergence of a curl is equal to 0 we know that so you can write down stream function like so.

And so since psi is 2 dimensional the direction is along the along the z hat direction. So, you can write down this kind of a equation for u.

And this effectively the stream function by virtue of its name and we also saw how you know a stream there are different definitions of a streamline, they are physically intuitively appealing definition of a streamline is 1. Where you are essentially just joining the little velocity vectors you like that you join them together and you form a streamline yeah.

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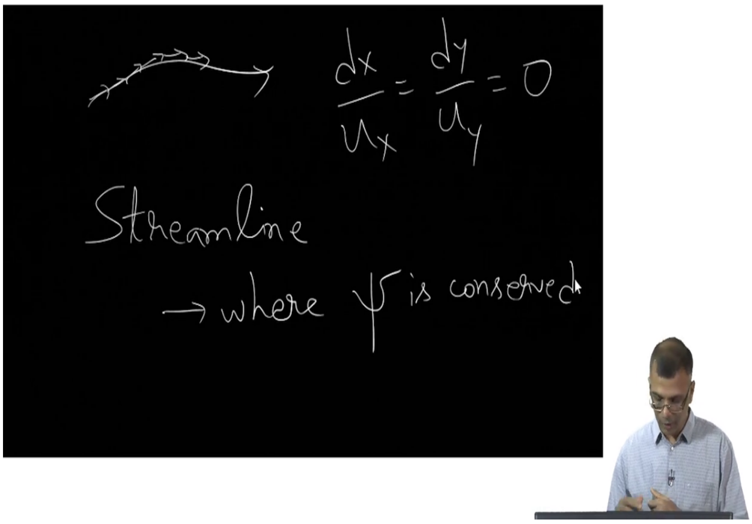

And you can also write it as like that. You can also write a streamline is a streamline is 1, where the stream function psi is conserved; in other words each streamline has a particular value for psi labels a streamline right. So, that is a stream function and taken together in other words taken in other words this equation and this equation taken together gives you this condition and I urge you to work this out.

And we will show why this is so also a little later on. Taken together this implies that the equipotential lines and streamlines are orthogonal ok. So, the gradient of the equipotential surface and the gradient of the stream function surface you dot this together and they are equal to the dot product is equal to 0. So, that is just what this is saying.

And you this is also something that is familiar to you from electrostatics right; you move from equipotential surface to equipotential surface and that is the direction of the an electric field line ok. So, if you want them to jump from one equipotential surface to another one you are moving orthogonal to the equipotential lines right. So, that is what this is saying there is also something that is familiar to you from electrostatics right.

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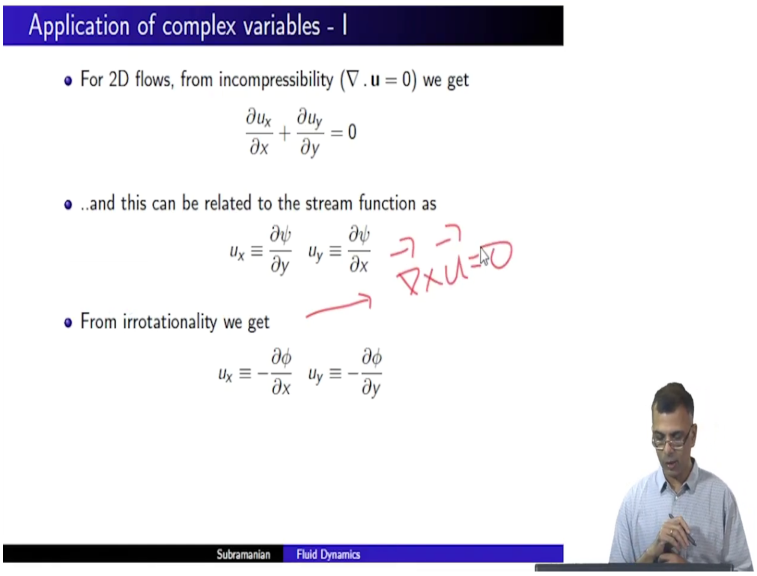

So, and we hinted at this early on you can apply the theory of complex variables to; let us go on and we will see why that is ok. So, for 2 dimensional flows from incompressibility remember we are talking about an incompressible irrotational fluid, incompressible inviscid fluid, but yeah. So, if you want to restrict yourself just to you know 2 dimensional flows this is essentially the same as this right.

It is simply the definition and this can be related to the stream function as this right. That simply follows from this simply follows from the definition of the stream function just follows from the definition of stream function here ok. You only have ux and ui and therefore, from this definition you just take the curl and this follows ok, this is from incompressibility.

From irrotationality you get this ok, ux in other words from the fact that from irrotationality is just a statement of; messing up here right.

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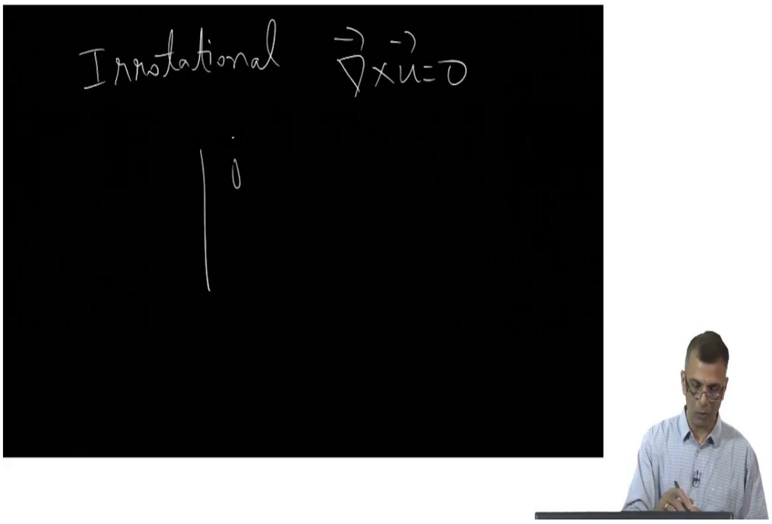

You merely write down you know irrotationality is simply saying that curl of u is equal to 0 and you remember the curl is just i we had written this down already no need yeah. So, this definition here right.

So, you do not have to; you do not have to take this last column; because you are only talking about you know two dimensional flows. So, if you just consider this much this follows right.

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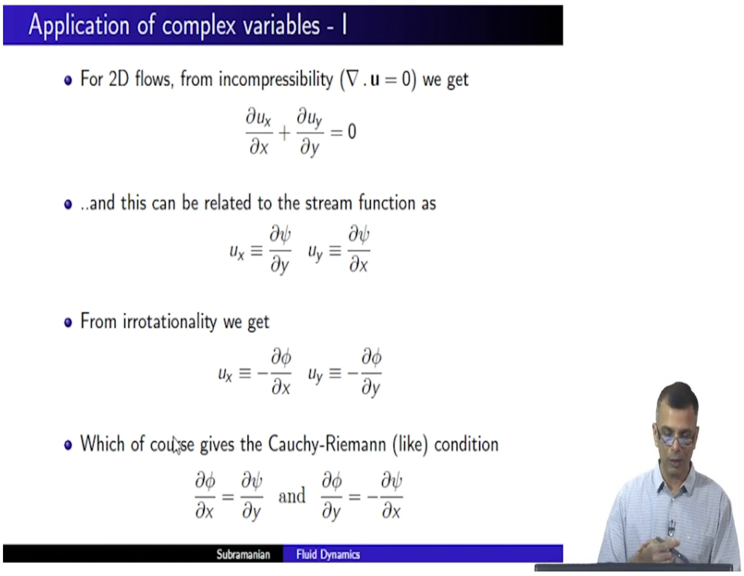

So, therefore, you look at this and this taken together taking both of these together you get you know d phi d x is d psi d y whereas, d phi d y is minus d psi dx right.

So, this is very familiar; I mean its essentially the same as the Cauchy Riemann condition for complex variables. Almost as if you know phi m psi are from phi m psi taken together form one complex variable ok. And this is the Cauchy Riemann condition for complex variables right.

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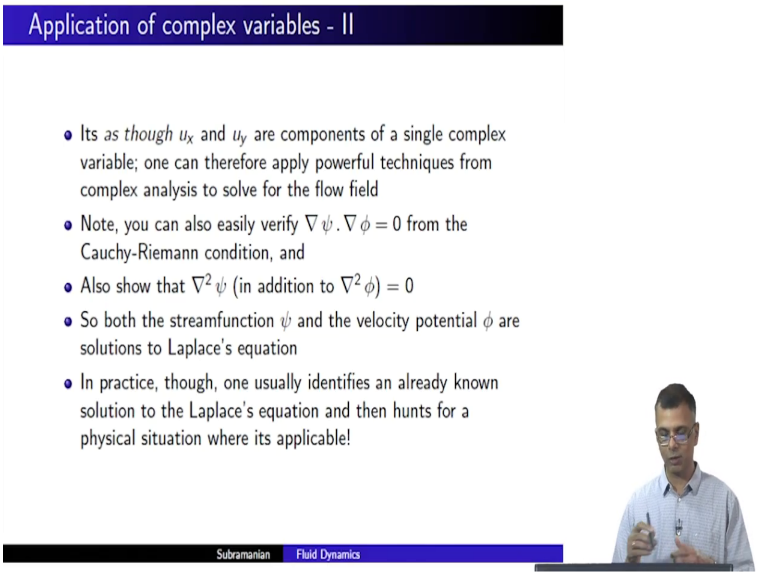

So, I just wanted to point this out and this yeah is as though ux and ui are components of a single complex variable one can therefore, apply powerful techniques from complex analysis to solve for the flow field 2 dimensional flow fields I must emphasize. Although you know a lot of this is done from mathematical convenience its also very relevant to real flows, these are the early days this is how much of the early progress in fluid mechanics was made.

And you can also easily verify you remember we remarked on this condition the fact that streamlines and potential lines are orthogonal to each other. You can this follows directly from the Cauchy Riemann condition ok. You can also in addition to the fact that del square phi equals 0; which you know which follows through incompressibility and irrotationality both ok.

You can also show that del square psi is also equal to 0. So, both are solutions to a Laplace equation with different boundary conditions of course, ok, but both are solutions to the Laplace equation. So, this is another remarkable yeah. So, both the stream functions and psi and the velocity potential phi both are solutions to the Laplace equation.

In practice and this is the trick in practice though one usually identifies an already known solution to the Laplace equation and then hunts for a physical situation where its applicable it. Turns out that there are very many its not simply playing games mathematics its there are very many interesting physically interesting situations where such solutions are applicable.

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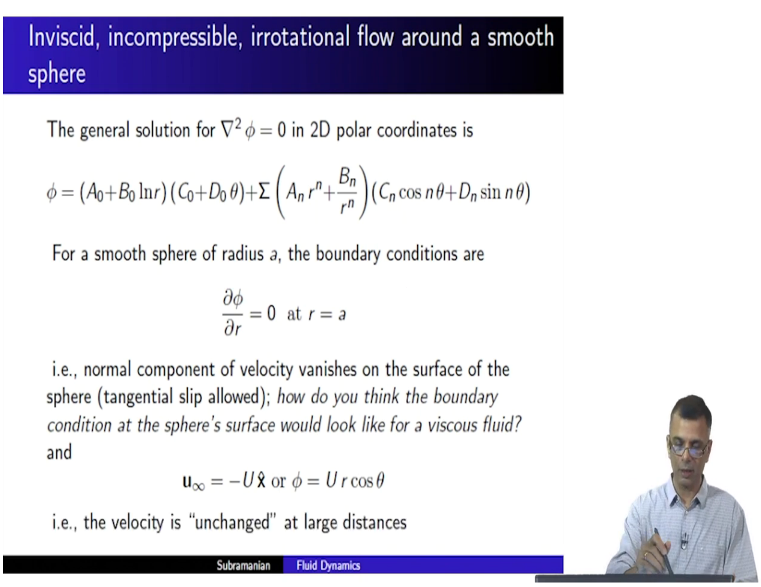

Let us look at 1 or 2 of them right. So, an example of an inviscid incompressible irrotational flow around a smooth sphere, this might seem like lots of adjectives inviscid incompressible irrotational smooth sphere. But, you know its a highly idealized situation to be sure, but its very very you look at these ideal solutions and you have a fairly good idea of how you know a real world solution will behave ok.

So, there is a lot of value to looking at these idealized solutions and figuring out at the general behavior, I will leave it at that right. So, now from mathematics the general solution for del square phi and I remember phi is the velocity potential in 2D polar coordinates, 2D polar coordinates as you know r and theta yeah like.

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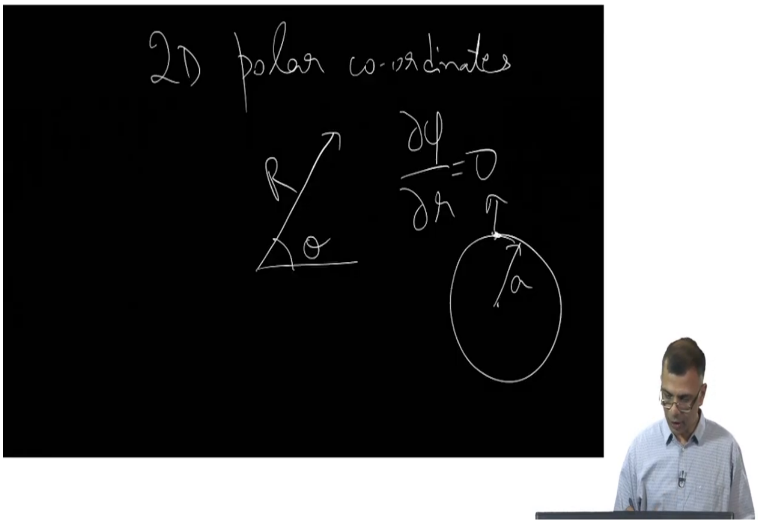

So, are just you know this r and theta these are the only two things that matter this is the general solution.

I am simply writing this down you can find this in many places you can think of this as it simply follows from the definition of del square in 2D polar coordinates, you can the this follows from Legendre polynomials its really a Legendre polynomial that is what it is; these are ylms and. So, you can see only two variables here you can see the r and you can see the theta right.

So, there are only two variables here. So, this is the general solution and you apply the boundary conditions appropriate to an inviscid incompressible irrotational flow around a small sphere, you apply the appropriate boundary condition and you get the solution immediately.

We will see how, we would not go through the gory details, but we will see how yeah. So, you see for a smooth sphere of radius a the boundary conditions are d phi; I claim that these are the boundary conditions d phi d r equals 0 at r equals a.

In other words at the; so, I claim that this is sphere of radius a and at the surface I am claiming that right, what does this mean? In other words you remember from the definition of phi and what was the definition of phi we will go back, we might have to go back a little ways this is how phi was defined remember u is a gradient of phi.

So, if d over d r of phi is equal to 0; that means, you are talking about the normal component of the velocity, but the normal component this component either you are really talking about the normal component of the velocity. So, the normal component of the velocity is equal to 0 this is what we are saying with this is what we are saying with this statement, but the tangential velocity need not be equal to 0.

In other words, you are allowing for infinite slip at the surface of the sphere ok, and this has to do with this adjective. So, this adjective has to do with this. How do you think the boundary condition at the sphere surface would look like for a viscous fluid? Think about it an inviscid fluid is one where there is no sticking, the that the fluid does not stick to the surface.

That is how you think about you know think of the flow for instance a flow of honey past a solid sphere as opposed to the flow of a less viscous fluid such as water past you know smooth sphere right.

Water tends to slip more easily at the surface whereas, honey tends to stick. So, tangential slip is allowed for an inviscid fluid not allowed for a viscous fluid ok. So, at any rate we are now talking about inviscid fluid. So, this is the first boundary condition we apply and you see this is the second order there you know the its like a d square over dx square or d square over d y square kind of situation.

So, you need two boundary conditions right. So, this is one boundary condition already the other boundary condition the other natural boundary condition is now I am mixing up between polar coordinates and rectangular coordinates, but really its a small thing to transform between 2 dimensional polar coordinates and x y coordinates.

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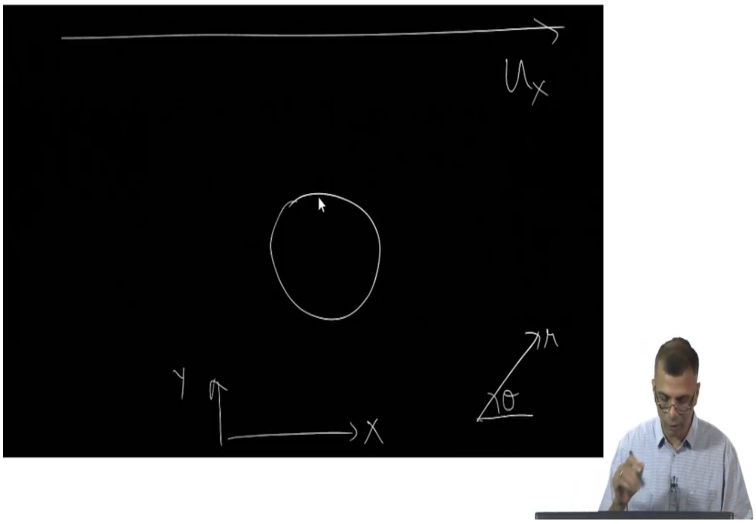

What this means, what this condition means is that, at very large distances from this sphere say like so you have a sphere and this is X and this is Y. Although we are really talking in terms of spherical polar coordinates we are we are talking in terms of r and theta what we are really saying is at infinity very far away from the sphere here or here let us just draw it here.

The sphere hardly has any if the fact that there is a sphere present hardly has any influence on the flow ok. If the flow was flowing in the x direction like so no question of any gradients or anything because we are talking about inviscid fluids. So, the sphere hardly disturbs the flow very very far from the surface of the sphere. So, we have two kinds of boundary conditions one very far from the surface of the sphere one at the surface of the sphere.

At the surface of the sphere we have this boundary condition very far from the surface of the sphere we have this boundary condition. In other words, we are saying that very far from this from the surface of the sphere the velocity is entirely in x direction with some amplitude U. So, these are the two boundary conditions; when you impose these two boundary conditions this and this on this general solution yeah ok

So, here is the other thing, so either you think in terms of the velocity or you can we are really looking at the potential are not we. So, it is useful to write down the potential and that turns out to be u or cosine theta and this is nicely you know written down in spherical polar coordinates. And so you have these two boundary conditions this and this on phi.

So, and this would be you know; I think that that is called the Neumann boundary condition this is called a direct Dirichlet boundary condition unless I have got them mixed up. At any rate this is a mixed you either have 2 Dirichlet 2 Neumann or a mixed. So, here is a here is a boundary condition on the variable itself and here is a boundary condition on the derivative of the variable normal derivative to be specific.

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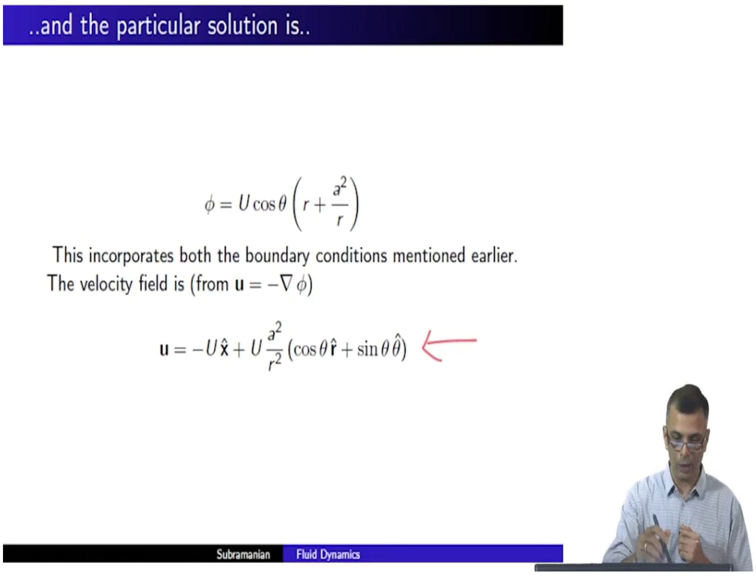

Both of these you apply that those to this particular solution yeah. So, that is what this means like I explained a moment ago and the particular solution is this ok. This incorporates both the boundary conditions mentioned earlier and you having gotten the potential it is a simple matter to just take its gradient, you would have to take the gradient in spherical polar coordinates; obviously, and that is what it looks like ok.

This is what the solution looks like this is the solution to the flow of an inviscid irrotational fluid in sorry the same thing really inviscid incompressible fluid around a smooth sphere there you go right.

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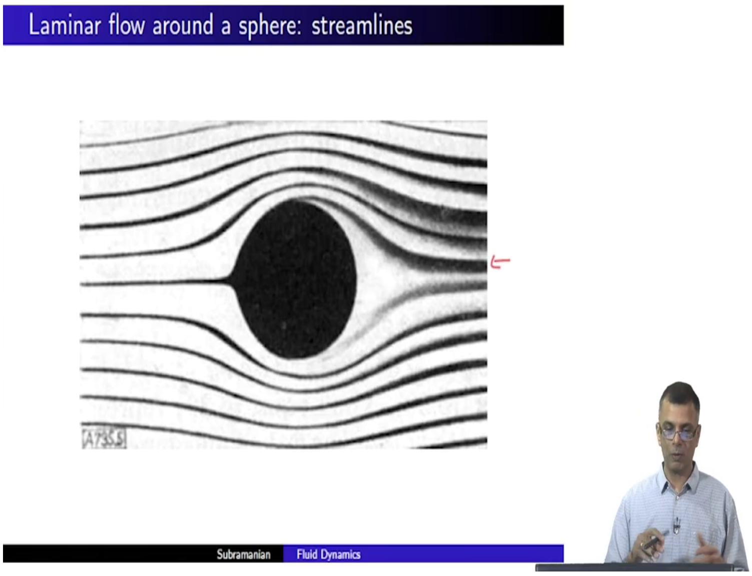

And what does it look like? That is what it looks like you plot, you plot this as a function of r and theta and that is what the streamlines look like ok.

And this is the actual photograph of streamlines; you actually inject a dye through you know a sphere through a flow that is flowing smoothly past the sphere and that is what it looks like these are the non ideal.

These are the kinds of this broadening these are the kinds of things that will not be captured with this with this nice elegant solution. But, you can see how closely you know it resembles the actual situation. So, this is an example of a situation where you know where these idealized solutions are very very useful right

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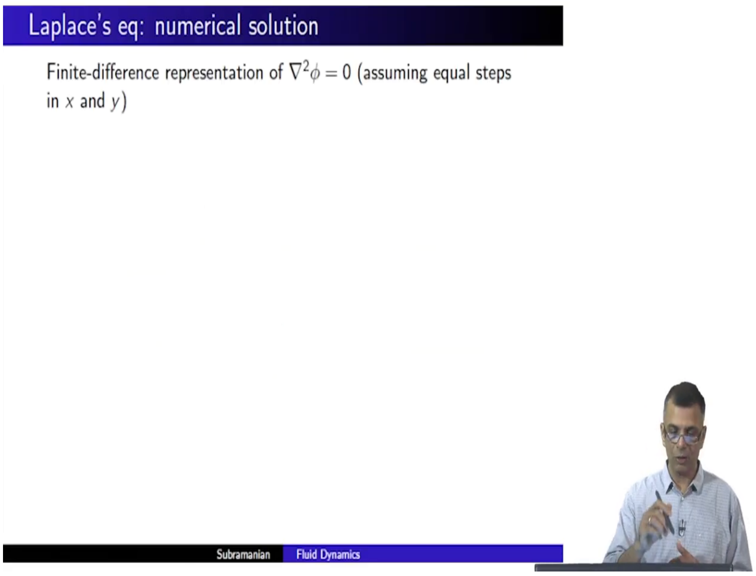

So, that is what I; this is where I will stop for the time being and we will go on and investigate a few other interesting consequence of the potential function and the stream function and before moving on to the Navier-stokes equation.

Thank you.