Fluid Dynamics for Astrophysics
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Lecture - 08
Conservation laws - Recap

Hello, welcome back. So, we will continue our discussion of Conservation equations. I figured we would go over what we have done so far a little bit in a blackboard fashion before moving on. Recall we were discussing the momentum equation.

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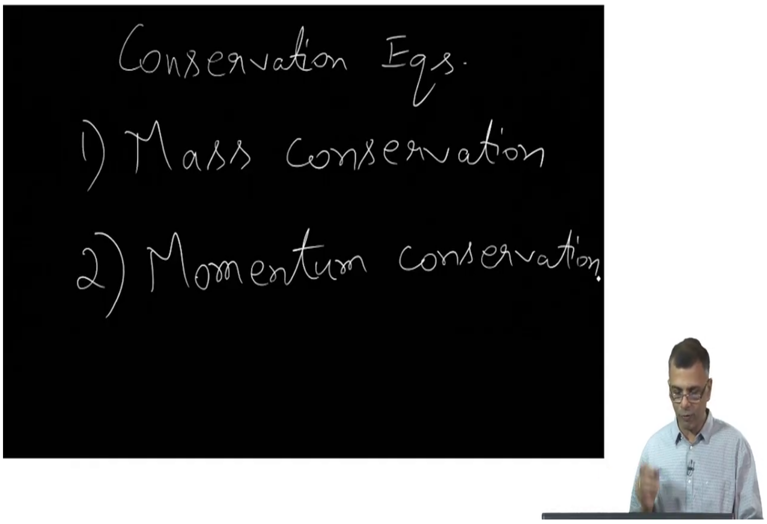

So, essentially by way of conservation equations, we discussed mass conservation and right now we are discussing momentum conservation this is what we are doing.

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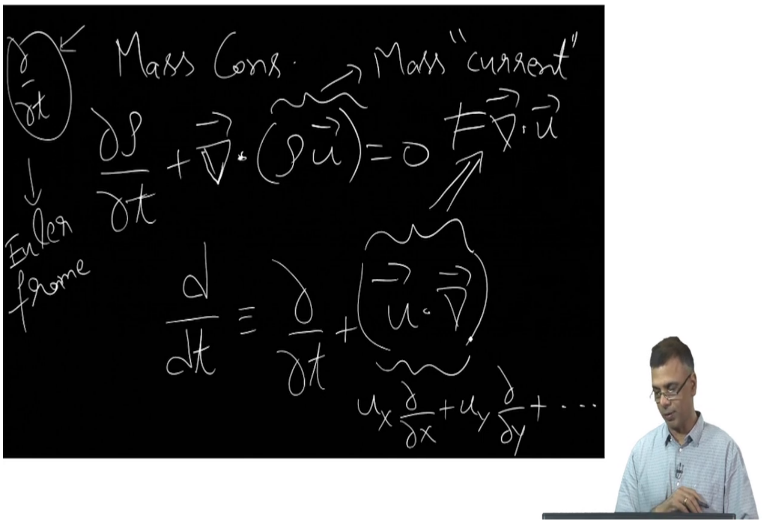

So, very very briefly when we talk about mass conservation, we are essentially saying mass conservation is essentially. So, this quantity would be something like the mass current. If you prefer think thinking in terms of you know drawing in an analogy with the electrostatics where this would be the rho would be the charge density and the and then the quantity rho u would be the j, it is the same thing. This is the very same concept.

So, one thing to keep in mind of course, is this d over d t this quantity is in the Eulerian frame ok. It is in the, it is in the frame of the person who is sitting in the lab and watching stuff go by right.

So, and you remember the relation between the Lagrangian derivative which is the straight d over d t not the partial yeah and this kind of this kind of Eulerian derivative is that this is equal to plus this is the. So, if for instance you had something. So, this operator is the same as this operator.

So, this time derivative would signify the time changes as discerned by a Lagrangian observer someone who is moving with the parcel of fluid ok. And in case mind you this is not the same as del dot u no, it is not. What this would be for instance in a rectangular I mean in Cartesian coordinates would be something like u x d over d x plus u y d over d y plus so on and so forth. That is what this operator is ok.

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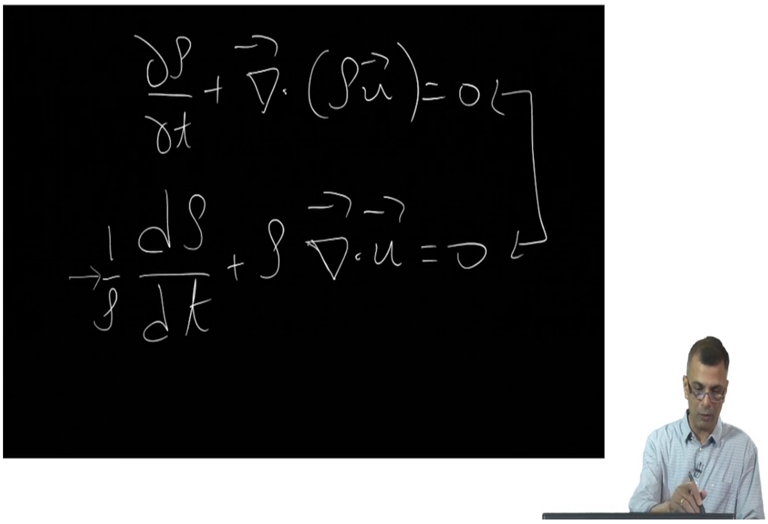

So, the mass conservation equation which is written as d rho d t plus divergence of equals 0 that in Lagrangian coordinates; I mean in by way of Lagrangian observer it would be this ok. So, these are the same thing, this and this the same. Same equation just a matter of the perspective.

So, this would be the lab perspective and this would be the perspective of an observer who is moving along with the fluid along with the fluid parcel ok. And there is an advantage there are different advantages in considering different perspective in and there is an advantage with looking at this perspective. And that is it is easy to talk about compressibility or lack thereof compressibility or incompressible you know while looking at this picture when you talk about.

So, you can you know divide this entire thing by rho and you can write one over rho here and you can this goes away right when you divide by rho. And when you are talking about compressibility you are talking about a situation where this term is important.

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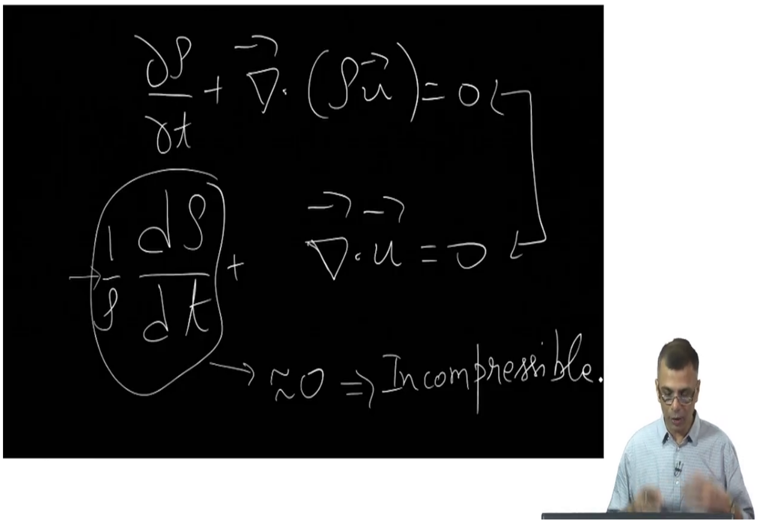

When you are talking about incompressibility if this is equal to 0 then it implies an incompressible flow right. And so, if in other words they are not as far as the I mean according to the person who is moving along with the fluid parcel, there are not too many density changes yeah.

So, in which case the incompressibility condition boils down to in which case if this first term is almost equal to 0; that means, the second term it better be equal to 0 because some of these two is equal to 0 in other.

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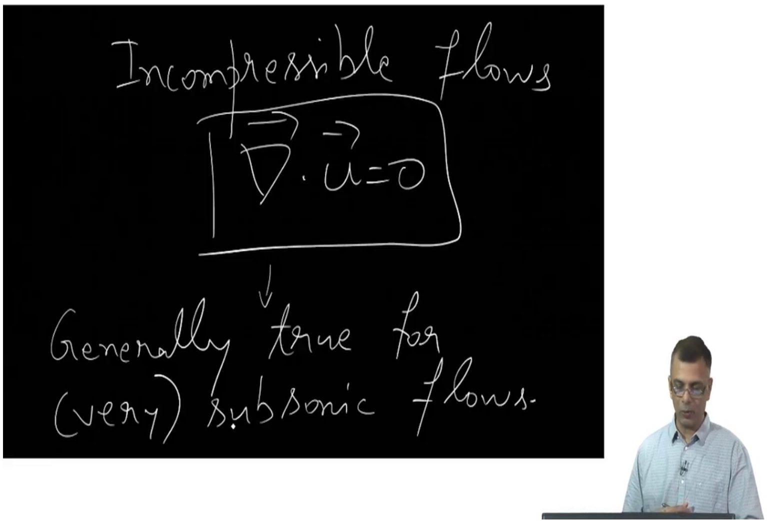

So, therefore, the incompressibility condition the condition for incompressibility the divergence of u, u does not have a source or a sink this is a condition for incompressibility right. So, now, you see this is an advantage of looking. So, it is easy to intuitively easy to understand the condition for incompressibility when one looks at the Lagrangian way of writing down the mass conservation equation right. So, then why bother with Eulerian way?

Well there is a there is a you know there is an advantage and we also we also remark that this is generally true for very subsonic flows. Let us simply keep this in mind for now and because we have not introduced the speed of sound, we have not said what a subsonic flow is, but let us just keep it in mind and we will revisit it when the time comes right.

And we also remarked that there is something called the Boussinesq approximation which is approximate, which is often boils down to the same as a as the incompressible approximation, but not always ok.

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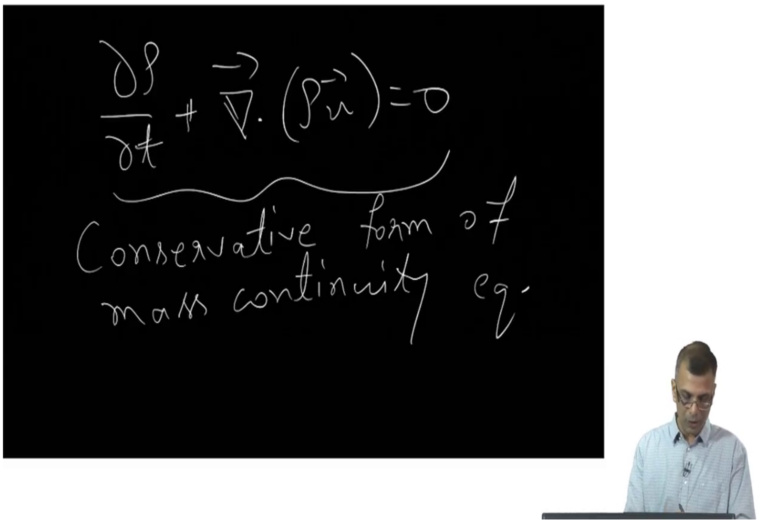

So, now why even bother about this way of writing things? Why even bother about this way of writing things then? Why even bother about this way of writing the mass conservation equation. The advantage will manifest itself when we start talking about the momentum conservation equation, but this is what is called the conservative form of writing this is the this is the conservative form, there is something that is conserved of the mass continuity equation ok.

And the conservative form of any quantity is written as the partial derivative in other words the derivative as seen by the lab observer as a partial derivative of that quantity ok. In this case the partial derivative of the mass density plus the divergence of the flux of that quantity and the mass flux is given by rho u ok. So, this is what is called a conservative from the mass continuity equation and by exact analogy you can write the conservative form of the momentum continuity equation.

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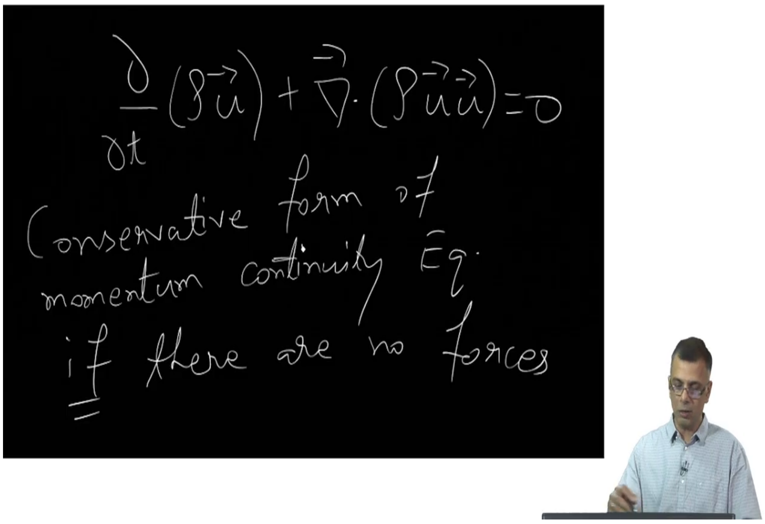

Which is if there are no forces acting on the parcel of fluid, it would be ok. This is the conservative form because this is a momentum and this is the flux of momentum. This advantage of writing down the conservative form of the momentum continuity equation.

If there are no forces and even if there are forces it is possible to bring those forces in right here into the flux of the momentum very easily and although it is a bit of a formal way of writing it, I you see that advantage you can immediately well here is a momentum density to be precise momentum per unit volume.

And here is a flux of the momentum and mind you this u u is not an inner product it is an outer product and we have seen what an outer product is ok.

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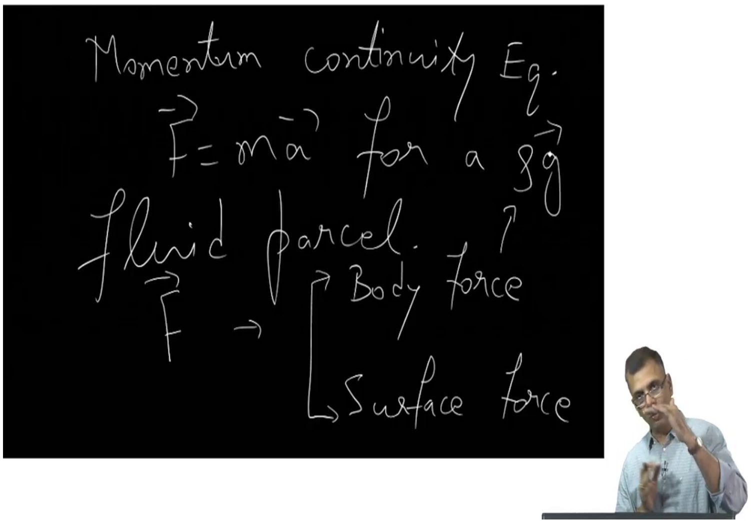

And if there are forces; so, really the momentum continuity equation this is nothing but F equals m a for a fluid parcel not for a point particle, but for a fluid parcel. In this case what we did here is this is simply m a this entire thing is just m a and we assume that there are no forces.

Now the thing is there are two there can be two kinds of forces for a fluid, one is a force F can be F can be a body force or a surface force ok. A body force is one which acts throughout the body of the fluid yeah. So, the body force density for instance the force of gravity this is one also you know if the fluid is charged then in if it is immersed in an electric field then and you know that is another, but that is not something we are talking about right now. We are talking about neutral fluids.

So, this is F I beg your pardon maybe I should erase this and write it a little more clearly yeah. So, example of a body force would be rho g why is it called the body force because this is a force that acts throughout the body of the fluid throughout the volume of the fluid ok. Whereas, a surface force is one that acts only on the exposed surfaces an example of a surface force would be one where you imagine a little you know a little an imaginary cubicle surface ok.

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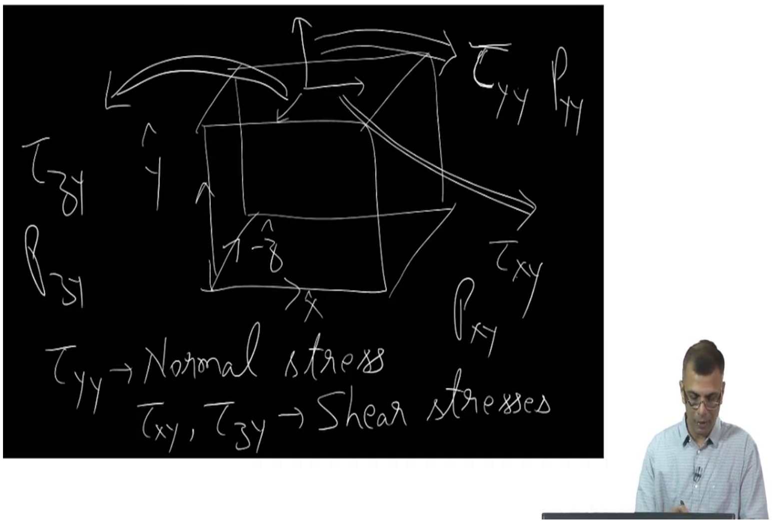

One of the surfaces can be free or not ok. So, these are the different surfaces and let us label the axis let us say this is the x axis, this is the y axis and this is the minus z axis because x cross y has to be z right. So, if you rotate from x to y a right handed screw would come out in this direction. So, that would be the minus z axis right.

So, now, you see there can be different kinds of forces on say you concentrate just on this top surface yeah. What direction is the outward normal of this surface pointing in it is pointing in this way yeah. It is pointing in the y direction ok.

So, there can be a force on this surface right in the y direction there can also be a force like this there can also be a force like this you agree. And so, and since there is a force and since we are talking about surface forces it makes sense to talk about forces per unit area in other words stresses. And this would be what is called a normal stress and we call this tau y y. The and we call this.

And I when I erase this is not very nice this is tau y y. This would be tau z y and this would be tau x y ok. Tau y y would be a normal force, this is what a normal thing normally thinks of as pressure the force that is acting normal to the surface. What this means is; it is the you know that the second y this y here. This denotes the direction of the outward normal or the phase you are talking about that is why you have a y here and you have y here also.

Whereas the first y here denotes the direction of the force; so, this is a force acting in the y y direction on a phase whose outward normal is also in the y direction. This is a force which is acting in the x direction on a surface whose outward normal is pointing in the yth direction and so on and so forth.

And tau y y is a normal stress and tau x y tau z y these are shear stresses. If you think of a surface like this, that would be a normal stress whereas, this and this are shear stresses ok, that is what these are. The reason we brought this up is because when you write F equals m a remember right. So, this is m a, but what is F?

F you can have two kinds of F, you can have a surface force which is what we just talked about and you can have a body force which is this yeah. So, therefore, you can simply write down the momentum equation in this way just by analogy with this a complete version of the momentum equation then would be.

Just by analogy with the you know surface forces and everything would be this the conservative form of the momentum equation with forces included of momentum equation with forces would be d over d t of the momentum, density plus the divergence of the momentum flux would be equal to minus the divergence of the pressure tensor.

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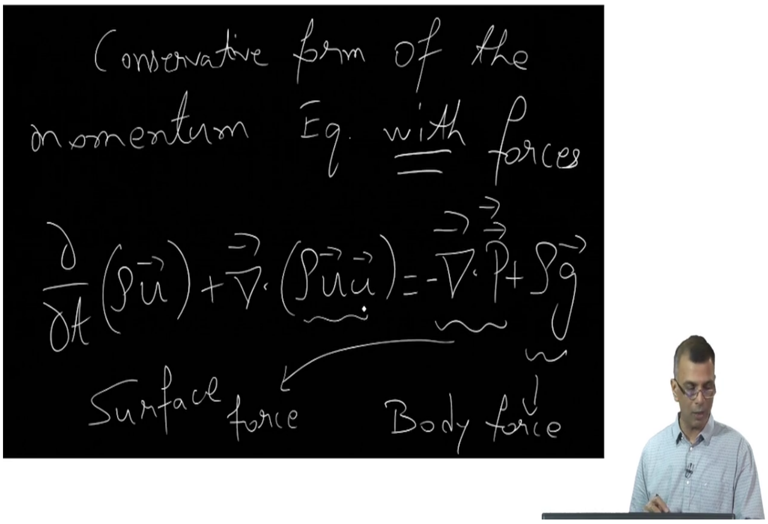

Where this would be representing the body force and this would be representing the surface force. And what is inside of this p? Yeah ok.

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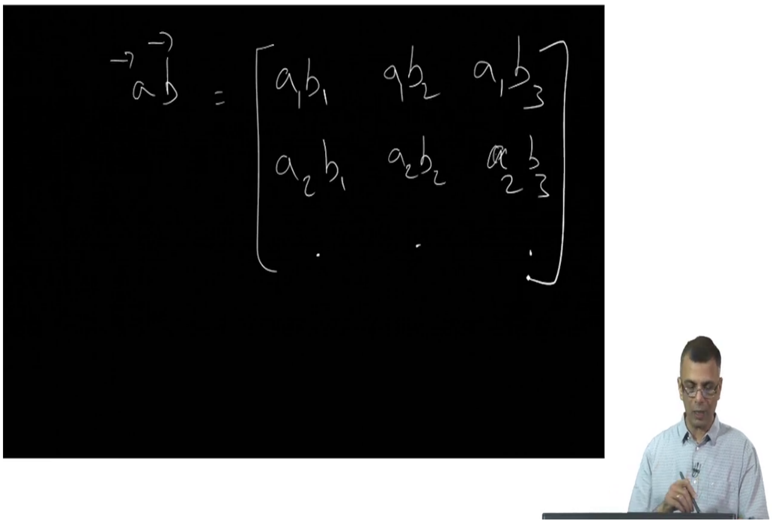

So, a couple of things what really is let us just you know this u u is a really a second order tensor and the u u would be the outer product of two. This kind of outer product is just like that a 2 b 1, a 2 b 2. So, this would be a 2 b 3 and so on.

That is what the outer product is. That is what this quantity is and the pressure tensor would essentially be this. In particular we are talking about the divergence of the pressure tensor right. So, the divergence I mean; so, the elements of the pressure tensor the P i j would just be when I write tau I really mean P. This is the same as P you can think of it as tau y y or P y y you can think of tau x y or P x y so and so forth yeah.

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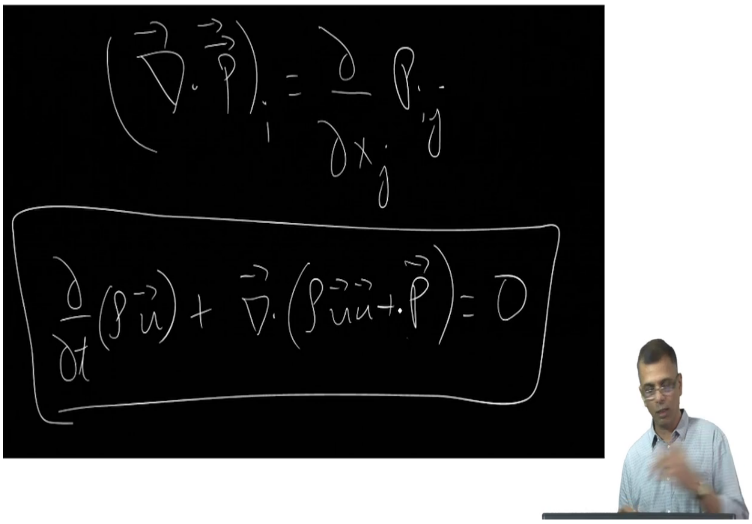

And, we need to understand what the divergence of a tensor is; the divergence of the tensor simply is this is simply this. The ith component of the divergence of this is simply where P i j is simply this or this or this, same thing. So, now, you see you have a neat way of a folding all this in and saying that you go from here you can actually absorb this guy in here; so, that you can write the momentum equation.

Finally, as plus divergence of the quantity we already have plus P equals if I want to include the body forces I can include it, but for the time being if we neglect the body forces I can write this as 0. And this really is a conservative form of the momentum equation with surface forces represent represented by the divergence of the pressure tensor with surface forces including.

So, this is the advantage of writing things in the conservative form. You can simply this is again remember why am I saying this is a conservative form? This because this is of the same form a partial derivative of the quantity in this case the momentum density plus the flux of the momentum ok.

Which also includes pressure forces they are on the same footing, I should not say if pressure forces I should say a surface forces, surface forces of these kinds. Either the normal surface force a normal surface stress or tangential surface stresses tangential or shear stresses same thing ok. So, this is one way of writing the momentum equation. If all this seems a little, ok.

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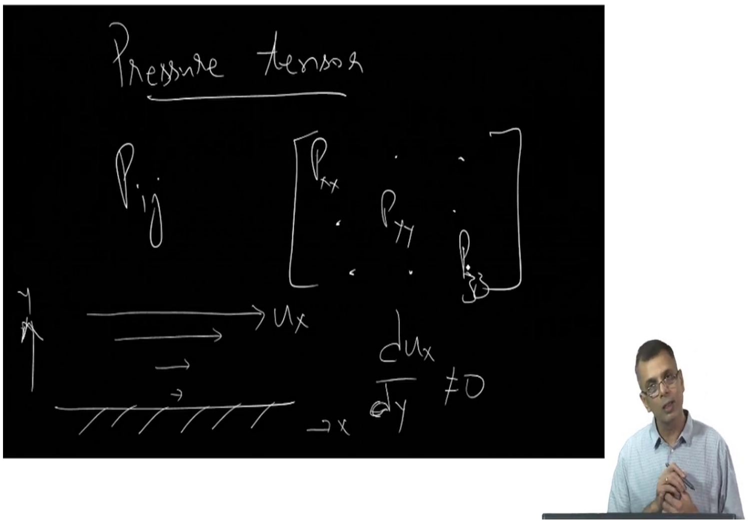

So, before proceeding I should say that in general we assume that there are you know the pressure tensor is P i j which is something like this yeah. So, it would that you would have a P x x you would have a P y y you would have a P z z and the cross terms here, here, here, here and here yeah. So, this would be P x y P x z and so on and so forth yeah.

Now if on the other hand and remember this cross terms P x y P these are all associated with the cross terms that are associated with shears right this one and this one. If there is no shear in other words and remember shear is associated with the viscosity.

Shear, the whole concept of shear is that you have the whole concept of shear is that you consider a fluid that is flowing like this yeah and if there is the x direction and that is the y direction. And if you have a gradient in the x directed velocity with respect to y yeah the x directed velocity is large here not so large here, not so large here, not so large here. Only here you would think of this as representing the flow of a viscous fluid would not you this seems like you know the flow of honey.

It is a sticky kind of thing and you know right at the bottom the honey is barely flowing whereas, right on the surface it is flowing somewhat vigorously in other words there is a non zero these cross terms are non zero ok.

So, this represents the shear and if on the other hand you have a relatively less viscous fluid say water. You would think that this gradient is not as large ok. It is almost as if this these vectors are almost the same length and right at the surface the fluid is able to slip rather freely ok.

In other words for an inviscid fluid. There is no such thing as a perfectly inviscid fluid it is all these terms are all relative for an inviscid fluid; these cross terms are not important it is only the diagonal terms that are important ok.

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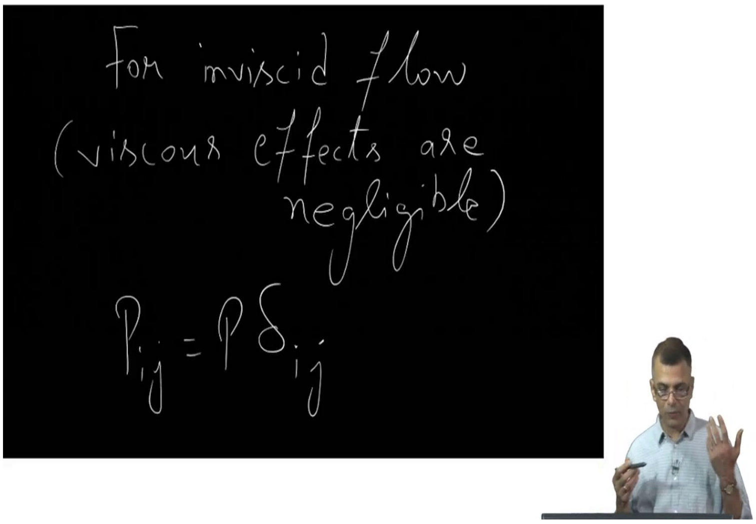

So, for an inviscid fluid i e viscosity or viscous effects are negligible. In this case the P i j is simply some scalar pressure times the Kronecker delta ok. In other words it is a situation where all of these are simply the same number P P P and the and off diagonal terms are all simply 0 ok.

So, this means you know a situation where the delta is 1 only when i is equal to j in other words only along the diagonal and off diagonal and when i is not equal to j as is the case for this element this element, this element and so on so forth. Then off diagonal elements it is simply 0 ok.

So, this is a situation which represents an inviscid fluid and I this is y I mean I and this is where I explained why there is so. I urge you to think this is my way of explaining it. I urge you to think about this a little more. And in this situation the momentum equation gets a little simpler and so, I write down the momentum equation once again the full momentum equation plus the divergence of this represents a tensor.

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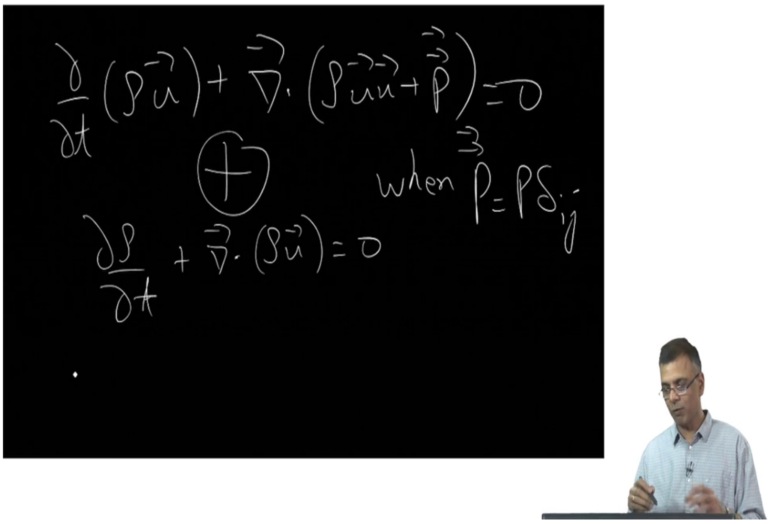

This is the full momentum equation you use this and plus the mass continuity equation you would have to you know essentially what you would do is you would expand out this one and that is where you will need the mass continuity equation here.

To use this both of these ok. In the situation when this is not only the diagonal elements of the pressure tensor or non zero in the rest is 0. In that case you use this and the mass continuity and essentially you will need the mass continuity equation when you expand out you know this quantity to get the Momentum equation for an inviscid fluid.

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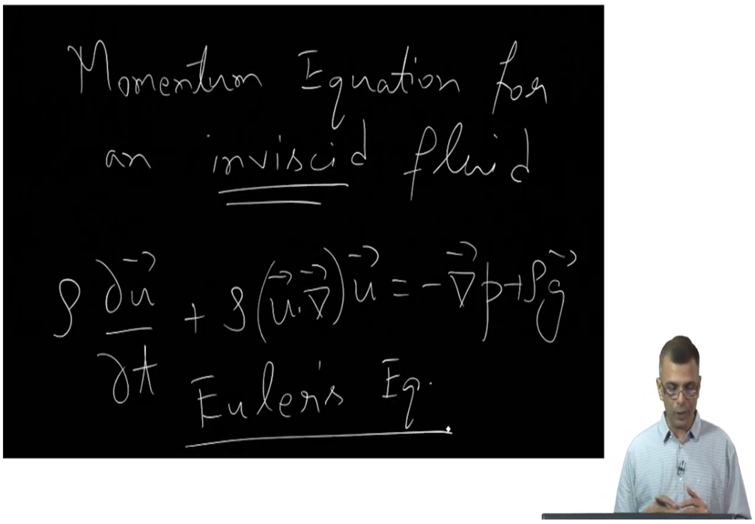

The momentum equation for an inviscid fluid you can think of where viscosity is not important you can write that down as rho. Again all of this is an Eulerian as a you know is equal to minus this is just a scalar plus if you want to include body forces you can ok. And this is called Euler's equation ok.

Euler's equation as it is a; it is a different thing than the Eulerian frame please you know keep this in mind it is an obvious thing, but I thought I would you know emphasize this. So, this is the momentum equation for an inviscid fluid in general this is true for any situation you can include body forces here if you want wish. But as such this includes a situation where viscosity is included ok.

So, as such we have already introduced what is called the Navier Stokes equation. This is it really we will discuss it we will discuss it separately in a minute, but this is how the Navier

Stokes equation which includes viscosity you realize right I mean when is viscosity important?

Viscosity is important when the off diagonal terms in the pressure tensor are important and off diagonal in this case we have not made any statement about whether the pressure tensor only has diagonal terms or not. No, we never said that only afterwards we are specializing to this and we are using them the you know mass continuity equation in that and getting the Euler's equation as this.

I urge you to carry out the steps and maybe I will assign it in your problem set. And so, this is the Euler's equation slightly. Now if you recall the difference between the in the Eulerian derivative and the if you recall this definition and you look at this ok. You can quickly see that it is possible to write down this equation in Lagrangian form also ok.

As in it is a little easier in this case for you saw the advantage writing down the momentum equation in the conservative form in an Eulerian frame. In other words in the frame or in from the perspective of a lab observer you could simply write down the conservative form in exact analogy with the mass conservation equation.

You can simply write it down of course, body forces will not included surface forces not included when you have to include those you will need a few more complications, but that is ok you could write it down formally mathematically it was very easy to write down. But there is a little bit of physical intuition missing there.

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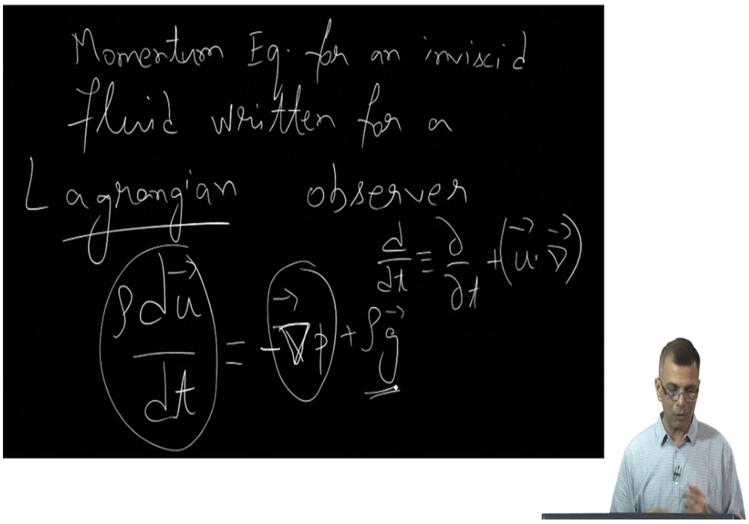

By way of physical intuition it is easier to let us see if momentum, allow me to write this equation for an inviscid fluid in other words the Euler equation written for a Lagrangian observer. In other words an observer; in other words an observer, who is moving with the fluid parcel.

So, what would be the momentum right for the person who is moving with the fluid parcel what would be the rate of change of momentum density. It would simply be rho that is it. Is not it? So, this is my m a yeah. And what is the force? Well as for an inviscid fluid I mean a fluid was is going to flow because there is a pressure difference ok.

So, the pressure is higher here and it is lower here there is a pressure difference in other words there is a gradient in the pressure that is why this fluid is flowing right. So, the F would be and there could also be body forces obviously you can immediately write this down simply from you know intuitive physical considerations.

So, this is an advantage of looking at the momentum equation in Lagrangian framework. And if you wanted to relate this kind of derivative to I mean in this equation to this equation, you can see that the relation between the two kinds of derivatives which we it immediately follows. The relation between the straight d over d t and the partial d over d t if you apply that to this equation this one immediately follows right.

If you simply apply the fact that yeah and you operate this whole thing on u right, this immediately follows. So, there are advantages to both kinds of descriptions I simply wanted to point this out. So, what we have done now is look at the momentum equation for when we really started in full general generality we did include viscous effects when we wrote down the momentum equation like this ok. Viscosity was indeed included because if yeah.

And then we so, viscous just to reiterate viscous effects are important when the off diagonal terms of the pressure tensor are non zero yeah. But then we specialized through a case where viscous effects are negligible in other words we specialized to an inviscid flow where only the diagonal terms of the pressure tensor are nonzero, the other terms of the pressure tensor are 0 and we wrote down the Euler equation.

It is a bit of a complicated way of arriving at the Euler equation because you know you had to also in order to arrive at the Euler equation you also had to use the mass continuity equation. So, you arrived at the Euler equation written in the Eulerian frame written in the frame of an observer who is in the lab frame ok. Hence these partial d over dt is right.

You can arrive at the Euler equation written in the Lagrangian frame in other words in the frame of an observer who is moving with the fluid either by looking by drawing the analogy between the partial derivatives and the straight derivatives like this. You can go from you can use that on this equation yeah or you can simply write down from simply from physical considerations I am moving with a parcel of fluid and I am noticing that the momentum density of that parcel of fluid is changing.

What is the rate of that rate of change of that momentum density? Well, it is this it is this. And why is it changing? Well, m a is this and why is it changing it is because of forces acting on it and what is the kind of force I am not I am not interested in viscosity at the moment I mean what causes and you know forced to act on a parcel of fluid.

Well, it is probably because you know there is this pressure difference pressure is higher here lower here. So, fluid flows more precisely it is a gradient of pressure if the pressure difference is larger over a smaller interval that matters more. So, it is really the gradient of the pressure of the you know pressured of the pressure essentially. And on top of that there can also be body forces acting on it.

So, I can write down the momentum equation in for a Lagrangian observer almost intuitively, but if you want to include viscosity it is I mean it is not that intuitively simple, it is better to stick to the formal math. And when you include viscosity this is the momentum equation this one on the top. And formally this is really the Navier Stokes equation, but we will discuss this separately.

And before moving on to the Navier Stokes equation I would want to discuss a few other things that have to do with streamlines and stream functions and things like what we defined in our study of kinematics.

So, I would like to do a few of those things and point out a few interesting effects before we move on to discussing the Navier Stokes equation and we will also say a few words about the energy equation. So, for the time being we will stop here.

Thank you.