Fluid Dynamics for Astrophysics
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Lecture - 06
Conservation laws: Mass conservation and incompressibility

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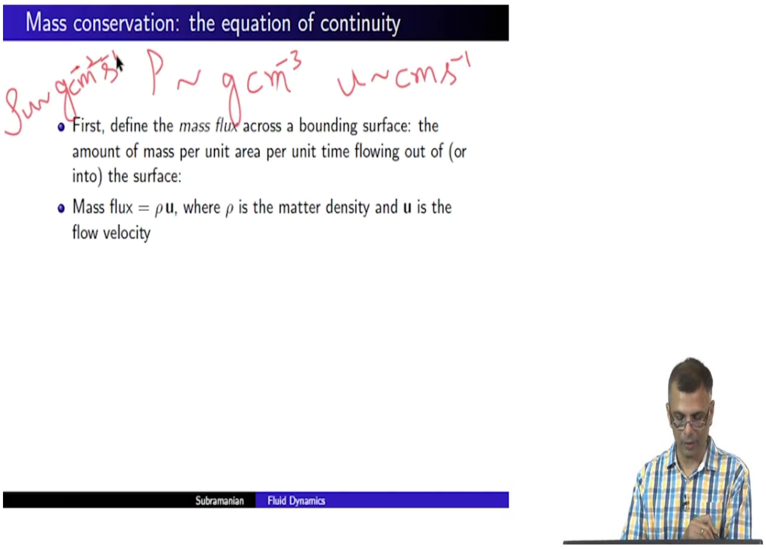

So, hello we are back and as promised we will start talking about Mass conservation equation; the equation of continuity right here. And so, before doing this; I want to define a quantity called mass flux or for that matter the flux of any quantity ok. It is defined like here as I have written here; it is amount of mass per unit area per unit time flowing out of or into a surface.

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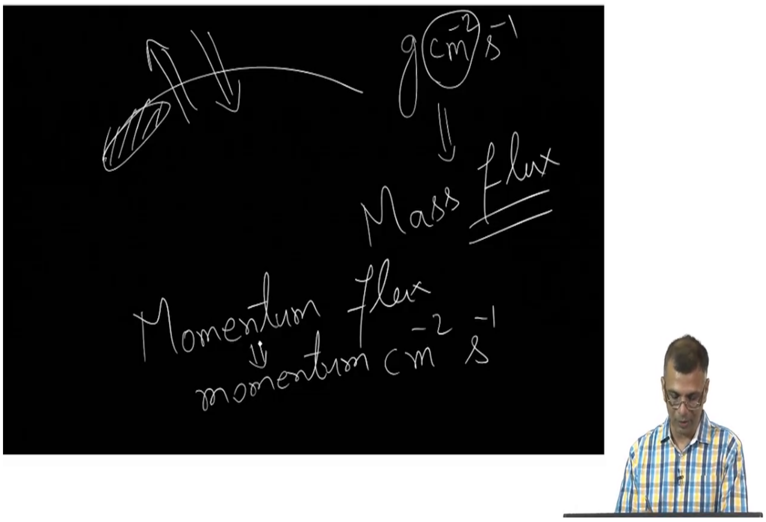

In order to appreciate this envisage a surface like this you know a surface like this; it might be open or closed that at, for the time being it does not really matter. And so you have you know mass either flowing out or flowing in ok. The units of mass are gram; if you adopt cgs units; I am most of the time you know astrophysicists work in cgs. So, so I am used to cgs, but not necessarily. I mean if you are more comfortable with kilograms that is alright.

So, the unit of mass per say is grams, but what we are talking about is the mass flux ok. So, this would be grams per centimeter square per second. So, this would be the mass flux and this makes sense only if we consider a surface because you know the centimeter squared here; what is it about? It is you know per it is a mass; per unit area you know per unit cross sectional area say some sort of a cross sectional area here ok; that is flowing out or flowing in per second of course, ok.

So, this is the definition of mass flux and we will come back to this kind of a definition over and over again. Turns out that one can define something called the momentum flux as well; the flux of any quantity for that matter which is momentum whatever the units of the though; that is essentially grams centimeter per second because this is MV, momentum per unit area per unit time; this is momentum flux ok.

So, this concept of flux will keep coming back over and over again during our discussion. So, I thought I would you know take the moment to explain or to you know introduce what this is and we will do so, once again also as a need arises right. So, getting back; so we have defined mass flux now and so the mass flux is simply rho u where rho is the matter density and u is the flow velocity. If you look at the units of rho you know rho is the units of rho are grams per centimeter cubed and the units of u are centimeter per second right. So, you multiply these two together and you get; rho u something like grams per centimeter squared per second right.

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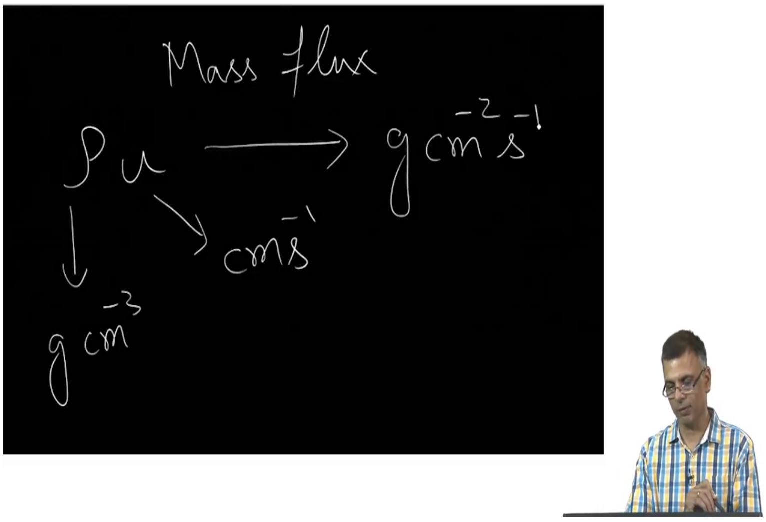

Just to make it a little clearer; mass flux is rho u, this is grams per centimeter cube and this is centimeter per second. And therefore, multiplied together the units of this are grams per centimeter squared per second ok; so, that is how it goes right.

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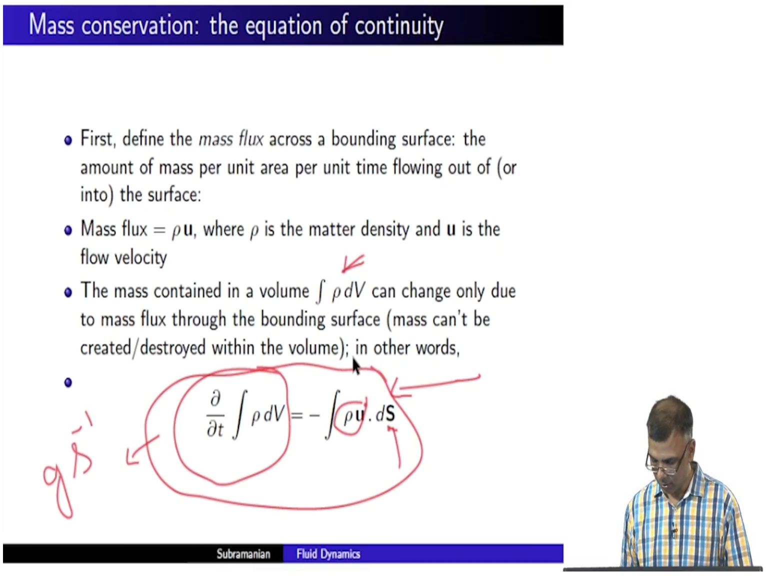

So, that is the definition of mass flux and why is that useful? Consider this statement; the mass contained in a volume which is the integral of rho d, where d V is an elemental volume. It can change only due to mass flux through the bounding surface because mass cannot be created or destroyed within the volume.

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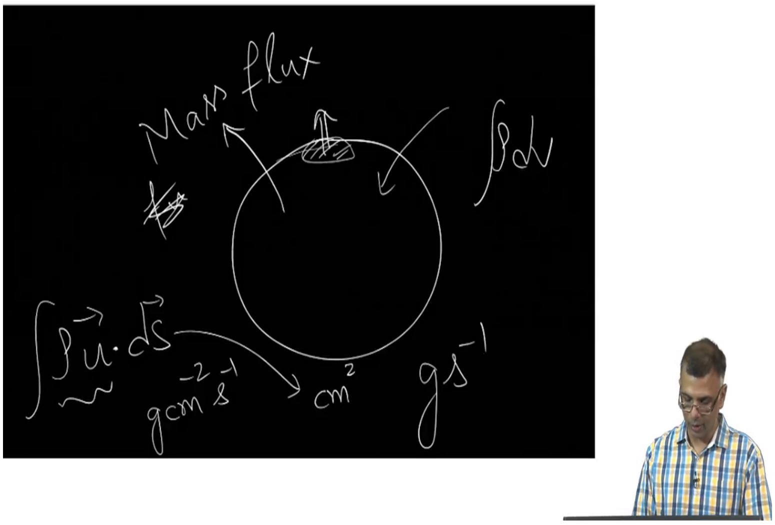

Yeah, in other words before looking at this equation; let us go back and do a little sketch. You see suppose you have some kind of a bounding surface like this and the whole point is you cannot magically create or destroy mass inside here yeah. So, if there is a change in the mass flux inside here that is that can either increase or decrease that can only be because of mass flux flowing out or flowing in; mass flux ok; either flowing out or flowing in like this.

There is a only way the matter density inside this volume can change right and so what is the total mass contain inside this volume? Total mass contained is simply rho; the mass density times d V integrated over this whole volume right. So, that is what this says here that is what this says.

And so the rate of change of mass flux of the total mass right; the rate of the time rate of change of the total mass and this is the total mass this integral is the total mass is essentially the negative of the mass flux which is rho u; integrated over the surface area. Which surface area? Well, the bounding surface ok; the first thing to notice is that the dimensions match up, I urge you to; I urge you to check that out, you just have a per second here.

So, this is grams per second and the dimensions of this are grams per centimeter square per second and the dimensions of this; of the surface area is essentially centimeter squared. So, you have you know; this was rho u dot d s ok, where d s is just this any small little surface element whose area vector is pointing outwards. So, this is by definition and this is of course, centimeter square.

So, this integral has units of grams per second and that is exactly what the units are here; this is in grams per second right. So, it is an extremely simple statement to make; mass cannot be cannot be created or destroyed inside this volume.

Therefore, the only way the mass density; the total mass inside this volume can change is because of mass flux because something is carrying matter away or inside this volume through the surface which is bounding this volume; that is what this equation is telling you; that is what this entire equation is telling you ok.

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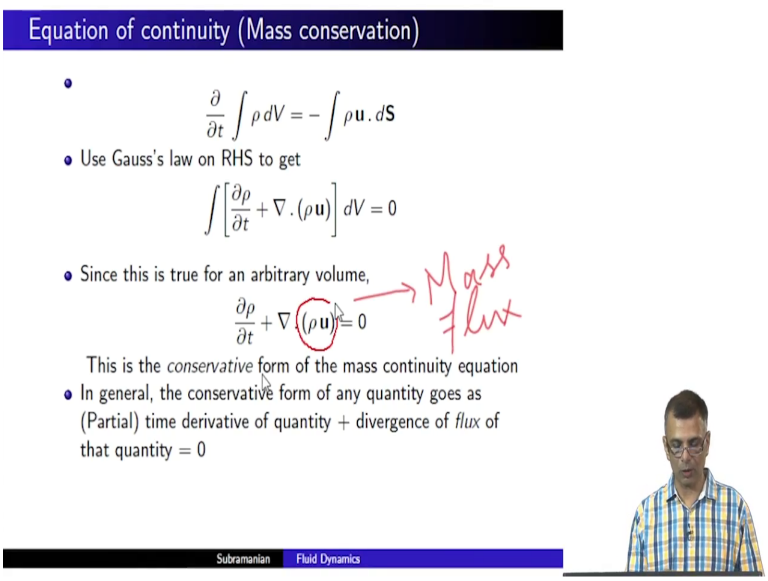

So, this is the same thing that we that we wrote here; it is exactly the same thing, no difference. And now we do a little bit of vector calculus; we apply Gauss's law to the right hand side to get the following. In other words, we convert the surface integral into a volume integral.

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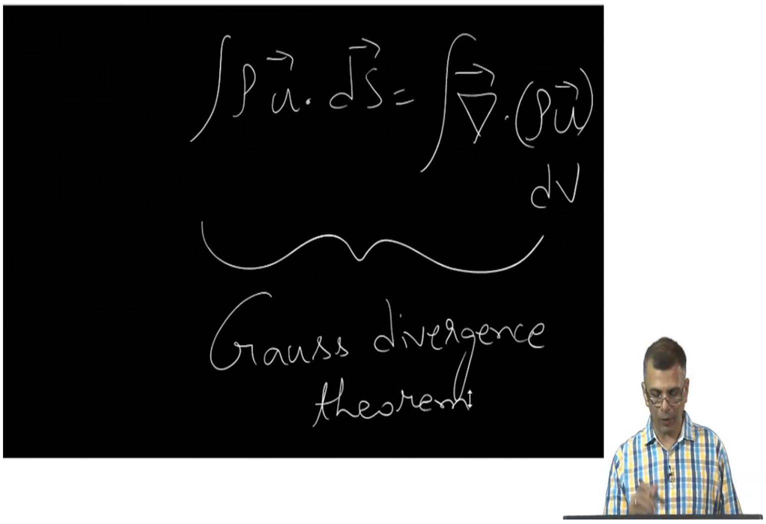

So, in doing; so we this becomes this; all we are saying is that is all we are saying and this is; so, we apply this here, here to get something like this. Why did we do that? That is because you know you have a volume integral on the left side and a surface integral on the right hand side, little awkward. So, you convert both of them into a volume integral and so you have a d V and it is obvious right, it is much more convenient.

So, you just take it to the left hand side and you write it like this and we are almost there. And since this is true for; since this statement here this entire statement is true for any arbitrary volume, no matter how small ok. And this is very very important since this is true for an arbitrary volume, I should probably have written arbitrarily small volume ok; you can take this d V to be arbitrarily small in which case the integral is irrelevant right.

So, we can say that the integrand itself is equal to 0; notice this is nothing but the integrand here and that is it. This is the statement of mass conservation this is the equation of continuity and you might be familiar with this from electrodynamics as well. It is exactly the same, while doing electrodynamics; this would be rho would represent charge density; here we are talking mass density.

So, that is all this is the equation of continuity, what is it really saying? It saying that mass cannot be created or destroyed, the mass contained in the volume can change only due to mass flux through the bounding surface; mass flux through this bounding surface right.

So, yeah this one; I am sorry I meant this figure. So, the only way the mass density or the total amount of mass inside here can change is because of mass flux through this bounding surface and that is what; at the end of the day yeah. So, that is what this equation is telling us and this is called the equation of continuity.

So, there you there you have it; we have you know derived from intuitive considerations, the continuity equation. And this is what is called the conservative form of the continuity equation ok; what you mean by that? I mean this is just a word such you know, but I will; so, remember this word conservative form. And really what this means? In general, the conservative form of any quantity goes as in this case it is a conservative form of the mass continuity equation.

In general, the conservative form of any quantity goes as the partial time derivative of the quantity; in this case mass density plus the divergence of the flux of that quantity is equal to 0. So, the divergence of the mass flux; remember, we said we spend a little bit of time in showing how rho u; this quantity is the mass flux right.

So, this is what is called the conservative form of the mass conservation equation and so, it takes the following form. The partial time derivative of mass density plus the divergence of the flux of mass is equal to 0 ok. We will have you know an equation to return to this particular conservative form for the momentum continuity equation as well, it is a little more complicated, but the concept is the same right.

So, there you have it; that is the equation of continuity, that is the equation of mass conservation and we are done with the first of the conservation equations that we promised we would look into.

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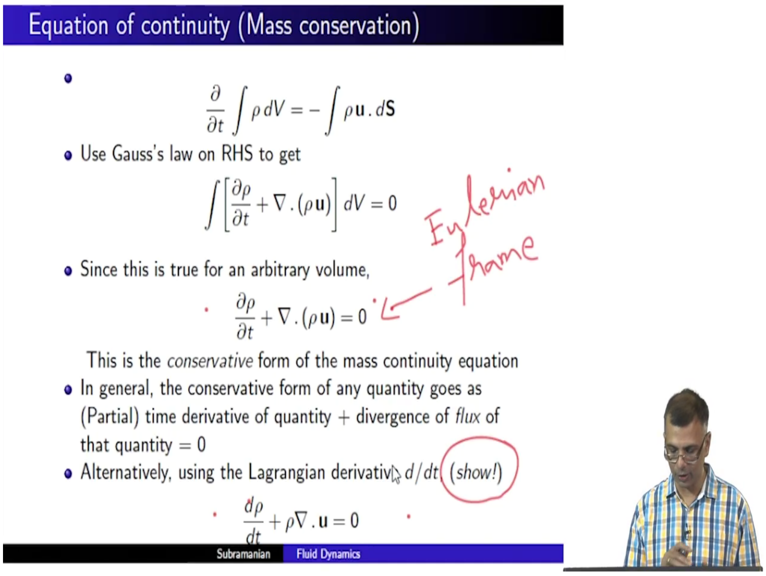

So, let us go ahead yeah ok.

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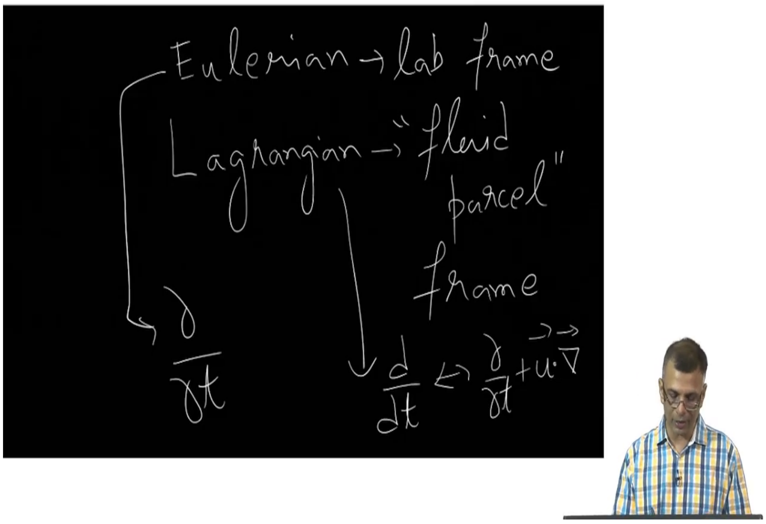

So, before that recall in our study of kinematics; we had two kinds of derivatives we had two kinds of; we had the Eulerian description which is the description of the fluid flow as seen by a lab frame observer. An observer, who is standing outside of the fluid flow and we had the Lagrangian description which is in some sense the fluid parcel frame; so to speak.

In other words, it is the flow of the fluid, as seen by an observer sitting on top of a fluid parcel and the time derivative while in this kind of description was mostly exclusively in fact, denoted by d over d t; whereas, in the Lagrangian description the time derivative was denoted with these regular straight d's right.

You might want a wonder why I am writing a partial here; why am I writing; why I am writing a total here? I mean the real reason is that; I mean you know in the Eulerian description things can change due to changes in time as well as changes in position. So, the flow field is a function both of time and position, hence the partial derivative. Here in the Lagrangian description, you know you are sitting on top of the fluid parcel; there is only one variable so to speak ok.

So, think about that and we also made; made the connection between the two kinds of derivatives, the connection between this and we had like that; you recall we make this connection right. So, this is how the time derivative in the Lagrangian frame and the time derivative in the Eulerian frame are related and that is exactly what we.

So, this entire thing is written in the Eulerian frame; in other words, the frame; the lab frame in which the observer is standing outside of the fluid flow and here she is watching the fluid flow by. And you can relate it using relation to the Lagrangian way of looking at things; this and this are both the mass continuity equation, except here the derivative is a Lagrangian kind of derivative ok.

So, this and this are exactly the same and I would urge you to show this; to explicitly show how this follows from that or vice versa ok; this is a useful exercise. Sometimes, it is more convenient to adopt the Lagrangian point of view, sometimes it is more convenient to adopt the Eulerian point of view; it is important to be familiar with both kinds right.

So, if you divide this entire equation by rho; on the right hand side you have 0, so that makes no difference; if you divide this entire equation by rho, you get this.

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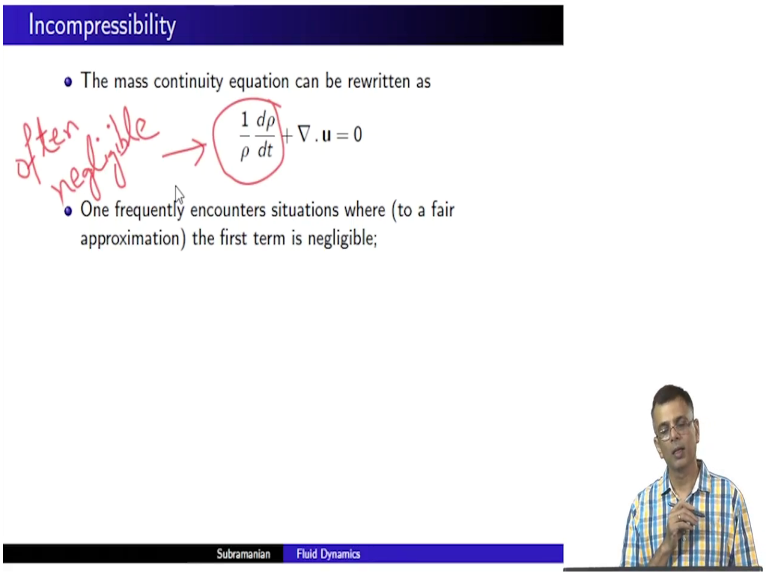

This is the same equation that we have written earlier; the mass continuity equation right, in the Lagrangian frame as seen by an observer who is sitting on top of a fluid parcel. And one frequently encounters situation where to a fair approximation; the first term is negligible, in other words this is often negligible ok.

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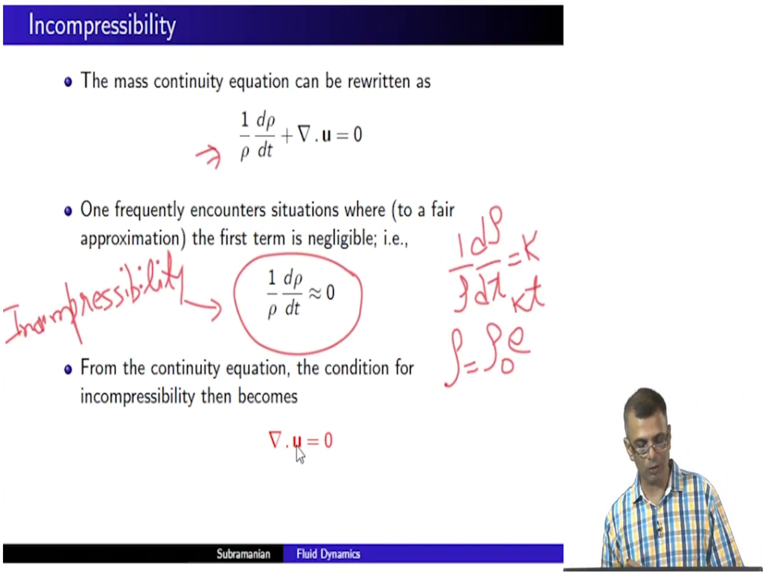

In which case, in other words this is what I mean; this is the 1 over rho d; rho d t is approximately equal to 0 in which case the; so and you can see this. So, suppose this was non zero right; suppose this one over was equal to some constant k right. So, the solution to this equation would be some sort of rho equals rho naught; e raised to k t right.

So, in other words the density is changing, as a function of time right. Suppose, that is not so, suppose the density is not changing as a function of time right. In other words the fluid is not squishy, you cannot change its density; the fluid is kind of it is what is called incompressible. This is the condition for; this is the condition for incompressibility that is what this equation is telling you ok.

If that is so, if the first term is if this term is 0, then automatically follows that the mass continuity tells you that the divergence of u is 0.

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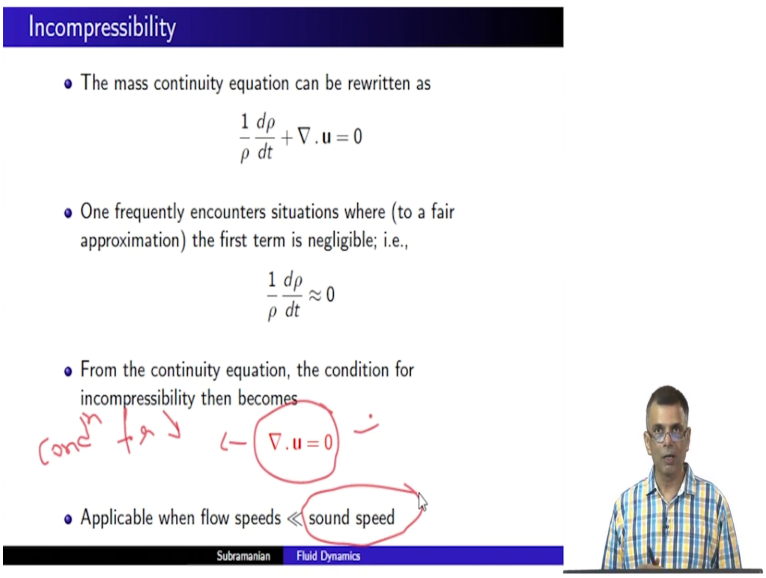

Many times, this is simply taken to be the condition; this is simply taken to be the condition for incompressibility ok. So, this thing; this divergence of u is 0. If the flow field is such that its divergence is 0, you immediately conclude that the flow is incompressible; for this reason that we just discussed.

And this is often now it turns out that there is these are not absolute statement; it is never the case that a fluid is definitely compressible, definitely incompressible; it all depends upon how fast the fluid is flowing.

And in physics, you never make this statement in an absolute sense; you never say how fast; well how fast with respect to what? With respect to the speed of light or with respect to some of the speed. Here, we are talking non relativistic dynamics; so the speed of light is does not enter the picture at all; everything all flow speeds are much much smaller than the speed of light.

Turns out that if the flow speed is much much lesser than the sound speed which we have not introduced yet. We will introduce this when we start talking about compressibility. If the flow speeds are much smaller than the sound speed, then the fluid can; any fluid can be regarded to be incompressible to a very good degree of approximation. Now, I simply I will simply make the statement of the for now and we will justify it later on when we when we start discussing compressibility, but this is something to be kept in mind right.

So, what we have done? For the time being is we have in a rather intuitive manner derived the equation of continuity or that of mass conservation. We have essentially derived two different guises of it; the Eulerian form of the continuity equation and the Lagrangian form of the continuity equation. And we have used the Lagrangian form of the continuity equation to derive what is called the incompressibility condition.

Now, this is small variant to this; this is small variant to compressibility or incompressibility and there is little bit of an advanced topic. But I figured, I would mention this; it is not totally central to the basic flow of what we are talking about, but let us do it anyway yeah.

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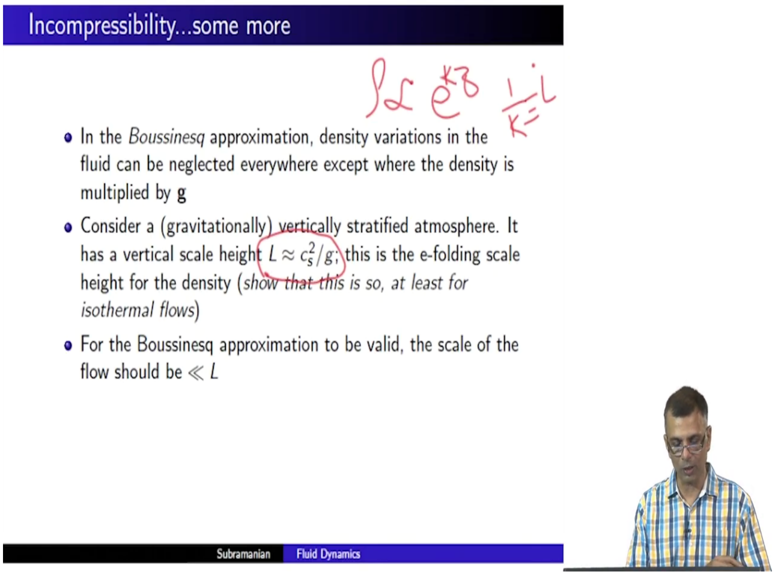

So, there is something called a Boussinesq approximation which is very closely related to incompressibility. You recall when we talk about compressibility or incompressibility; we were really talking about this. This is really what incompressibility was all about, in other words the flow not being squishy; this is really what it was about. But turns out that the you can; in other words when you can neglect density variations, you say that the flow is incompressible.

There is a related approximation called the Boussinesq approximation where density variations in the fluid can be neglected everywhere except where by everywhere; I mean in all terms of the equation except where the density is multiplied by the acceleration due to gravity ok.

So, this is a little bit of a weaker condition than incompressible than strict incompressibility ah. So, density variations can be in all terms of the equation whichever equation you are considering can be neglected. So, it is pretty much incompressible except where the density is multiplied by g, you do not neglect density variations ok.

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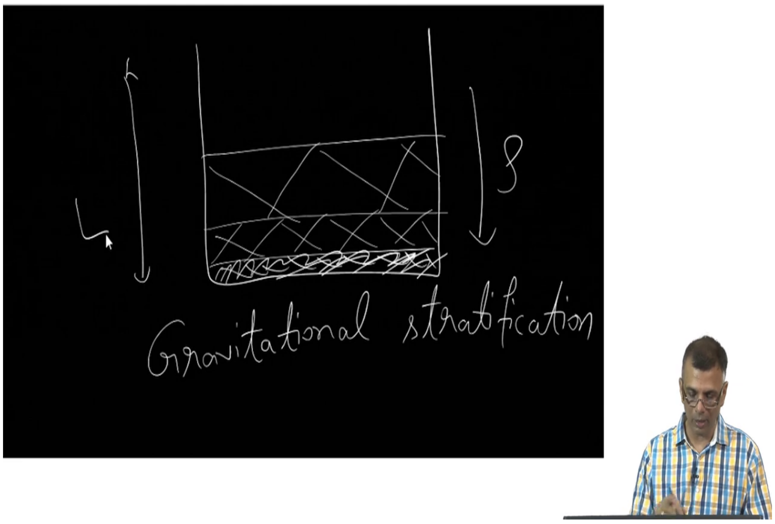

For instance, consider a gravitationally vertically a vertically stratified atmosphere which is vertically stratified due to gravitation; an example for instance would be you know a container, a fluid where the fluid is a mixture of various densities ok; oil, water various other things ok.

So, what will naturally happen; simply because of gravity, there will be a stratification and you will have the densest material at the bottom; this would be the densest material yeah. And the up next layer would be relatively less dense and the next layer would be even less dense like that. So, this would be an example of so the density would increase like this. So, this would be an example of a gravitational this right.

So, if that is so then it turns out that and this I will simply state without proof, for the time being. As and when we start discussing isothermal flows, I would like you to show this fact which is that; the scale height of such an atmosphere, in other words the scale height is essentially the height over which the density varies by a factor of e ok.

So, suppose the density is varying as e raised to some k; z then the scale height would be something like 1 over k ok, that is what the scale height is. So, and we call that L and that is equal to c s squared over g where c s is the sound speed. I simply state this right now, but I want you to keep this keep this in mind and derive this when we come to you know discussing isothermal flow.

Now, the thing is this Boussinesq approximation which is a slightly weaker form of incompressibility because density variations can be neglected in all terms of the equation except that term where the density is multiplied by g ok. Turns out that if this approximation is to evaluate the scale of the flow has to be much much less than a density stress stratification height.

You know let say if the density stratification height was something like; how about this? Say about L yeah and this L is essentially is essentially this quantity. So, if that was the case; then you want the scale of the flow, the whatever flow is taking place should be much less than L this flow should be confined to a vertical dimension that is much smaller than l. So, in under such circumstances the Boussinesq approximation is valid ok.

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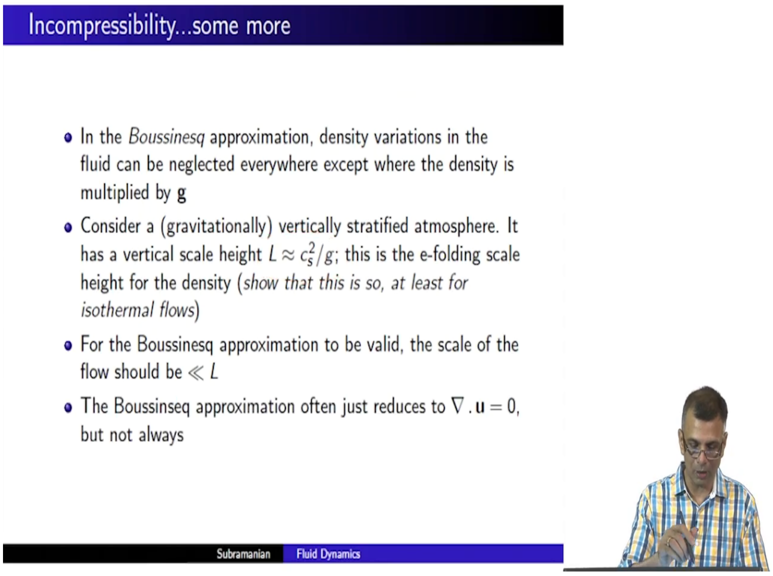

Turns out after having said all this, you will be a little disappointed when I make the statement the Boussinesq approximation often just reduces to divergence of u equals 0 which was just what we said earlier; it was just what we said here this is the incompressibility condition anyway right. So, the Boussinesq approximation often just reduces to a divergence of u equals 0, but not always.

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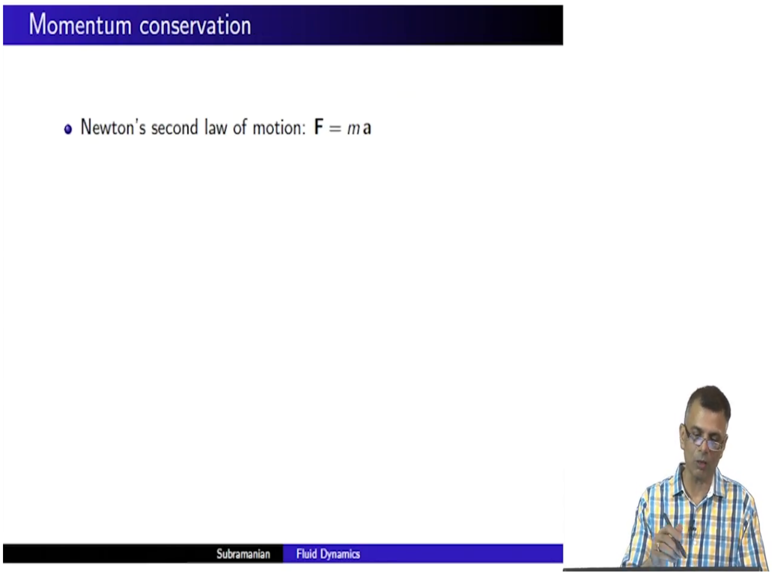

So, that is about mass conservation and let us now march right ahead into momentum conservation.