Fluid Dynamics for Astrophysics
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Lecture - 57
Magnetohydrodynamics [MHD]: Waves in MHD - Magnetosonic waves

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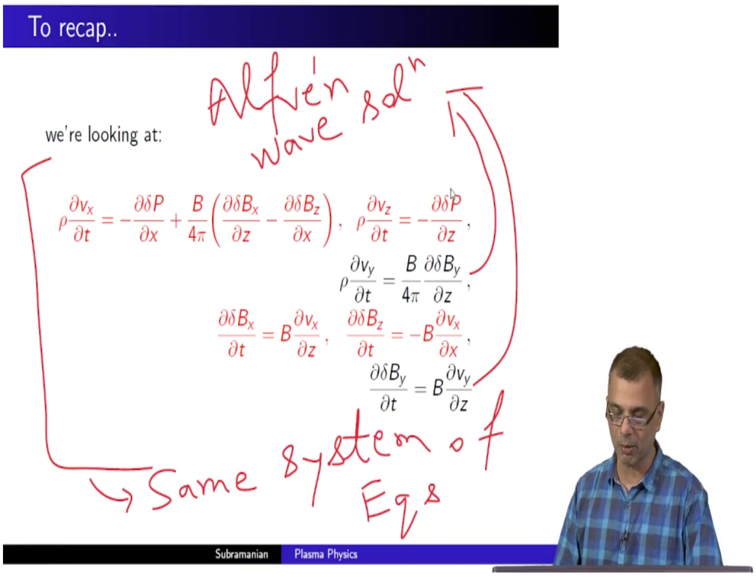

So, I am just to recap, we are now looking at the non-Alfven wave solutions or the solutions that arise from the equation that were not considered for the Alfven waves which are these; these equations in red ok ah. The equations in black were the once which we considered earlier, and they gave the Alfven wave solutions where the delta P, the perturbation in pressure or equivalent in density played no role. So, in this case, clearly the delta p is of central importance.

So, the solutions we will arrive at we will involve a density and pressure perturbation. So, in that sense, they are somewhat similar to or rather they have character that is similar to the sound wave and hence, they are called magnetosonic waves right.

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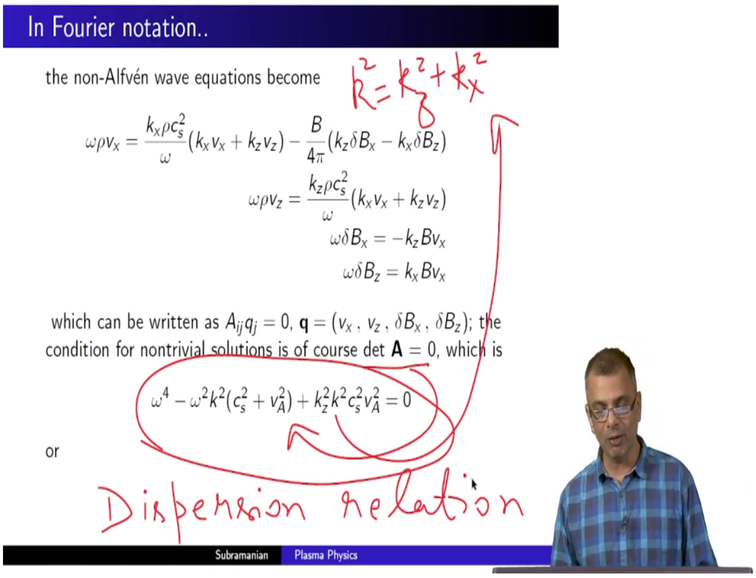

So, the non-Alfven wave equations which are just these equations in red ah, when written in Fourier language become this ok. So, you see wherever you see omega, you know that there used to be a d over dt, wherever you see a k, you now that there used to be d over dx and so on and so forth ok.

But mind you because of the presence of the magnetic field, the k is not the same in all directions ok. The propagation vector is not the same in all directions we have the hence, you have the appearance of a case of x and a case of z ok and the case of y is already taken care of right. So, yeah.

So, now, these equations can be written as a matrix times a vector equal to 0 where the vector comprises velocity v x, v z and magnetic field perturbations and the important difference is that none of these involve y, y directed perturbations, we have already taken care of that with the Alfven wave solutions right. So, they involve only x directed perturbations, x and z directed perturbations of course, in velocity as well as magnetic fields ok..

So, the q is just this and so, you know in order for to have non-trivial solutions, you have to have the determinant of this matrix to be equal to 0 and so, that is what is going to give us the dispersion relation and we will be as we will see the dispersion relation is now this, this arises from simply you know determinant of A equal 0, this gives rise to this whole thing right.

So, now, what we have done is instead of writing k z and k x, we write k z and k itself where k is square root of k z square plus k x square right. So, this thing k square is simply that is what this k means ok.

And we have made use of the sound speed as well as Alfven speed both of these quantities we have defined earlier right. So, this is the dispersion relation. So, this thing, this whole thing is the dispersion relation. Any dispersion relation is of the form is a relation between omega and k and this is a relation between omega and k like the once we saw earlier just little more complicated that is all ok.

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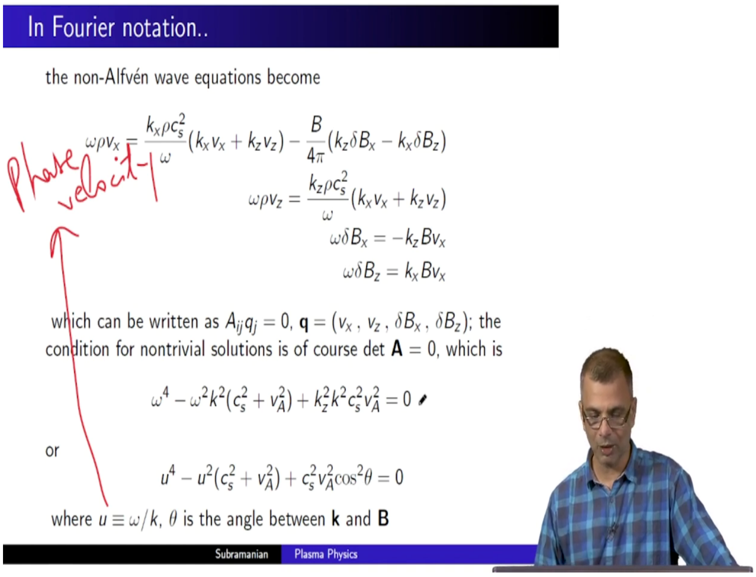

Anyhow, let us now look at the properties of these dispersion; of this dispersion relation instead of writing it in this form, instead of writing it as a function of both omega and k ah, you can choose to write it in this form where u is simply the phase velocity, u over k is just a phase velocity right and theta instead of writing k z and k separately, we have simply written you know, we have simply invoked a theta where the theta is a angle between k and B ok.

So, yeah so, essentially it is k z over k is tan theta yeah ah. However, you want to look at it I mean so, k theta is simply the angle between k and B and so, this is the same as this ok, these two questions are just same, this just looks a little simpler ok. This a way of simplifying this a little more or rather casting this in a slightly friendlier formalism and that is to right.

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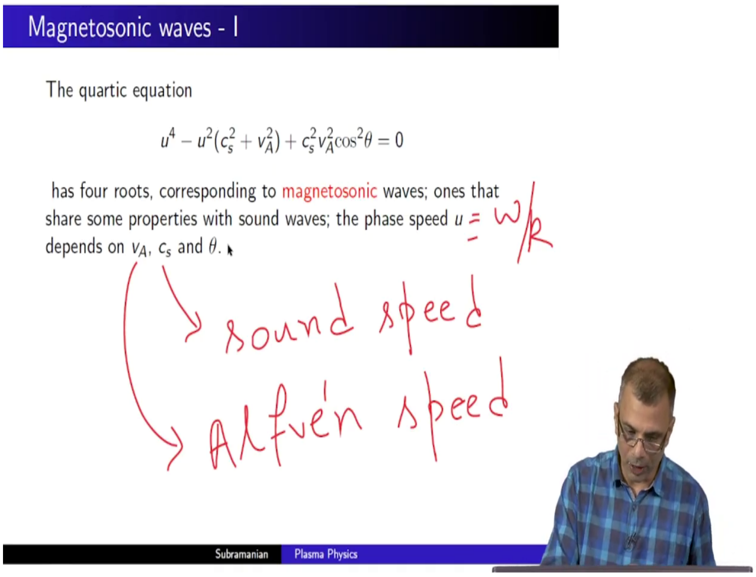

And so, this is a quartic equation as you can see. Thus, the highest power is u raised to 4 ok. So, this is the same as this, I have simply repeated this this on the next slide just for convenience and so, this equation has four roots clearly, simply because it is a quartic equation ok. It has four roots instead of two roots, a quadratic equation would have two roots in this case there are four roots.

And these four roots correspond to magnetosonic waves and as the name implies, magnetosonic since there is sonic in the name, it shares these waves share some properties with sound waves and as we remark at the very beginning of this discussion that is no surprise because delta p and by implication delta rho plays a central part in these waves and so, that is just like sound waves right. So, these magnetosonic waves in that respect, they share some properties with sound waves.

However, the phase speed u which is omega over k right, it depends upon not only the sound speed C s right. So, this is a sound speed, and this is the Alfven speed right so, it depends upon not only the sound speed, but also the Alfven speed and theta which is the angle between the direction of propagation of the wave and the magnetic field ok. So, it depends upon these three quantities.

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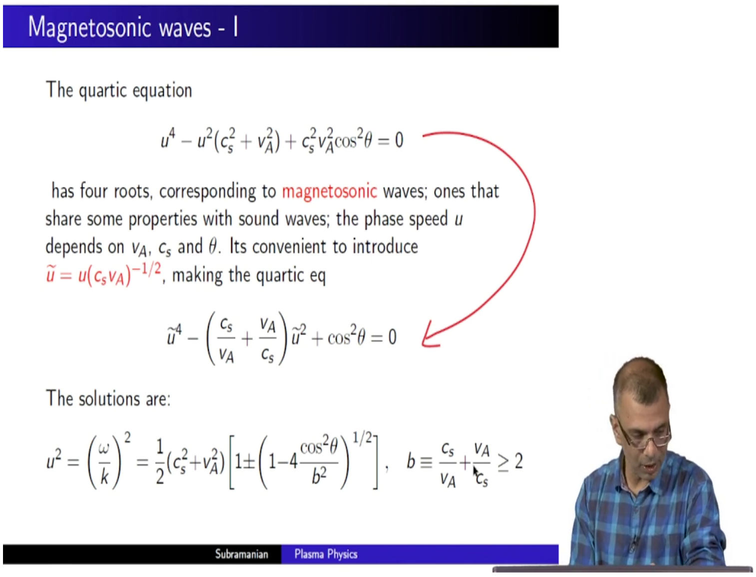

So, another you can cast this in a slight in a in an even simpler form by invoking a new variable u tilde which is u over square root of C s times V A and so, this no this equation essentially turns into this equation now. It is just a little bit of algebraic manipulation just to make things a little simple.

And now, from this, you can write, you can find out that the solutions to this quartic equation as is this alright. So, there are two roots corresponding to u plus and u minus, roots for u squared ok not just for u, roots for u squared themselves are double valued ok, you can have a plus branch in a minus branch ok where this quantity b square b is defined by this C s over V A plus V A over C s ok. So, let me this is a little complicated. So, I will repeat this in the next slide right.

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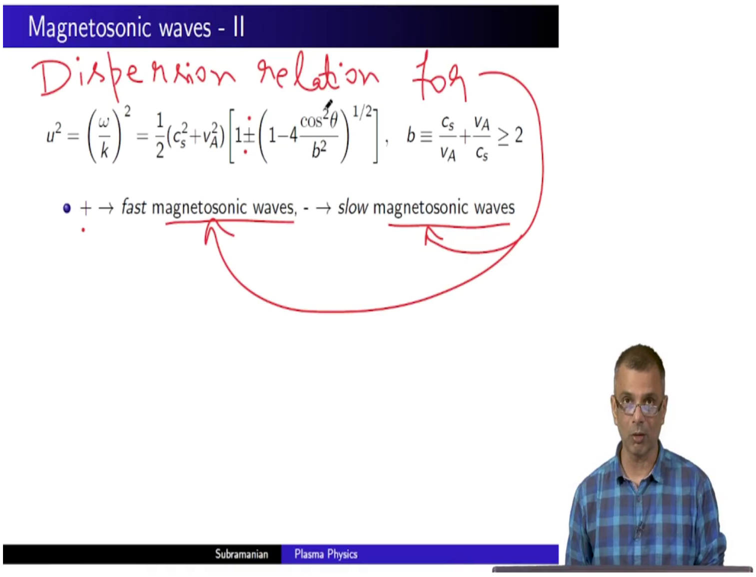

This is what that equation looks like. This is the dispersion equation; this is the dispersion relation for magnetosonic waves ok and the other interesting thing is that for magnetosonic waves so, both kinds of magnetosonic waves in fact, I would say ok, both fast and slow.

So, the plus if you take the plus sign here right this one, you get the plus branch gives you what are called fast magnetosonic waves and the minus branch here gives you what are called slow magnetosonic waves ok.

The dispersion relation is a little complicated, it depends upon C s, V A not just upon the ratio, but also individually upon C s and V A themselves and on this angle theta ok. Let us look at some limiting cases before going into the full generalities always good to look at limiting cases and right.

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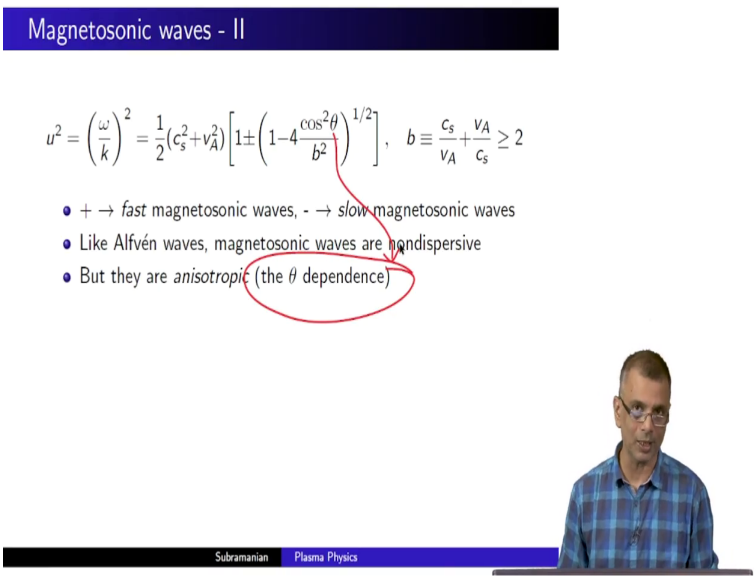

So, one of the first remarks we will make is that like Alfven waves, the magnetosonic waves are also non-dispersive. In other words, the phase speed and the group speed do not depend upon omega ok, they are non-disperse. Just like sound waves, just like a Alfven waves so, in that respect, the magnetosonic waves are similar to sound waves as well as Alfven waves, but they are an isotropic very importantly because why?

You see the theta dependence, see this is theta dependence right. So, there is a theta dependence here right.

So, clearly, the phase velocity omega over k depends upon the theta ok. If there was no theta dependence, there would be like sound waves and so, the direction does not matter, that is what; that is what having no theta dependence means. But here, the direction does matter which is to say there is a very significant an isotropy and the an isotropy is introduced by the due to the presence of the magnetic field ok. The magnetic field renders a medium, the properties of the medium and isotropic ok.

So, that is one important difference with the sound waves. However, Alfven waves are also similar in are similar in this respect in that the dispersion relation for Alfven waves depend on k sub z not in k itself in other words, they depend on k cosine theta. So, they also have a cosine theta dependence.

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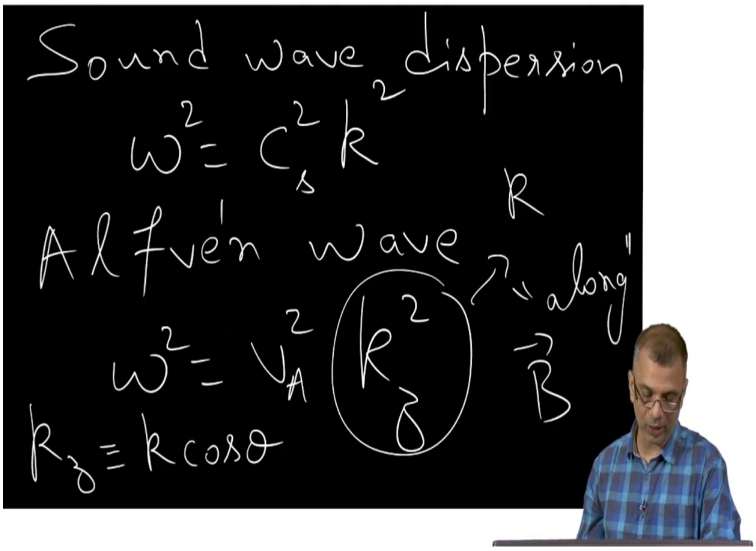

Although, if you remember here, you see the Alfven wave dispersion relation look like this right where the k sub z is really k cosine theta so, there is a theta dependence in the Alfven wave dispersion relation as well ok. In so, in that respect, a magnetosonic waves are like the Alfven waves, there is an isotropic, the exact the actual dependence might not be the same, but they are still an isotropic none the less right.

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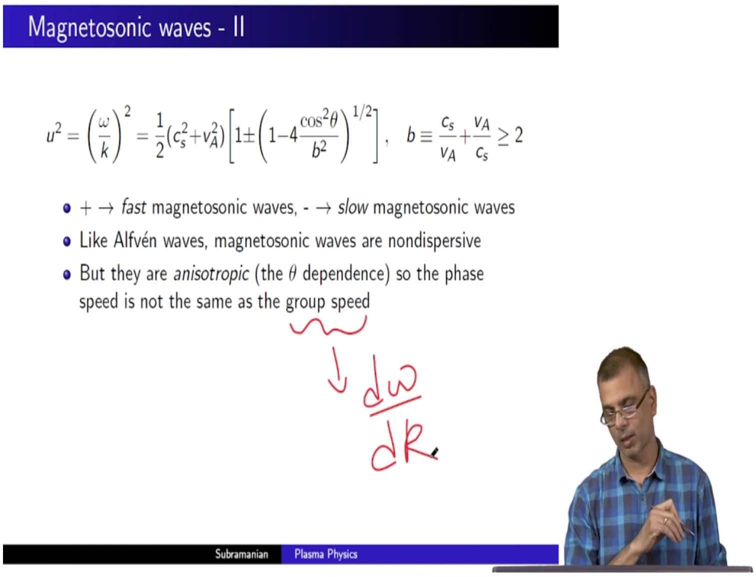

So, the other thing is the phase speed omega over k is not the same as a group speed which is the group speed would be d omega dk ok.

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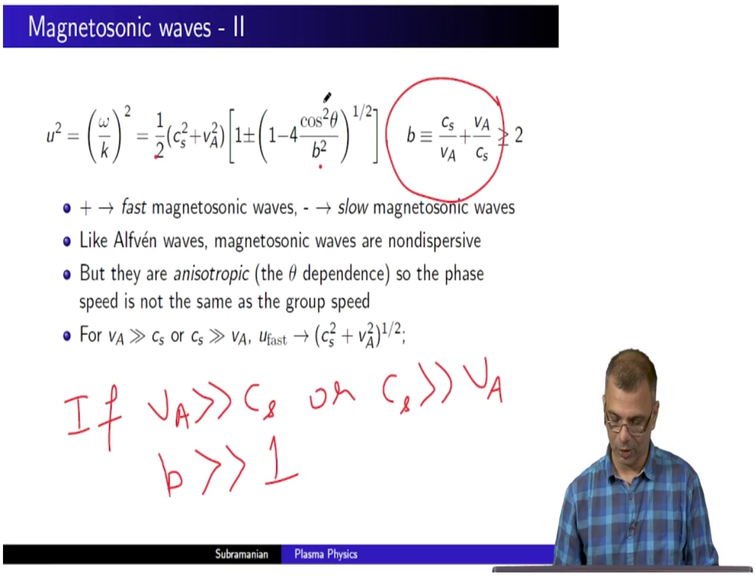

Now, I mean as such you know this equation is a little hairy so, let us look at one important limiting case. Let us look at the case where either the Alfven velocity greatly exceeds sound speed or the sound speed greatly exceeds Alfven speed ok.

In either case, if you look at this b ok, if V A much much greater than C s or C s much much greater than V A, what happens to b? b is much much greater than 1. For instance, if C s was much much larger than V A right so, which is essentially to say that this term is much larger than 1 and this is simply the reciprocal of this term so, this would be negligible, none the less, b is much much larger than 1 if C s greatly exceeds V A.

If this opposite is true, if V A greatly exceeds C s, then this term is much larger than 1 and this is negligible, either way b is much larger than 1 and where does b appear here? b appears in the denominator here, you see b appears in the denominator.

So, when this is much larger than 1, this entire term becomes negligible ok, this entire term becomes negligible and then, what happens is the phase velocity looses iits an isotropic character so, this entire term is, this entire chunk is not there and you if you look only at the positive branch in other words, if you look only at the first magnetosonic waves, the u fast which is the phase velocity of the fast magneticsonic wave is simply square root of C s squared plus V A square here out here.

In other words, only this 1 is important, this entire term becomes you know unimportant right. So, you have 1 plus 1 it is 2 and that cancels with this 2 in the denominator and you have this a fairly simple looking expression, but this is valid only in these two limits ok otherwise, the expression is a little more complicated than that, very important to keep in mind.

We are now examining certain interesting limits ok. So, in this limit, the fast magnetosonic speed is given by this and the fast magnetosonic wave is no longer an isotropic, the theta dependence goes away simply because this b is much larger than 1 and this chunk becomes negligible in in in comparison to 1 so, you might as well neglect this chunk, you might as well neglect this term which appears with the 4 alright.

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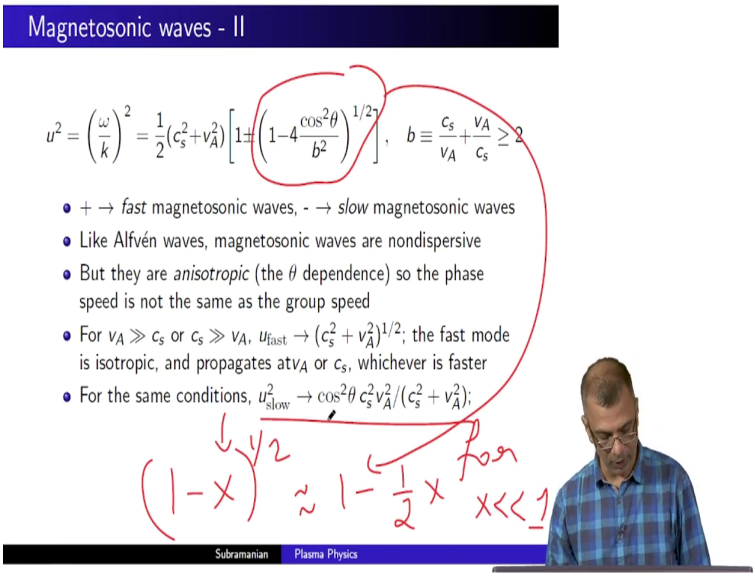

So, in other words, as we just said in only in this case, only when this Alfven speed greatly exceeds sound speed or vice versa, the sound speed greatly exceeds Alfven speed, the u fast is given by this and the fast mode is isotropic and it propagates at V A or C s whichever is faster, what this that is also obvious right.

Suppose C s is much much larger than V A, in which case the V A squared is negligible and so, the fast magnetosonic speed is simply the sound speed. If the reverse is true, if V A much much larger than C s, in that case, the C s is negligible and so, the velocity of propagation of fast magnetosonic wave is simply the Alfven wave. So, that is what I mean by the statement. It propagates at V A or C s whichever is faster ok. So, this is one interesting observation.

For the same conditions however, in other words, for the same conditions where b is much much larger than 1, this the cosine square theta over b square term is still negligible, but you say you are now considering the negative sign here so, you get 1 minus 1 right, this is still negligible.

But what you need to do is that you need to this does not you cannot simply take 1 minus 1 to be equal to 0, what you need to do is in this case, you need to this entire thing needs to be expanded like a 1 minus X raise to 1 half is approximately equal to 1 minus one-half X for X much much less than 1, it is kind of a binomial expansion ok you.

So, you see the second term is very small in comparison to 1 so, that would represent X right and o, you expand 1 minus x rise to one-half and you get this and from that, you arrive at this expression for the slow magnetosonic speed ok.

So, the only difference is that in this case, you are taking the minus sign where as for the fast magnetosonic speed, you were taking the plus sign, there is only difference, but that makes all the difference in the world when it comes to for instance an isotropy ok no matter what, the slow magnetosonic waves are still an isotropic.

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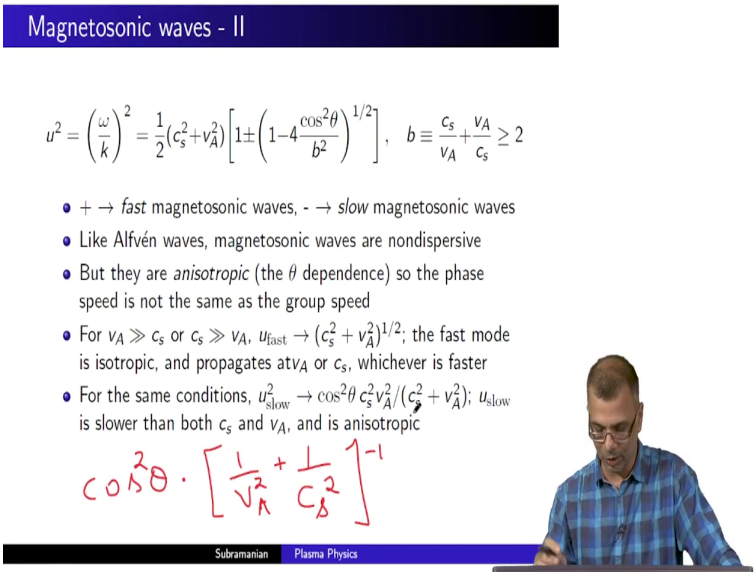

U and whats more u slow is slower than both C s and V A because of this cosine theta term ok. So, this is essentially, you know cosines theta square times you know a C s squared over V A squared over so, it would be 1 over V A squared plus 1 over C s squared raised to minus 1 that is what this is. So, that is why, the u slow is slower than both C s and V A ok.

However, this particular expression is valid only for this condition, but the slow magnetosonic speed is slower than the slower of these two ok. Either V A or C s whichever is slower, it is slower than even the slowest of these two which is why it is called the slow magnetosonics speed and it is an isotropic as is evident from the from the appearance of this cosine square theta right. It is total, there is no way then an isotropy is going away ok.

So, these are two limiting cases of this dispersion relation ah. In general, the dispersion relation is quite complicated ah, but you know it is exactly in such cases that one seeks to examine the limiting cases and try to see you know what one can get out of it.

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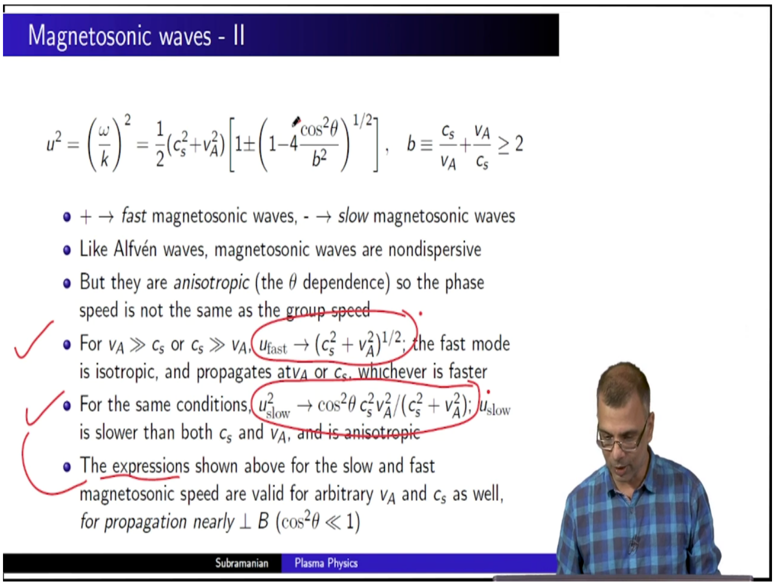

Now, the other thing is these two expressions for the fast and slow magnetosonic speed are valid for arbitrary V A and C s and other words, this expression and this expression, this is what I mean by these expressions. This, and these two expressions for the slow and fast magnetosonic speed of course, they are valid when V A is much much larger than C s or C s is much much larger than V A, they are valid in this case.

They are also valid when these conditions are not satisfied, but if the propagation direction is nearly perpendicular to B. In other words, cosine; cosine square theta is much much less than 1. In that case also, these two expressions are valid. The reason we making such a big deal out of it is because these two expressions are you know relatively simple, they are definitely simpler than this as you can see ok.

And so, you know those why is that why is that true? You see what was the main thing? The main thing was that when V A is much much larger than C s or C s is much much larger than V A, the main thing was that b is much greater than 1 ok so, b is much greater than 1 in which case you can neglect this entire term or rather this entire term multiplying the 4 is much much smaller than 1.

In the case of the fast magnetosonic speed, you could completely neglect it. In case of the slow magnetosonic speed, you take that to be a small parameter for expansion, for expanding in a binomial expansion right.

Now, if that is not the case, if b is not much larger than 1 right in other words, if C s is not much larger than V A or V A is not much larger than C s, if b is not much larger than 1, but on the other hand, cosine square theta is much smaller than 1, it demands the same thing mathematically, you see it demands the same thing.

Therefore, these expressions are valid for arbitrary V A and C s as well as long as the propagation direction is much and is nearly perpendicular to the magnetic field direction. In other words, cosine square theta is much smaller than 1 ok. These were two limiting cases.

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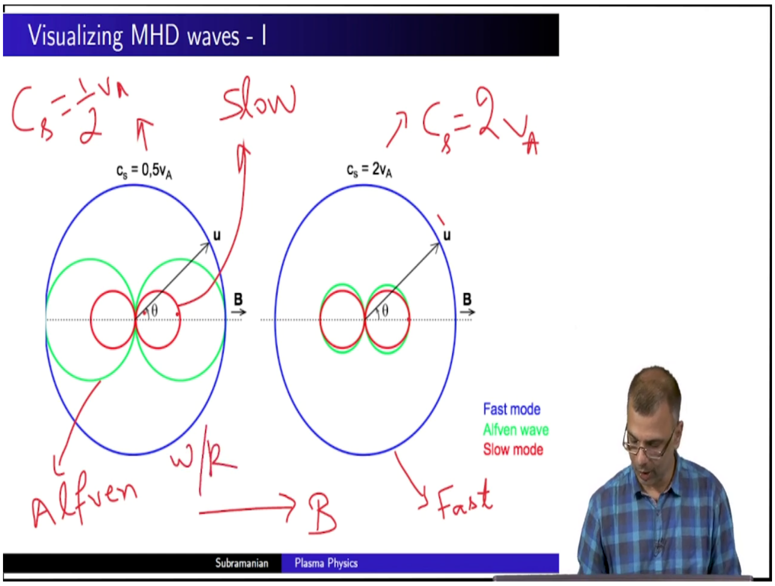

And since this is so we discuss the limiting cases a little bit. But now, it is time to now discuss the general case little bit and where you cannot assume that either the propagation is nearly perpendicular to b or you cannot assume and or you cannot assume that either the sound speed or the Alfven in speed is much larger than the one is much larger than other.

So, in the general case, you really you have to consider this full equation and here is kind of a diagram which helps in visualization. So, what this is? So, here this in this case, the sound speed is equal to one-half V A and in this case, the sound speed is equal to twice V A, two cases. In neither case, can we make the simplification that you know the sound speed is much smaller than or much greater than the Alfven velocity ok, they are comparable one-half and 2 are comparable. So, how do these two cases look right?

So, the B is along this direction ok ah, this is the direction of B right and fast mode is given by the blue line ok in all cases, this is the vector representing u ok and so, the blue line is represents a fast mode, the green line represents Alfven, the Alfven mode right and the red line represents the slow mode ok that is what I have said here.

The fast mode is represented by the blue line, the green line represents the Alfven wave and the red line represents a slow mode. So, what is this say right and this theta is essentially the direction between the angle of B the direction of propagation of the wave ok.

So, you see in this case, where the sound speed is slightly lesser than the Alfven speed, this is what the surface if you will for the fast mode looks like ah, the slow mode looks like this right.

So, there is a very and you see this one looks very much like a cosine theta or cosine square theta kind of plot whereas here, there is not much evidence for a cosine theta, there is not much evidence for there is some an isotropy, but not much ok and the Alfven wave also looks fairly an isotropic, it also looks like a you know cosine theta, this also seems to have a cosine theta or a cosine square theta kind of dependence the Alfven wave ok.

And clearly, the amplitude of the velocity for the slow mode is slower than the amplitudes of the velocity for the fast mode as well as a Alfven mode and that is true here too ok.

But the important difference between this case and this case where the sound speed is smaller than the Alfven's velocity and here, the sound speed is larger than the Alfven velocity, then the important difference between these two cases is that the fast mode is more nearly isotropic here ok, here there is a little bit of flattening here, whereas here, it almost looks like a perfect circle, the theta dependence is not there at all for the fast mode ok.

And as far as an isotropic goes, in this case, you know the Alfven mode in the slow mode are pretty much they preserve the same kind of cosine square theta dependence except the speed difference between Alfven mode and the slow; and the slow mode is has become much smaller ok.

The Alfven wave is still slightly faster than the slow mode except for propagation exactly along the magnetic field line ok, along exactly along the magnetic field line, the Alfven mode and this slow mode seem to have exactly the same speed, exactly the same phase speed ok.

Away from that, there is some the there is some difference, the difference is much larger in this case, but it is much smaller in this case ok. So, we can sort of surmise that you know the as the sound speed becomes progressively larger than Alfven speed, the difference between the Alfven mode and the slow mode goes down even more ok.

Both are quite an isotropic ok ah, but the difference in magnitudes; the difference of the magnitudes of the Alfven speed and the slow mode speed becomes much smaller as a sound speed becomes larger in comparison with Alfven speed. With the more that this asymmetry as in other words, the larger the sound speed in comparison the Alfven speed, the more isotropic the fast mode becomes. So, this is the kind of you draw these diagrams and you make these kinds of inferences.

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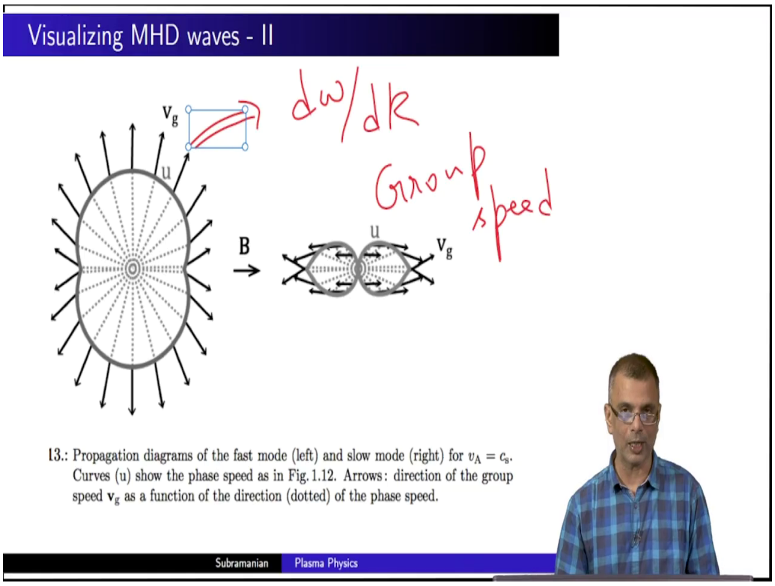

The other kind, other way of visualizing this MHD waves are these for instance, this would be the propagation diagram for the fast mode and this would be the propagation diagram for the slow mode and the curves, these show the phase speed as in this diagram and arrows, the arrows denote the direction of the group speed.

So, these are the phase speeds right omega over k, these are omega over k and in this case, the arrows these denote d omega dk ok. So, these contours are still omega over k and the arrows denote you know d omega over dk and the dotted lines denote the direction of the phase speed.

So, you can clearly see that the d omega dk, these solid arrows are not in general the same as the dotted lines. The group speed, this is the group speed, and the phase speed is simply omega over k which is given by these dotted lines. The dotted lines and the solid lines do not necessarily coincide sometimes they do, out here sometimes they do ok when you are exactly perpendicular, the magnetic field directions like so when you are exactly perpendicular the magnetic direction, they do, but not always.

Especially, when you are going along the magnetic field directions, there is a considerable a difference between the direction of the group speed and the phase speed ok. So, this is another way of visualizing these things ah. Suffices to say that magnetosonic waves or the dispersion relation from magnetosonic wave so, is considerably more complicated than that of just the Alfven waves or the sonic waves themselves. So, this is where we will stop for now.

Thank you.