Fluid Dynamics for Astrophysics
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Lecture - 55
Magnetohydrodynamics [MHD]: Waves in MHD- Alfven Waves

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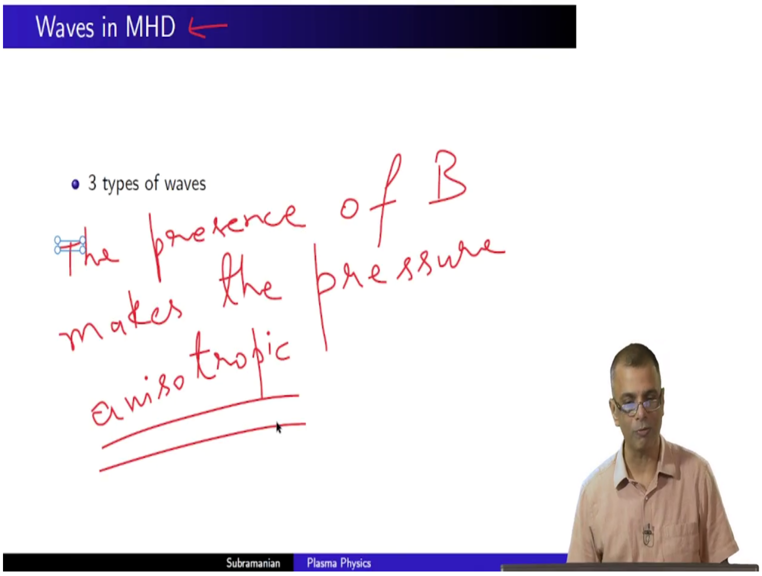

So, from today on pretty much further rest of the course until the course ends, what we are going to be discussing is, waves in magnetohydrodynamics which by now we abbreviate as MHD as you know right. So, why I mean why talk about waves? We have already talked about sound waves of course, in fluid dynamics and we will briefly review that right now. But the whole point of magneto hydrodynamics is that, its hydrodynamics plus a magnetic field is not it.

And we have seen all kinds of curious aspects of magnetic fields, we have seen that magnetic fields act like robber bands and so, on so, forth. So, simply from the fact that as we have noted earlier magnetic fields the presence of a magnetic field, presence of B makes the pressure anisotropic right. In other words the pressure tensor the components of the pressure along the magnetic field and perpendicular to the magnetic field are different right.

So, you would imagine that things would be different in the presence of a magnetic field right. The kinds of waves that can be supported by a medium containing a magnetic field will be different right that is why we are starting to examine the phenomenon of waves in MHD.

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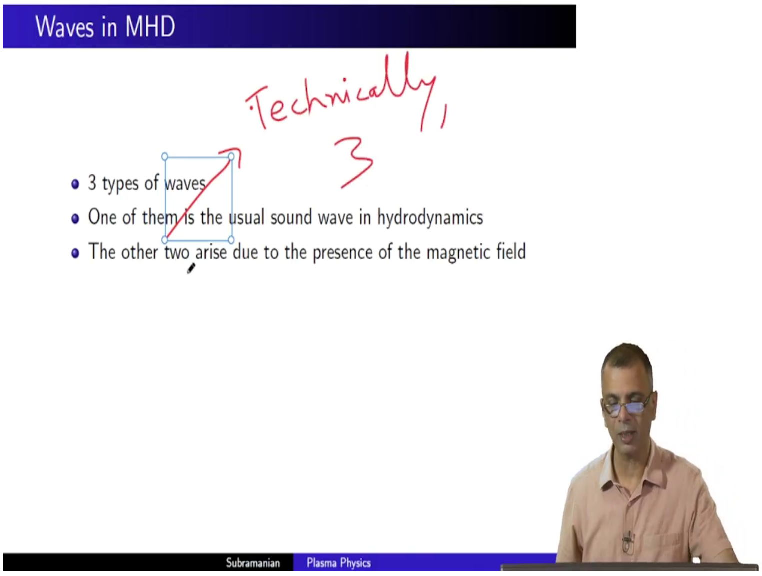

As it happens there are 3 types of waves, one of them is the usual sound waves even hydrodynamics which we have already encountered. But we will discuss it very briefly just for sake of completeness, I will hurry over it because we have already discussed it. But you know it is good to discuss it right before the other kinds of waves, you will see that that this certain kind of certain amount of completeness that is accrued by a discussion right.

So, the first one is the usual sound wave in hydrodynamics, the other two which are actually technically other 3.

Technically 3, I call this 2 but a one of them is the Alfven wave and the other one I call magneto sonic, but actually there are two kinds of magneto sonic waves ok. So, really there are 3 kinds of there are 3 more waves in addition to the sound wave. And so, the waves other than the sound waves are essentially hybrids between the sound speed and what are called what is called the Alfven speed.

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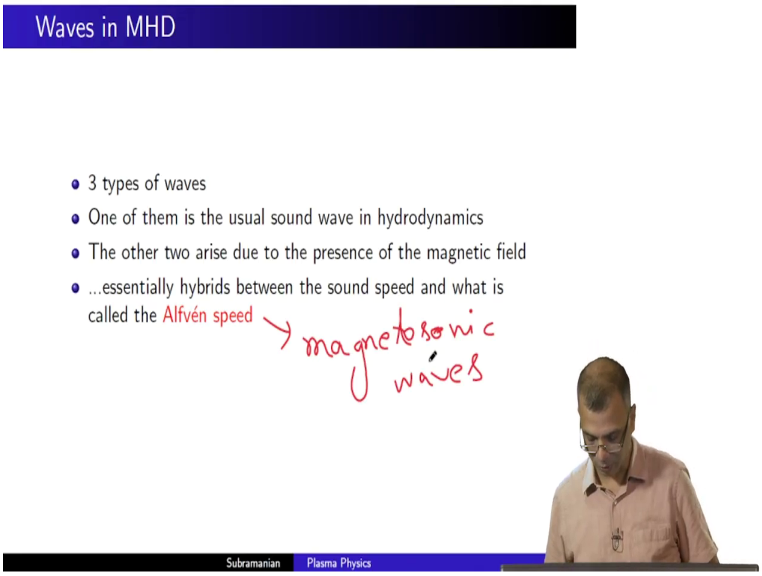

These are what are called magneto sonic waves ok. These are hybrids between the Alfven speed and the sound speed ok. So, we will discuss these 3 kinds of waves as we go along.

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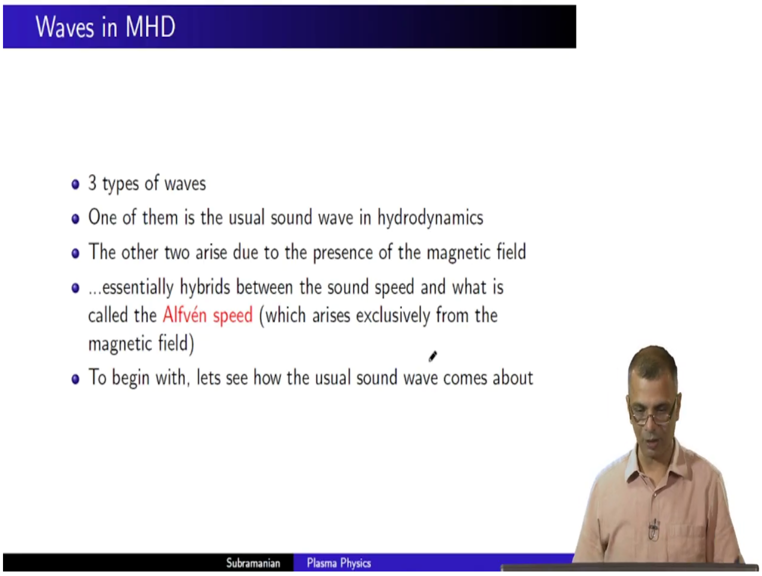

The Alfven speed on the other hand, the Alfven wave one other hand arises exclusively from the presence of a magnetic field. It is a transverse wave much like the kind of waves that would be excited on a string, on a stretched string like guitar string for instance and you immediately see the analogy right.

We talked about magnetic fields as rubber bands of sorts, strings of sorts. So, these kinds of rubber bands and you can imagine that they will support transverse waves. You take a rubber band, you stretch it, and you twang it in a transverse manner and there will be waves that will propagate along that and those are exactly what Alfven waves are ok right.

So, to begin with let us quickly review the sound wave and. So, that is what we will do for the remainder of this particular session right.

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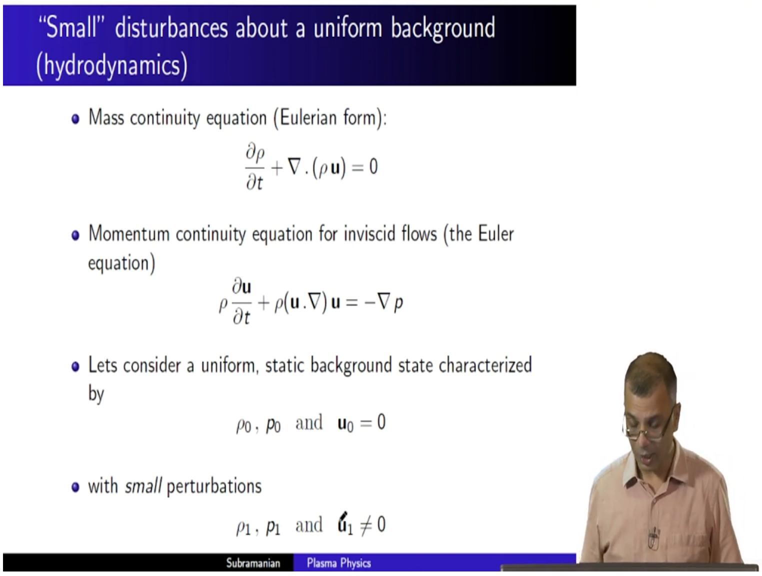

So, you have already seen this, we have the mass continuity equation written in Eulerian form and the momentum continuity equation, we will not bother about viscosity for the time being. This is only for inviscid flows, which is the Euler equation right.

So, this is the momentum continuity equation for inviscid flows and that is the mass continuity equation. And what we are now going to do is, we are going to perturb the background right. So, let us consider a uniform static background characterized by some background density rho naught, some background pressure u naught and without loss of generality we can consider background velocity u naught to be equal to 0 right.

No background velocity, no breeze flowing through the room, the air in this room is static ok. Now what we are going to do? If you remember we are going to perturb this background, we are going a the perturbations will be denoted by subscripts 1, rho 1, p 1 and some sort of a perturbed velocity which is obviously, non 0 ok.

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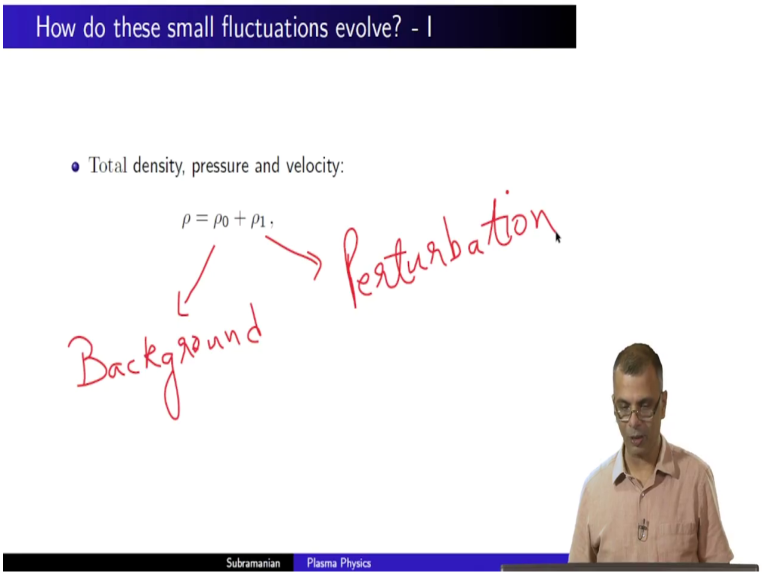

And you remember the main thing. So, the total density would be the background density plus the perturbed density. So, this is back ground right and this is the perturbation right. So, similarly the total pressure would be the background pressure plus the perturbed pressure so on so forth.

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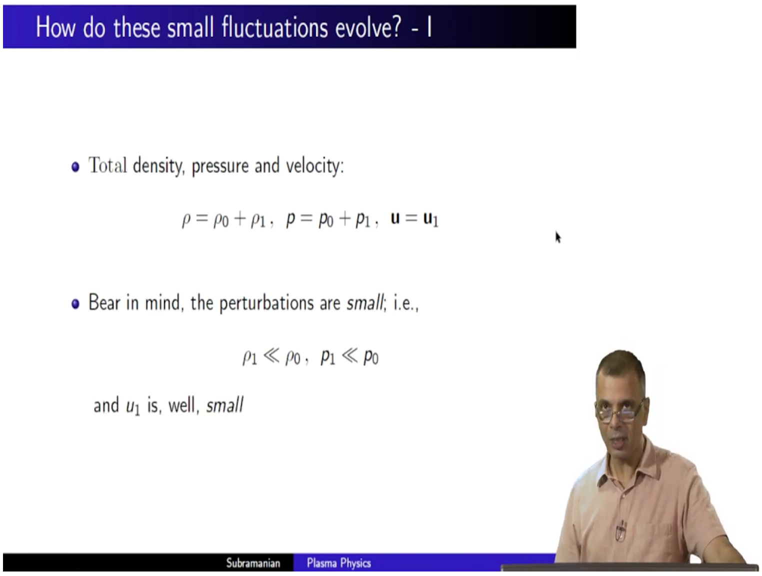

And the total velocity simply the perturb velocity because the background velocity was taken to be 0 without loss of generality. The other thing of course, is that the perturbations are small. In other words, rho 1 is much much smaller than rho naught, p 1 is much much smaller than p naught and so, on so, forth.

And u 1 is just small you know the background was exactly equal to 0. So, there is no there is nothing to compare u 1 with ok. So, it is simply small.

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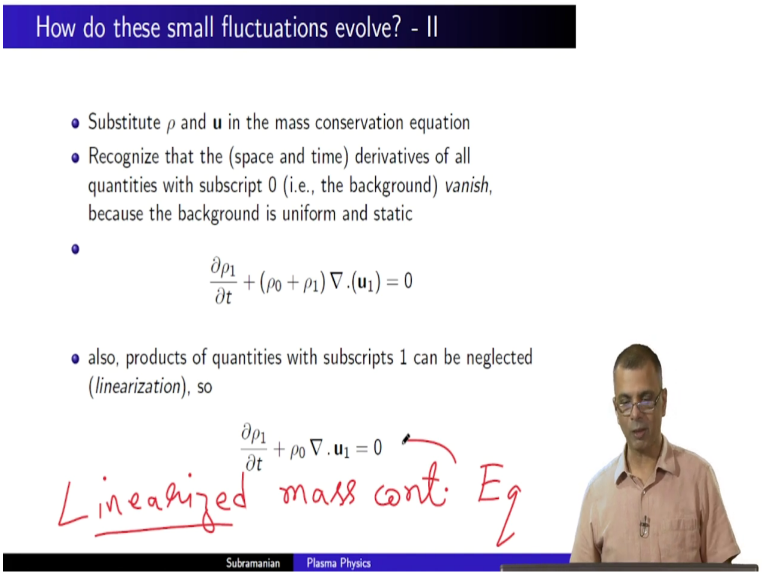

Now, what we do is, we substitute these total quantities the background plus the perturbations in the mass conservation equation right, in this equation right here ok. And recognize that the since the you know background is taken to be to be uniform and static we recognize the space and time derivative.

So, static means the time derivatives of all quantities with subscript 0 become 0 right and the space derivatives in other words d over d x of rho naught or d over d x of p naught would be 0 that is because the background is uniform. So, the background is uniform and static right. So, that is what so, you substitute rho equals rho naught plus rho 1 anywhere you see d rho naught d t you said that equal to 0. Anywhere you see a d rho naught over d x you said that equal to 0 so on so forth right.

And so, after you do that this is what becomes of the mass continuity equation. I am going a little fast because you have already seen this before ok. I am simply you know redoing sound waves very very quickly for the sake of completeness. So, this is the perturbed mass continuity equation. What is more, since the perturbations are taken to be small, products of small quantities rho 1 times something of the dimensions of u 1, it is actually divergence u 1, but that is ok, it involves u 1 ok.

So, it is like 0.1 times 0.1 is 0.01. So, products of small quantities can be neglected. So, when you expand this, this becomes rho naught divergence of u 1 plus rho 1 divergence of u 1 right. So, the rho 1 times divergence of u 1 is neglected in favor of the first quantity right. So, this is the linearized. So, this is what is called the linearized mass continuity equation this one ok. Similarly, one can write down a linearized momentum equation right.

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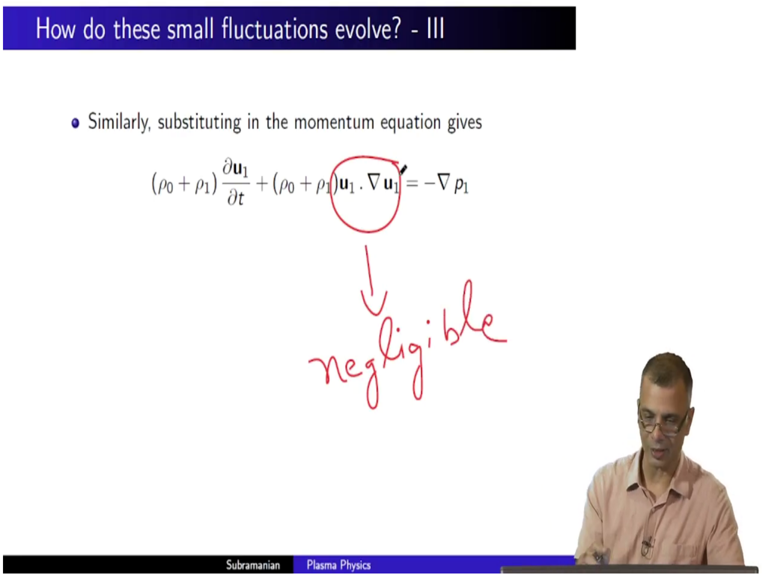

Similarly, substituting in the momentum equation using the same things. So, wherever you see d rho naught d t or d rho naught d x where you neglected and obviously, quantities like this, this are this are negligible because this involves you know products of small qualities right, u 1 times gradient of u 1.

So, you neglect this.

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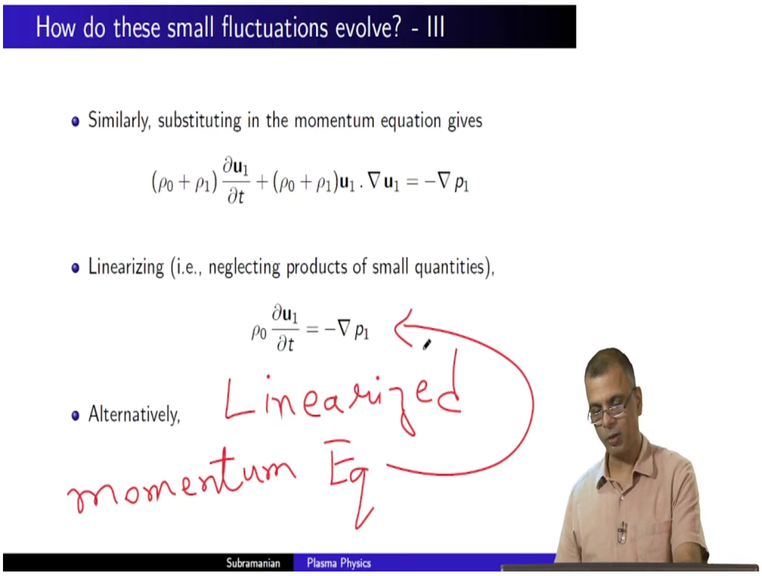
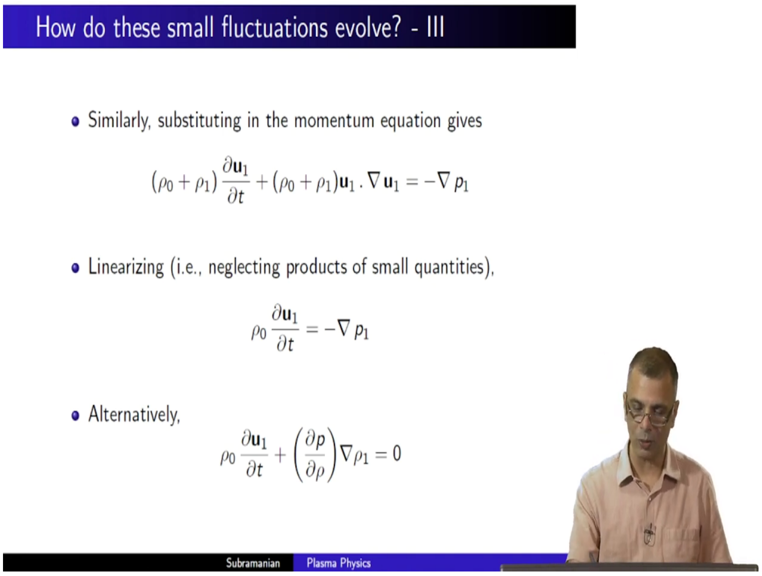


So, this is entirely neglected. So, similarly rho 1 times d 1 d t, although it is a derivative, it is still product of small quantity. So, that is neglected. So, you are left with this very simple equation which is the linearized momentum equation. So, this is the linearized momentum equation this one ok. So, you have this.(Refer Slide Time: 09:51)



Alternatively, instead of using the pressure we can use we can write it in terms of the gradient of a the perturb density ok. Gradient of rho 1 and we have to make use of the chain rule of course, and this quantity is the sound speed as we know right.

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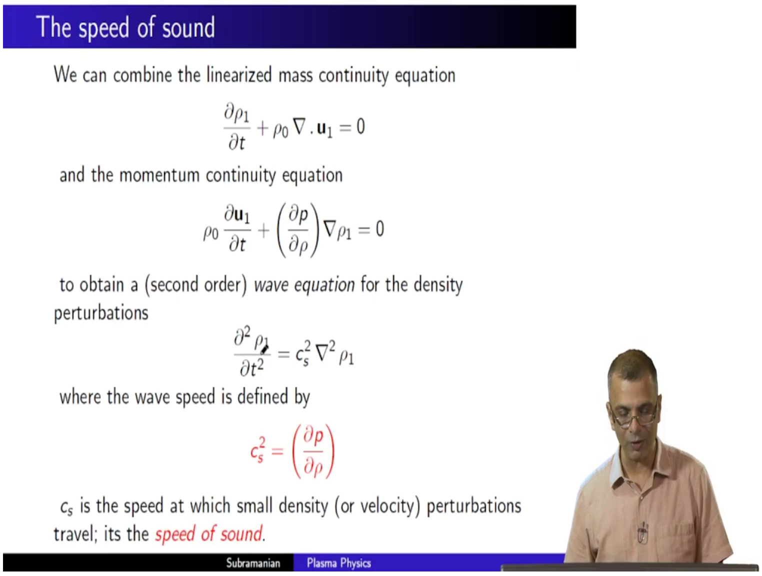

And so, you combine the linearized mass continuity equation and the momentum continuity equation to obtain a second order wave equation for the density perturbations, we have seen this before this is the speed of sound.

So, this guy which appears. So, this is a wave equation and whatever appears here is a speed of the wave and this is the wave the speed of sound. So, by definition by simply by virtue of writing down this equation, we can identify C sub s as the speed at which small density or for that matter velocity you can obtain an equation that looks exactly like this for the quantity u 1 ok. So, C sub s is the speed at which small density or velocity perturbations travel and it is a speed of sound.

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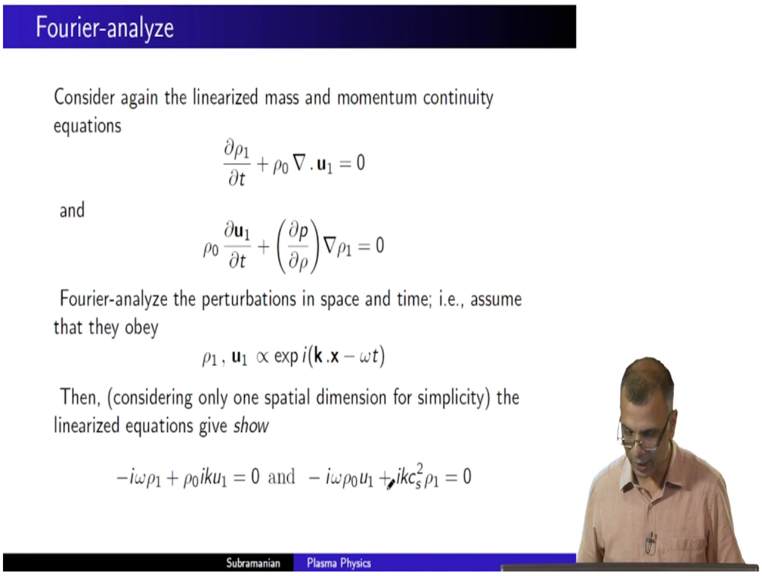

And we also Fourier analyze the equation. In other words what we do is we instead of a d over d t we can write an e raised to i omega t, instead of a d over d x we can write an e raised to i k right. So, you linearize these two equations and that is what it looks like.

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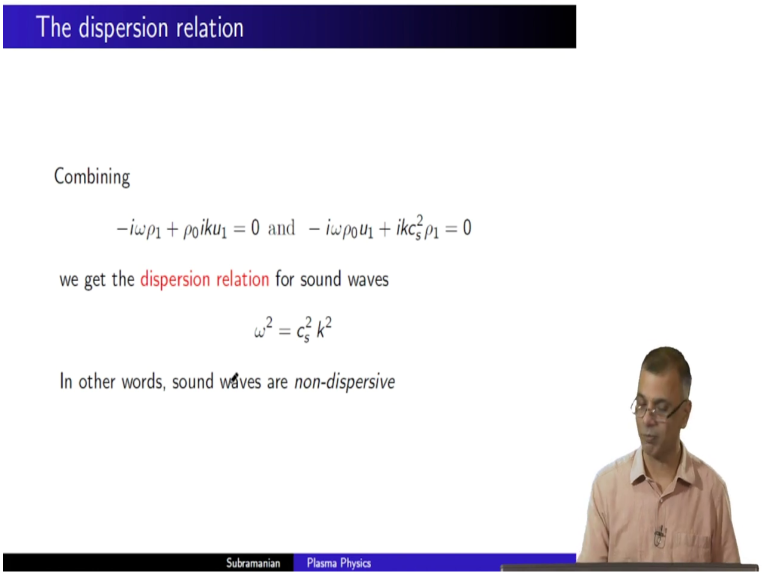

The only reason I am doing this is to get the important dispersion relation. So, this is the dispersion relation ok. The omega k relation and what is evident from this dispersion relation? It is just the main thing is that sound waves are non-dispersive. The omega over k is the same no matter what the omega is ok. It is not like light passing through a prism where different wavelengths have different velocities ok. All wavelength have the same velocity for that matter even d omega d k which is a group velocity is the same.

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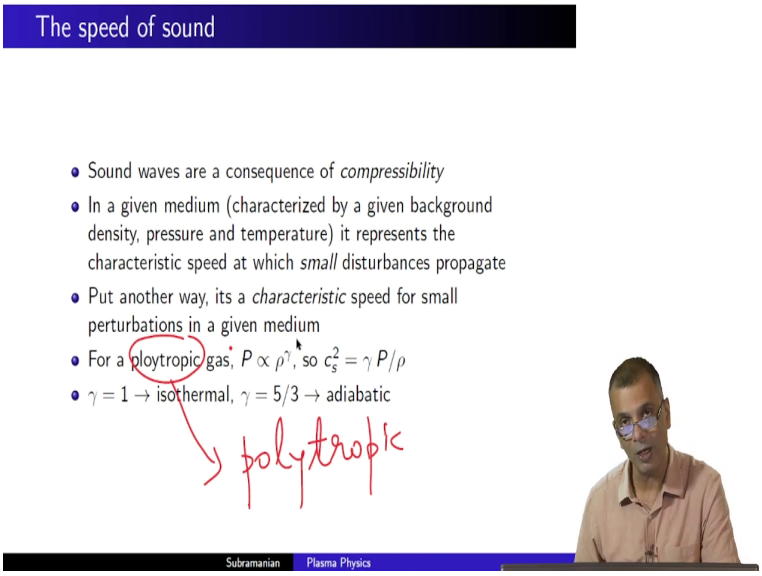

So, this is the other thing. Sound waves are non-dispersive, in other words low frequencies, high frequencies they all travel at the same velocity. And you know sound waves are consequence of compressibility and in a given medium characterized by given background, density, pressure and temperature, it represents the characteristic speed at which small disturbances propagate. Very very important ok there is only one speed at which small disturbances can propagate.

I mean you know, given a certain temperature in the room or given a certain background density and pressure ok which is the same thing as saying temperature and because you know if you take an equation of state, these three quantities are related to each other right. So, given a certain temperature there is only one speed at which small perturbations can propagate, a small unmagnetized perturbations can propagate.

And so, put it another way it is a characteristic speed for small perturbations in a given medium ok. And the actual speed of perturbations can depend upon the manner in which this is actually this is a spelling mistake, it should be polytropic I beg your pardon for that ok, it is not ploytropic, it is polytropic. So, essentially what this is saying is depending upon the thermodynamics of these perturbations ok, these perturbations can be adiabatic or isothermal right or a something in between.

So, the thermodynamics of these perturbations is all hidden, is all contained in this gamma parameter. So, depending upon the thermodynamics of these perturbations the sound speed can be different ok.

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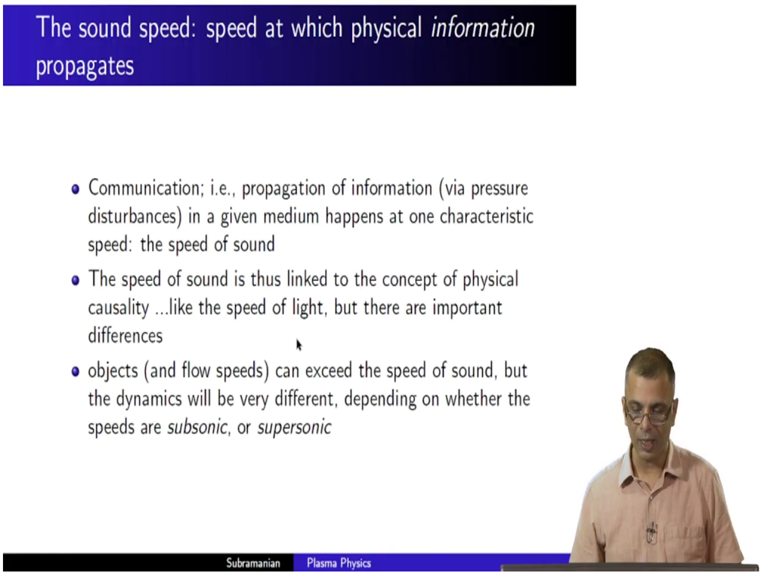

So, it is a speed at which physical information propagates via pressure disturbances. The speed of sound is thus linked to the concept of physical causality in that you cannot hear what I am speaking until the sound waves have had time to reach from me to you ok.

And as we have seen depending upon whether you know an object is subsonic or supersonic or something the dynamics would be very very different ok.

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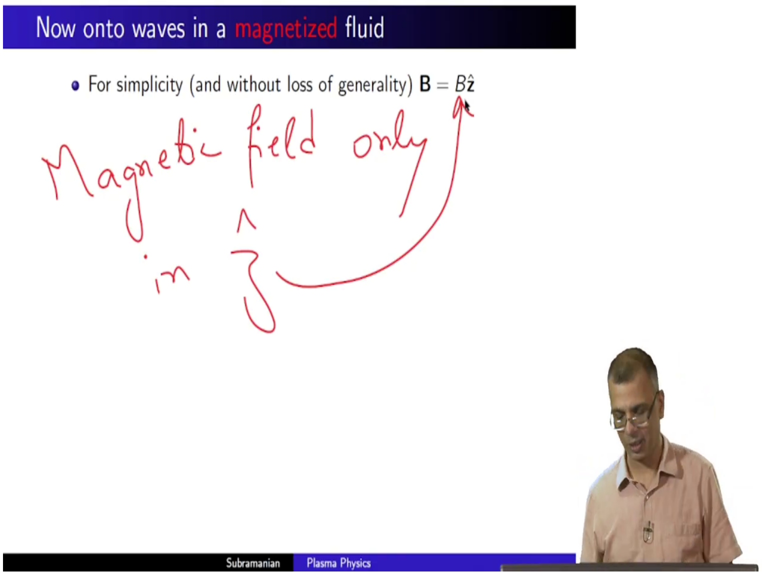

Now, imagine now how much more complicated. So, essentially the point the real reason I wanted to do this recap right now is to emphasize once again the fact that the sound speed is a characteristic speed ok. Now owing to the presence of the magnetic field ok there can be other characteristic speeds as well. In fact, to be precise three more characteristic speeds.

The Alfven speed, the fast magneto sonic speed, and the slow magneto sonic speed. The magneto sonic speeds as the name implies are mixtures between the Alfven mode of propagation and the regular sound propagation that we have seen right now. So, now, imagine now you have four characteristic speeds. So, now, this really makes the dynamics of a magnetized fluids very very interesting and very very rich ok.

We have already seen that depending upon whether you know a the speed at which an object or the fluid is flowing depending upon whether it is sub sonic or supersonic, the dynamics are very different. Now just imagine you have got three more characteristic speeds ok. So, depending upon whether or not the flow speed or object speed is below or above any one of these characteristic speeds, the dynamics will be different now ok.

So, this is. So, just to give you a hint of how interesting things can become ok. So, now, let us start to discuss one of the most important modes in magneto hydrodynamics that of the Alfven mode ok. So, for simplicity and without loss of generality let us consider that the magnetic field is exclusively in the magnetic field only in the z direction that is what this is saying right.

And if I have any other direction that is ok right I mean I can always superpose my solutions. So, that is why I say without loss of generality ok. So, from now onwards I consider the magnetic field to be only in the z direction ok.

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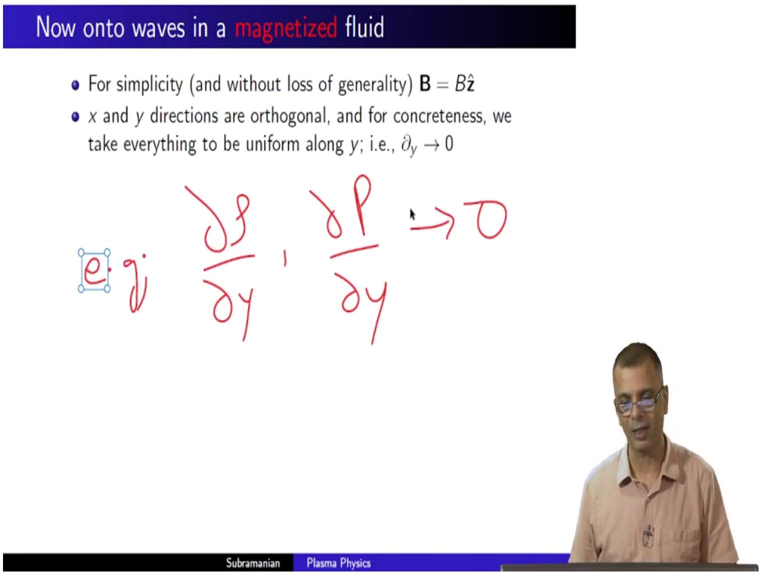

The x and y directions are orthogonal to z obviously, and for concreteness, we take everything to be uniform along y. In other words all derivatives with respect to y or 0 e g d rho d y, d p d y these are all equal to 0 ok.

So, the medium is completely uniform along y ok and x is the only orthogonal direction. Again this is completely without loss of generality if you have variations along y, we can next consider everything to be uniform along the x direction and consider variations along the y direction and superpose these two solutions together, linear superposition ok.

So, this does not in any way take away the generality of what we are discussing. It just simplifies a math which we definitely want to do as you will see the math does get a little hairy ok right.

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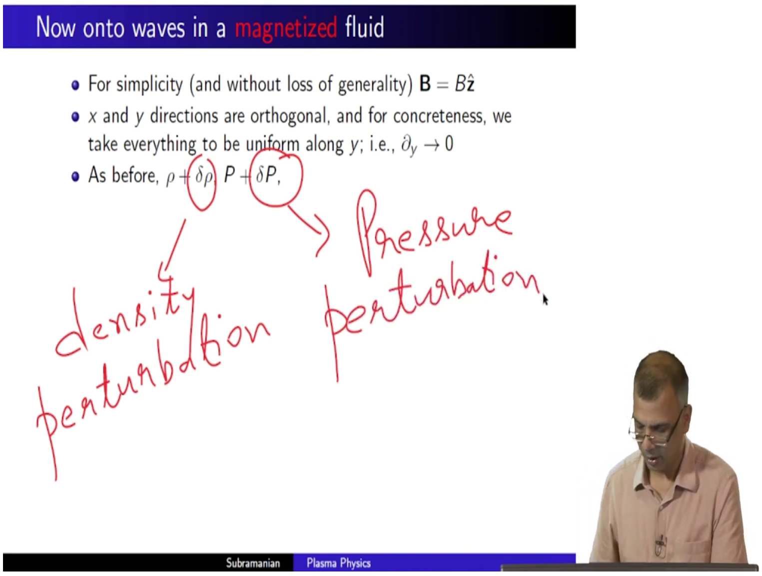

As before, now this is what we called as before we consider the background density plus perturbation in density. So, this is the perturbation, this would be density perturbation this is what we used to call you know rho subscript 1. Earlier now we are calling it delta rho it is the same thing right density perturbation and this is the pressure perturbation ok right.

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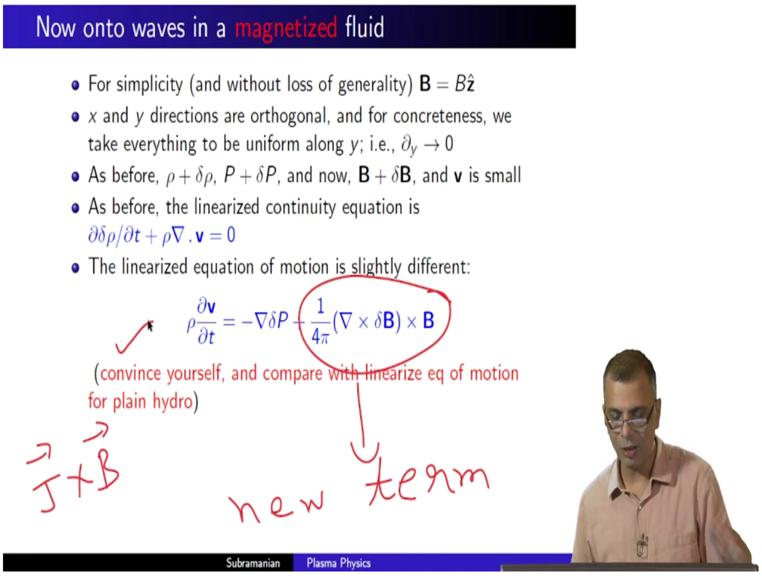

So, now and in addition to pressure and density of course, you have the magnetic field is not it. So, I perturb the magnetic field as well with a delta B ok. And whatever velocities there are the background velocity was 0 and so, whatever velocities there are, are the perturb velocities and therefore, v is small. As before the perturbed quantities are all small in comparison to the background quantities. In other words, delta rho is very very small in comparison to rho, delta p is very very small in comparison to P and delta B is very small in comparison to B ok right.

Now, as before we will not repeat the linearized continuity equation, the mass continuity equation it looks like this. It is exactly the same as what we have derived there is the reason I just recapitulated you know this regular sound wave a little earlier. So, the linearized continuity equation is this.

The linearized momentum equation now, you remember we only had this chunk and this chunk in the momentum equation right. These two chunks were the only ones that were there and this now is a addition due to the fact that there is a Lawrence force ok. So, this is all important addition to the momentum equation ok.

Well in fact, this is all important addition period as far as Alfven waves are concerned, the linearized continuity equation is exactly the same as what used to be there for sound waves, the only new thing is this ok.

And compared with the linearized equation of motion for plane hydrodynamics as we just said this is the new chunk this is the only new term ok. And I have simply written it down I urge you to you know look at the J cross B essentially it arises this whole thing arises from linearizing, J cross B where J is nothing, but curl of B is not it and in place of B, you right B plus delta B that is what you do right.

So, and you do the usual thing, you neglect the space and time derivatives of the background and you neglect products of small quantities. Once you do that you will arrive at this extra chunk ok. So, which is why I am saying convince yourself this is very very important I have not given you the steps, the steps are that the philosophy of arriving at the steps are is exactly the same as what was done for the sound waves and so, yeah.

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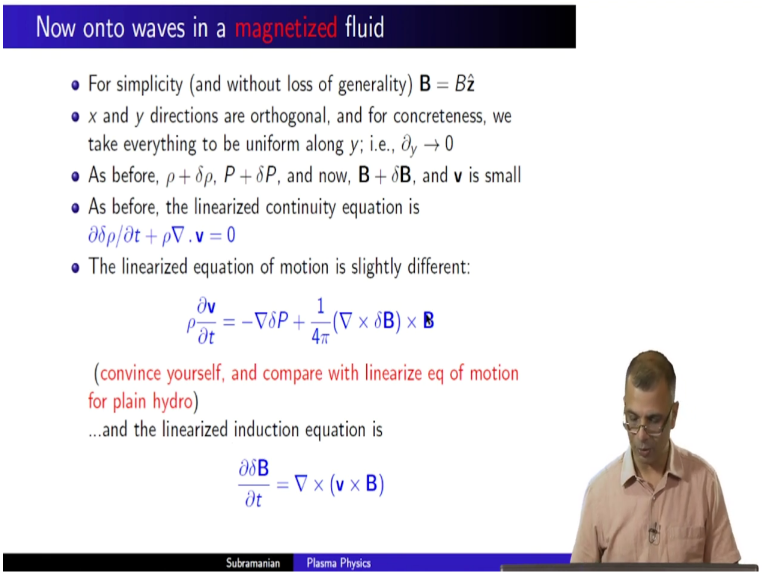

So, this is the only new term now and of course, when you are talking MHD, you cannot escape the induction equation of all important induction equation. So, the linearized the induction equation again linearizing any equation the same philosophy neglect take the background to be spatial uniform in space and time, and neglect products of small quantities that is all. So, the linearized induction equation is now this right v is already small, v is a delta like quantity already ok right.

So, that is it. You have these three equations this this the ones shown in blue. These are the three equations that we have to work with and you know the one thing to do is these are vectors right, v is a vector right.

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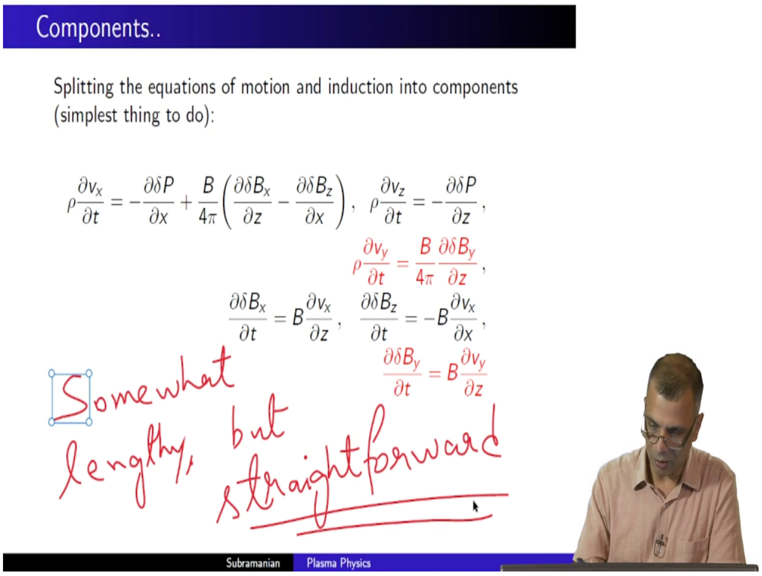

So, what we do is you split the equations of motion and induction into components. No need to worry about this, just this and this you split it into components you split it into x y and z components ok.

And it is a little it looks a little you know hairy, but this is all I mean you know it is all quite straightforward ok. All this is, is splitting up the equation of motion and induction into components ok. So, you have this, this, this and this. So, these are all quite straightforward you know somewhat lengthy somewhat lengthy, but straight forward which is what I want to emphasize ok.

So this, these equations are nothing, but this guy and this guy written down in component form that is all ok. Now the thing to realize I there is a reason I have marked this and this in red ok..

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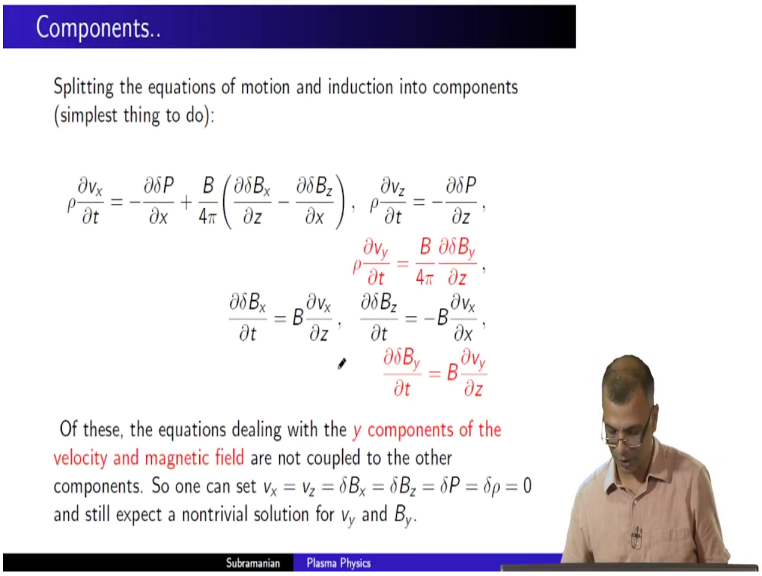

The reason is you can see that the equations marked in red are the y components of the velocity and magnetic field and they are not coupled with the other components. You see you have V y, you have delta B y, delta B y, V y there is no appearance of a V z or V x or B z or B x anywhere here ok.

Therefore, if you can it is perfectly acceptable to set V x, V z delta B x delta B z and delta P and delta rho all equal to 0 ok. In other words, no perturbations, no perturbed velocities in the x or z direction, no perturbed magnetic fields in the x and z direction. In fact, no perturbations of pressure or density either and still expect a nontrivial solution from these two equations why is that? They only concern each other they only talk to each other ok.

Delta V y and delta B y yeah. So, V y which is the same as delta B y because v is small V y and delta B y only talk to each other, they do not care about the rest of them. The rest of the components might as well be simply set to 0 which is what we will do in a minute. And so, you just you know set all of these to 0 which is to say I just neglect the other equations I am written in black, I only look at these two equations right.

And these are coupled first order equations and what does one normally do? With coupled first order equations you try to see if there is a way that you can combine them into one second order equation right. See, you say you have a d V y d t and you have a d V y d z. So, suppose I took time derivative of this entire equation yeah. So, I would have d square delta B y d t square equals B times d over d z of d V y d t and in place of d V y d t, I substitute this right.

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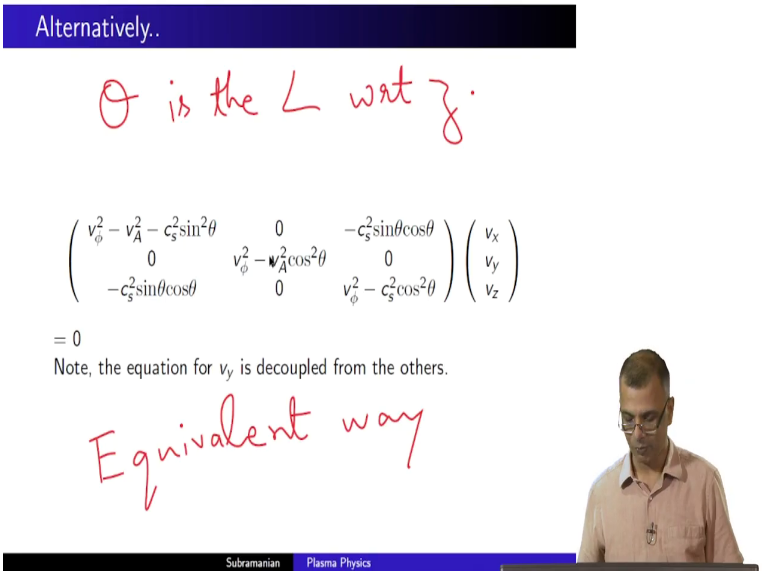

So, alternatively you could write down this system of equations as something like this where theta is essentially is just the angle theta is just simply, theta is the angle with respect to z ok which is where the sin theta and cosine theta come from. But, anyhow you mean now do not worry about it, this is in all I want to say this is an equivalent way of writing this system of equations.

And here too you can see that the equation for V y is decoupled from the other ones alright and there is an appearance of this C sub s which is the sound speed and the appearance of V A which we will see what it is. All I want to say is this is an alternative way of writing down this, if you wish you can skip this entire slide do not worry about it you can convince yourself about it later on.

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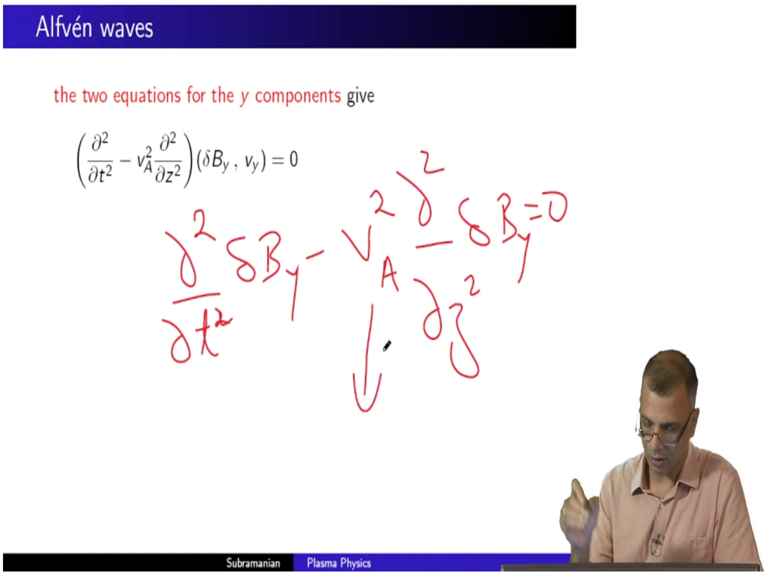

So, simply moving on from the two red equations ok the two coupled first order differential equations and so, you combine them into you know a second order equation like this ok, you get this alright. What this is saying is that, when I say delta B y comma V y what this is saying is that it is the same equation for delta B y as well as V y. In other words, d square over d t square delta B y minus some speed ok d square over the z square delta B y is equal to 0.

And same thing instead of delta B y you can substitute V y and you will get the same equation, where this all important V A squared is where this all important V A squared is B squared over 4 pi this thing.

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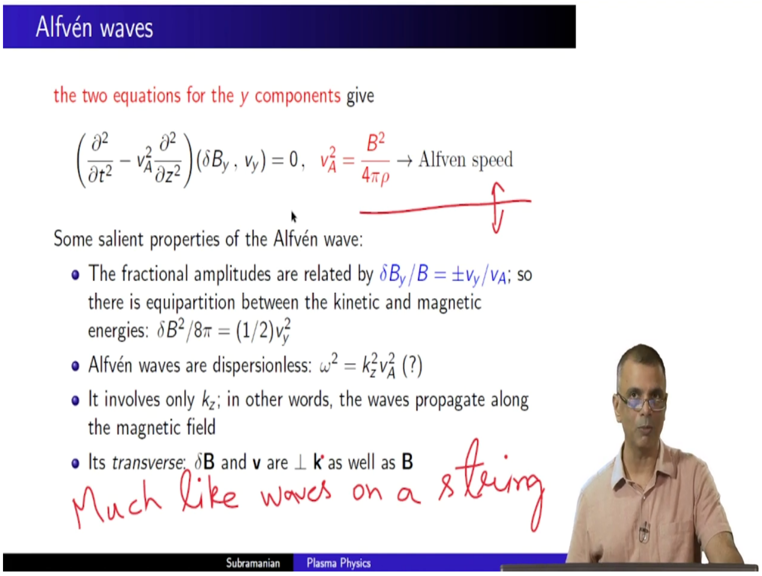

This is the wave speed ok. Now having done this it is important to sort of see what kind of wave this is ok. This is very very important we will talk about these two points, but the other you know the most important point that I want a emphasize right now is that this involves only case of z only a d square of d z square.

In other words, the waves propagate along the magnetic field. Remember the magnetic field is pointing along the z direction and this propagation is only involves d square over d z squared and otherwise the propagation vector only has a case of z ok. So, the waves propagate along the magnetic field that is one thing. The other thing is that, you see this the delta B and the v have only y components and the y we now is orthogonal to z right.

That is why so, the delta B the magnetic field perturbations and the velocity perturbations are perpendicular to the wave propagation vector as well as you know the magnetic field. Much like waves on a string, transverse waves on a string to be precise it is exactly like that you have a string and you twang it like so.

You have you know a string like this and you give it little velocity perturbations like this yeah. So, the perturbations are transverse to the direction of the string and these perturbations also you know propagate along the string like so, ok. So, these Alfven waves are exactly analogous to transverse waves on a string. And when we meet next, we will take up this you know the properties, the other properties of the Alfven wave in a little more detail.

Thank you. That is it for now.