Fluid Dynamics for Astrophysics
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Lecture - 50
Magnetohydrodynamics [MHD]: Magnetic stresses and magnetic buoyancy

Yes. So, when we finished last time, we were talking about Lorentz forces, if you remember correctly.

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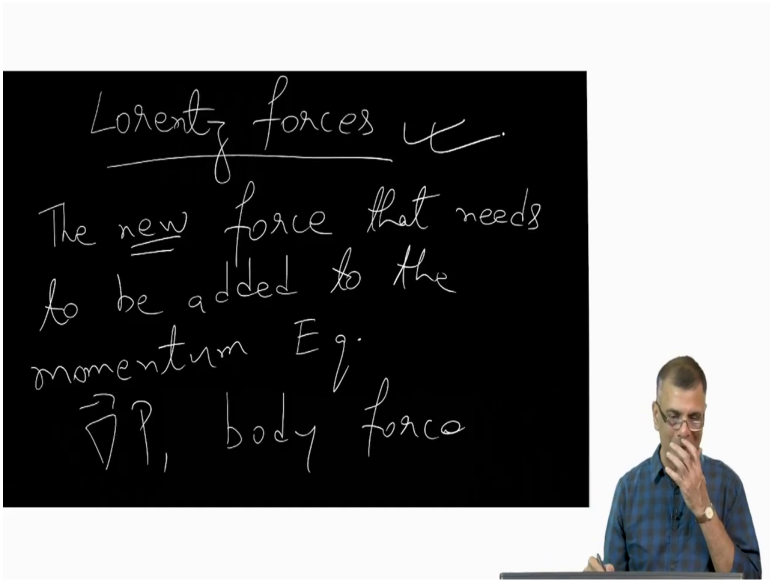

Lorentz forces which are the essentially the new force that needs to be added to the momentum equation. And when we say new force, what do you, what do we mean new? With regard to our previous discussion of neutral fluids, is not it. Everything else was the same. We had the gradient of pressure, we had the gradient of pressure, body force, and everything, right. But the Lorentz forces were the new thing. And what did the Lorentz forces look like?

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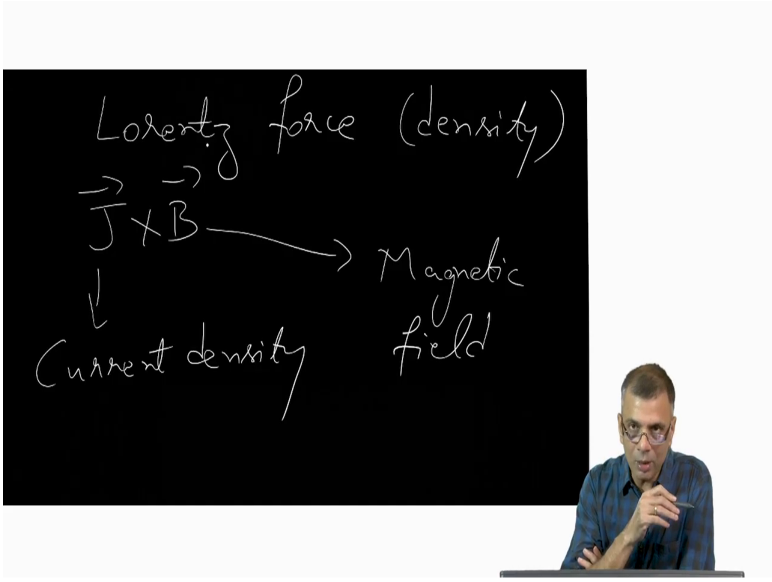

The Lorentz forces it essentially the Lorentz force density, ok which is force per unit volume and it look like a J cross B, where this is current density and this is the magnetic field. I am neglecting; there is a factor of 1 over 4 pi or something out front, but in the all that present in the actual, in actual slides that I will be showing you. But this is the main part, right J cross B.

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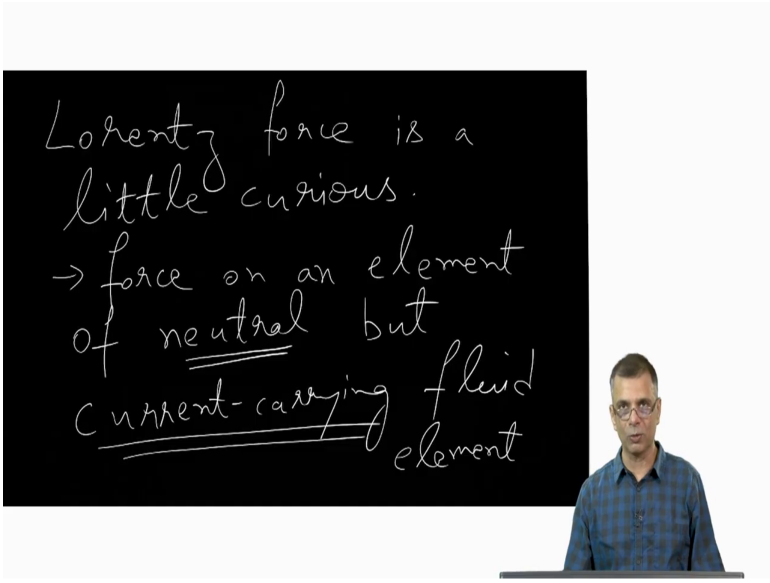

Now, you remember that this is a bit of a curious force, right. The Lorentz force is a little curious. Why? Why do we call it curious? Because it is the force on an element of neutral, but current carrying fluid. So, this is little you know at variance with our usual ideas of fluid element, fluid element really, ok.

This this neutrality is a little bit at variance with our ideas of Lorentz forces which is which we normally think of as a Q times V cross B kind of force, is not it; where you know the Q is an explicit charge, right. So, the fact that you know now we are suddenly saying that this fluid element is neutral is a little funny.

So, how can you have a Lorentz force? This is something that we need to come to grips with. At the same time you are saying the we are saying that you know the fluid element is neutral, but it still current-carrying there is a J, right. So, these are issues that we need to sort of come to grips with and these are related to you know the curiosities with regard to the idea of current itself in magneto hydrodynamics, which we have talked about.

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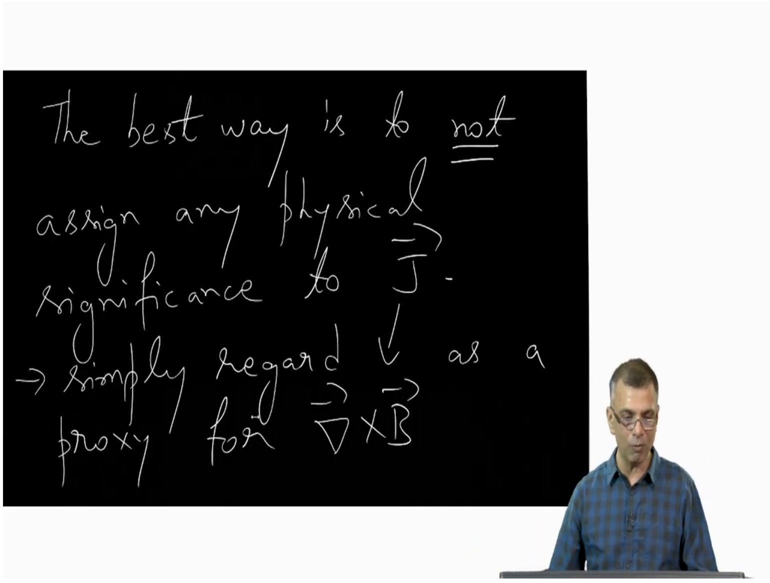

The best way out as we have seen as I have remarked earlier, the best way is to not assign any I we say physical significance to the current density, ok. Simply regard the current density, this as a proxy for the curl of B. As in the current density although I mean it has the dimensions of current density it is perfectly valid to think of it in terms of current density and so on so forth. But you know it is because of all these oddities that we encounter the fact that the fluid is uncharged you know.

And yet you need to think of it in terms of current density and so on and so forth, it is best to sort of think of it as a proxy or a as a short hand for curl of B. Wherever you see, wherever you see J, whenever you see J you plug in a curl of B there, ok. So, this is the thing.

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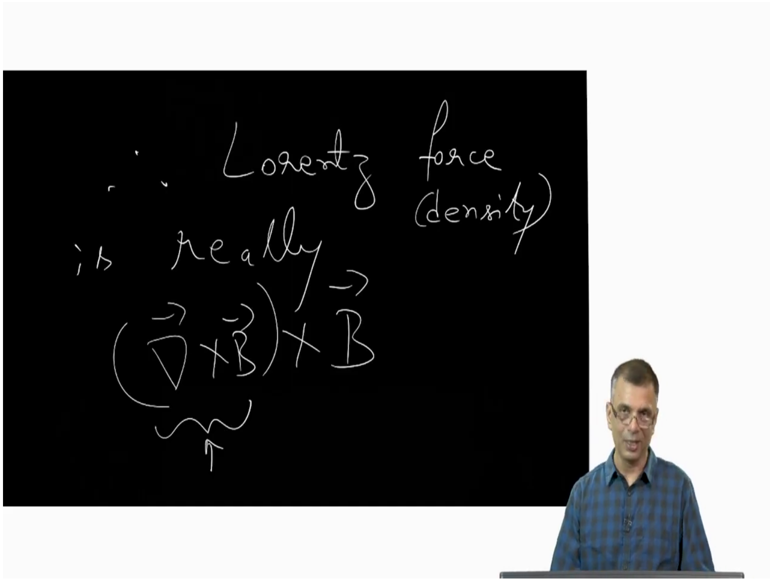

Therefore, the Lorentz force, the Lorentz force is really the Lorentz force density, Lorentz force density is really curl of B cross B this is the Lorentz force density, ok. So, again if, so if someone asks you what is a new thing what is a new element of force that appears in magneto hydrodynamics or rather what is the one new thing that appears in magneto hydrodynamics, as opposed to regular neutral hydrodynamics tell me one new thing.

Well, your answer should be well magneto hydrodynamics deals with a magnetized fluid, a magnetized and charged neutral fluid. A fluid which is somehow carrying a magnetic field along with it that, is it, ok. There is no scope for an electric field you know in the fluid because as we know the fluid is infinitely conducting and so, any presence of an electric field will be immediately shorted out, ok by the fact that you know that the fluid is highly conducting.

So, there is no scope for any electrical charges. But there is no scope for any electrical field as such. However, there is no such thing as shorting out a magnetic field because there are no free magnetic charges. Therefore, even in an infinitely conducting fluid a magnetic field can exist.

And the main thing about magneto hydrodynamics is that this is a manifestation of a magnetized fluid. So, point number one, there are magnetic fields and so, the new element of force and in the magnetic fields and what is more there are magnetic fields which can have a curl, ok. There can be a curl to the magnetic field, ok. And therefore, there is here is the new term that needs to be added to the momentum equation and that is curl of B cross B, ok.

So, here is the thing. And now you see this curl of B cross; and so, this is the Lorentz force, this is the Lorentz force density, curl of B cross. So, this is the new element that needs to be added to you can say J cross B, yes, but it is really curl of B cross B, ok.

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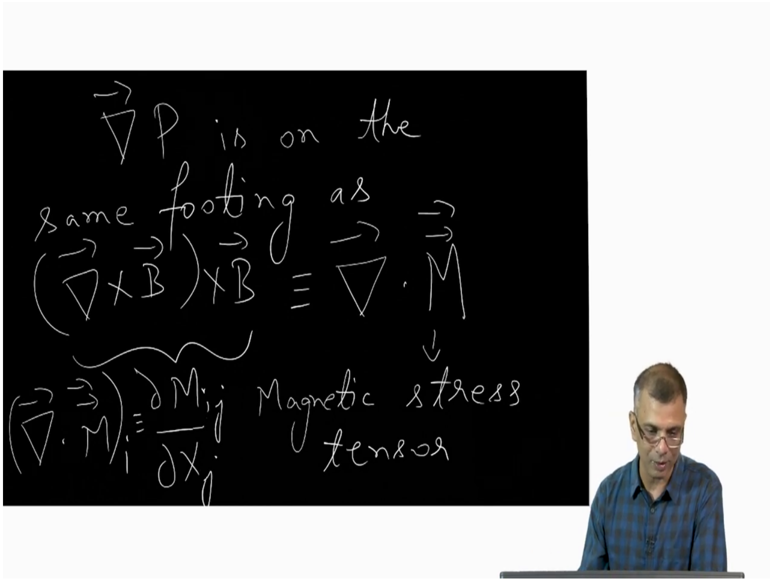

Now, you see in the momentum question, what are the other; what are the other terms? The other terms, well, apart from body force the other term in the Lorentz equation in the force equation are things like the gradient of a scalar pressure, ok. And this is on the same footing as the Lorentz force density because it appears in the I mean you know it is just one more term on the right hand side, on the F side of things and therefore, it as such it is the same as this, ok.

So, now, you can sink of this as the it is the dimension wise it has to be the same as a gradient of a scalar pressure. Finally, when you do this operation it is on the same footing, yes. So, you can think of this as the divergence of the magnetic stress tensor. So, this is the new tensor that we are introducing, this M, in order to understand what the B looks like. And what are the, I mean essentially what does the divergence of a tensor look like?

Well, the ith component of the divergence of this is just d M ij, dx j, where as we have we have discussed many times before you know the appearance of this repeated index simply means summation. So, the ith component of this divergence is simply, say let us be concrete.

Let us say this i is x, ok. So, this would be d M xx dx, ok where now my the i is x in any case. So, first of all I say j is equal to x, right, so it would be d M xx dx plus since I need to sum over j, I would have d M xy dy plus d M xz dz, ok. Let me write this down explicitly. So, that is what the divergence of this.

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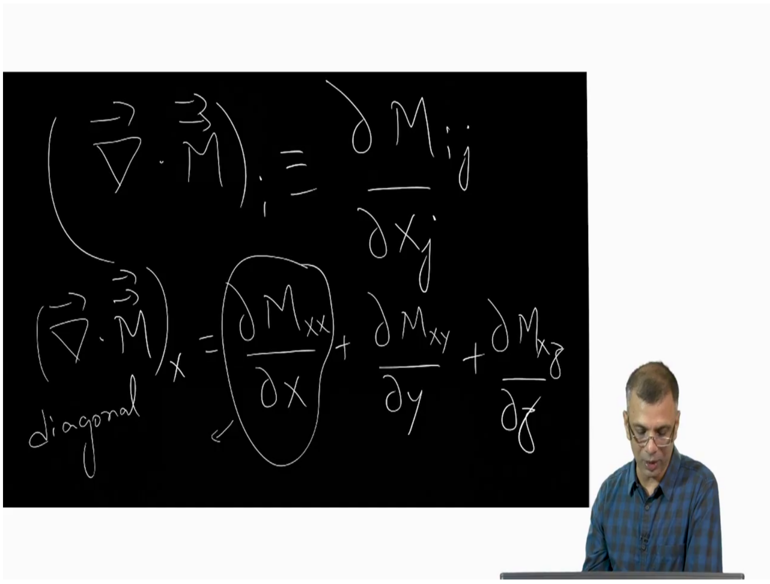

So, I will repeat this. I will repeat this here. So, sorry the ith component of this is where there because of the appearance of the j as a you know it is repeated the summation over j is implied. So, therefore, saying the x component of I of always seem to make this mistake.

I do not write the M properly for some reason it is just this pen I guess I do not know. So, the x component would be d M xx dx plus d M xy dy plus d M xz dz and the yth component would be well, it is just the second index that starts you know sorry. So, the yth component in the in that case the first index would now be y. So, you would have dy x dx plus dy y dy plus dy z d z that is how it goes. So, that is all, right. So, now, let us go back to our slides. And let us look at this.

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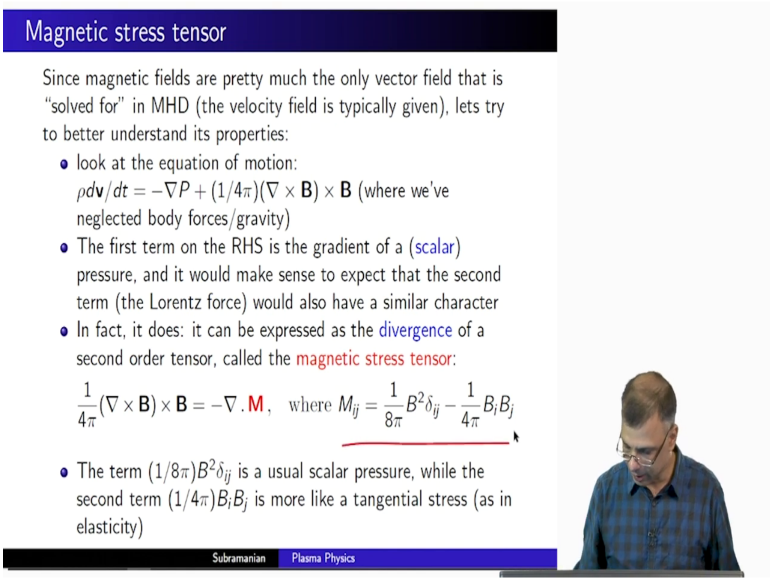

So, this is what I was saying. So, this is the magnetic stress tensor and where the ijth component of the stress would be written as this 1 8th B, 1 over 8 pi B square delta ij minus 1 over 4 pi B i B j. So, if you look at this, this would be the diagonal component; sorry this would be the diagonal component. This would be the diagonal term. So, terms like d M xx dx, d M yy dy, d M zz d z, these would be the diagonal components and those would be these, right.

And these would be the off diagonal components. So, and we know the diagonal components are much like the usual scalar pressure whereas, this is more like a tangential stress. And let us look at this a little more, ok. So, the point I wanted to motivate this a little more. The reason we are now talking about this magnetic stress tensor is to move away from assigning too much of a physical significance to the current density, I you know.

People do talk about j in MHD a lot, but really j does not have that much of a physical significance and really it is all about magnetic fields, ok. And the fact that you have a nonzero curl to the magnetic field and so, this curl cross B cross B makes the magnetic field behave in a certain way, ok.

And we are trying to understand and it make it behave like a pressure tensor and we are trying to understand what it means. That the reason we are starting to talk about the magnetic stress tensor, ok, right. So, yeah.

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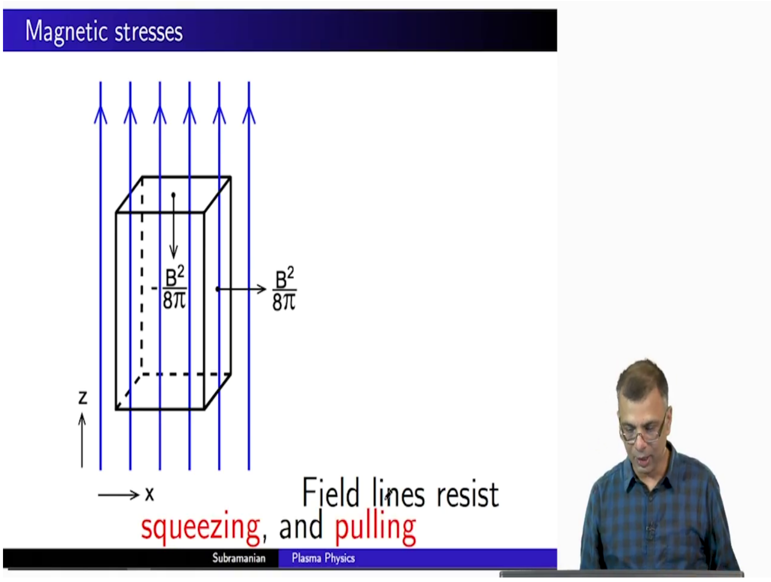

So, the tangential stresses is much like elasticity. And let us try to understand this. It is something like this, and we will come back to this diagram in a minute once again.

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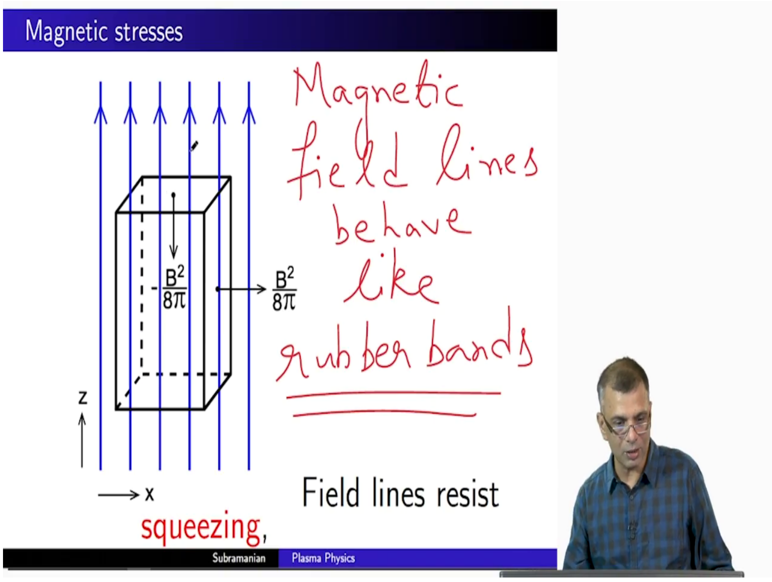

Essentially, the bottom line is that the consequence of this equation is that magnetic field lines behave much like a bunch of rubber bands. Magnetic field lines behave like rubber bands like a bunch of rubber bands. Suppose, you these were magnetic field lines, these blue lines were magnetic field lines.

And so, they were a bunch of rubber bands. Now, what do you expect from a bunch of rubber bands? Right. You try to pull these rubber bands, you try to pull them along the same direction and you would expect a tension along the rubber bands, right. It would they would resist pulling you try to lengthen them, and they would make they would try to resist the act of lengthening them and so, they would try to contract. So, that is number one.

So, that would be, so this tendency that we just described would be manifest in this, ok. The other thing is that a bunch of rubber bands they would resist squeezing, ok. You take bunch of rubber bands, and you try to squeeze them you try to squeeze them in the perpendicular direction they will resist squeezing. They will not like to be squeezed, ok. They will have a pressure that resists this squeezing. So, turns out that the character of this curl B cross B is much like that.

So, they resist squeezing and pulling. So, magnetic field lines behave much like rubber bands and we will see how.

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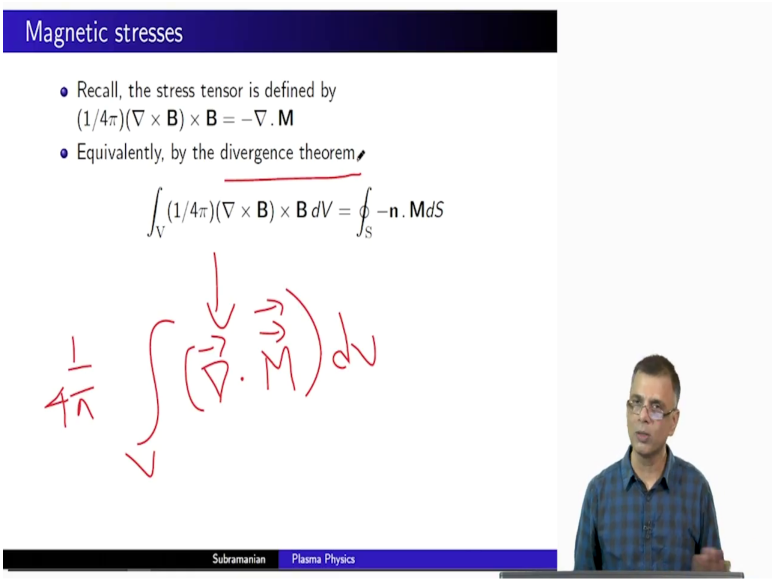

Yeah. So, recall the magnetic stress tensor is defined by this, yeah, where the fact that I have you know denoted M in bold phase means that you know M is a tensor. Now, what we do now is by the divergence theorem what we have done here is we have you know this is essentially the volume integral of well there is 1 over 4 pi outside, but there is the volume integral of the divergence of that is this. And we use you know the usual divergence theorem to make it look like this, yeah.

So, we want to transform the volume integral into a surface integral like this. And so, n would be the outward normal of the surface, ok. So, you would have an n dot M d S, yeah. So, we are using the divergence theorem here except the divergence theorem for a tensor. Same things, exactly the same concept, no difference, ok, right.

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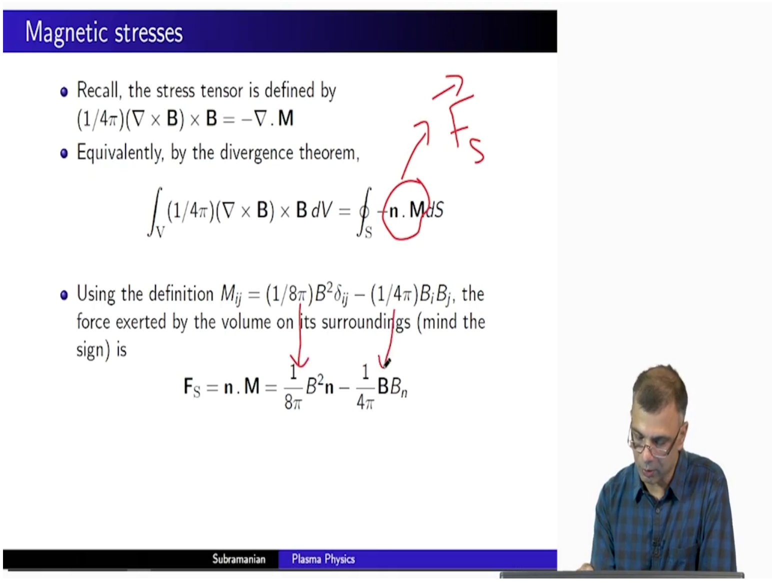

So, now using the definition M ij. So, this is the definition the ijth component of M is this, yeah. So, I use this definition in here. Mind the sign is very important, the negative sign is very important. So, now, what happens is this n dot M is essentially a force, is not it. That is what this is saying.

And what is the force? The force is n dot M which is B square, 1 over 8 pi B square because there is a dotting the this term is in the direction of whatever n there is whether the outward directed normal and this fellow becomes you know B n, ok. So, this comes from here and this comes from here.

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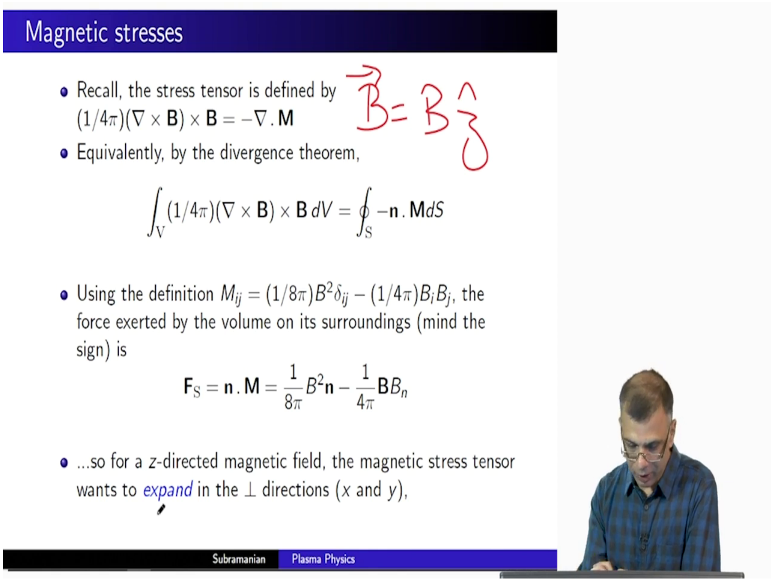

So, now let us say that B like that, a z hat I mean a magnetic field that is exclusively in the z-direction, ok. And let us consider a magnetic field that is exclusively in the z-direction which means that the n will be in the x and y directions, is not it.

So, for a z directed magnetic field what is happening? How is this force directed? That is the question we are asking. So, this is exclusively in the z-direction, right. So, so this B is just exclusively in the z-direction and that has a negative sign. You see this, right.

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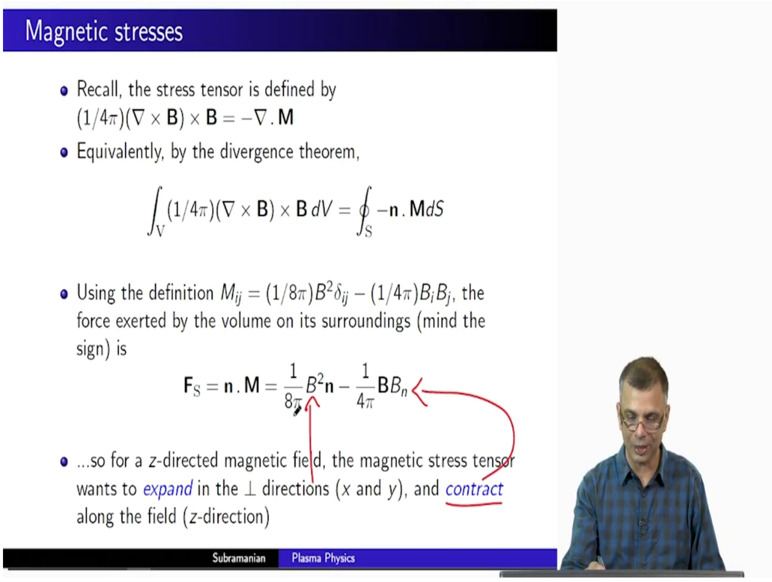

So, therefore, for a z directed magnetic field never mind this, because this B is exclusively in the z-direction and there is a negative sign the there is a force along the negative z-direction. So, this says that the force wants to contract the rubber bands, contract the magnetic fields in the z-direction. In other words, that there is a force along the negative z-direction and this comes from this; whereas, and the contraction the adjective contract is because of this negative sign that is evident from here, is not it.

On the other hand, here there is a positive sign. And what direction is this it is 1 over 8 pi B squared n? In the direction of n hat which is anything perpendicular to the z-direction which is along the x and y direction, ok. So, it wants to expand in the x and y directions, ok.

So, it is very much like a bunch of rubber bands which likes to have a you know a minus z directed force, in other words it tends to want to contract along the z-direction as evident from here and it wants to expand in the x and y directions as evident from here.

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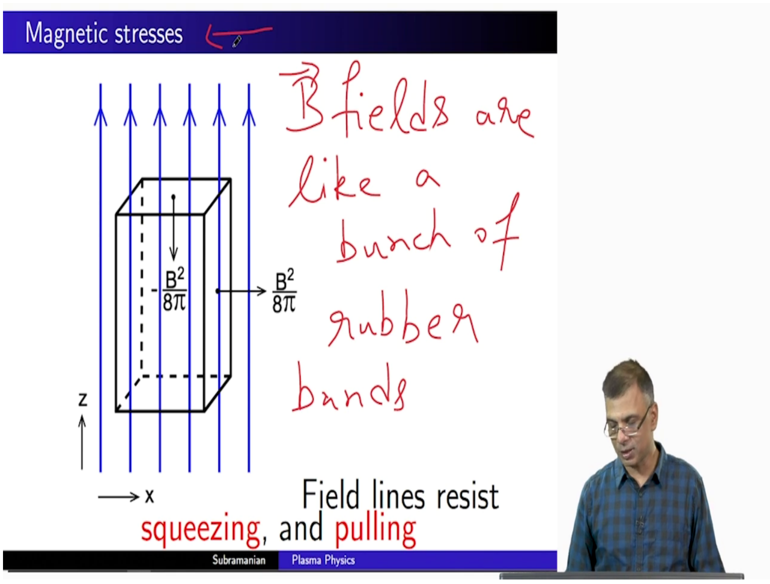

So, it is very much like this; it is very much. This is why this justifies the fact that B fields are like a bunch of rubber bands. Just like a bunch of rubber bands do not want to be pulled. If you try to pull them they will try to contract and that is the minus z directed force that you saw here, that you saw here.

And just like a bunch of rubber bands they do not like to be squeezed. If you try to squeeze them they will try to expand, and that expansion force is in the, is in the directions that is perpendicular to z, right, in other words in the x and y directions. Just like here, just like here, ok.

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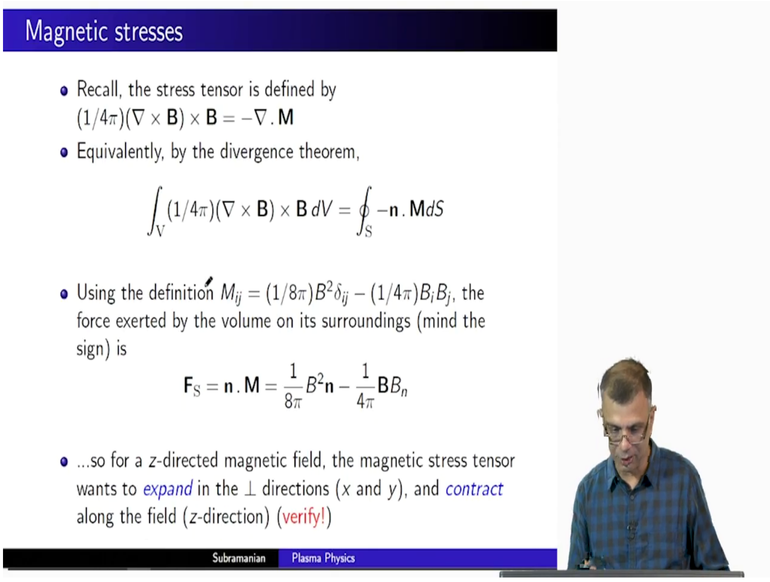

So, this is the nature of magnetic stresses which is a direct consequence of the fact that there is this additional term in the momentum equation which is the Lorentz force term, ok. So, this is I thought I would spend a little bit of time explaining this before going ahead because it is central to much of the applications in astrophysics that we will we will come across. So, this is about magnetic stresses, yeah.

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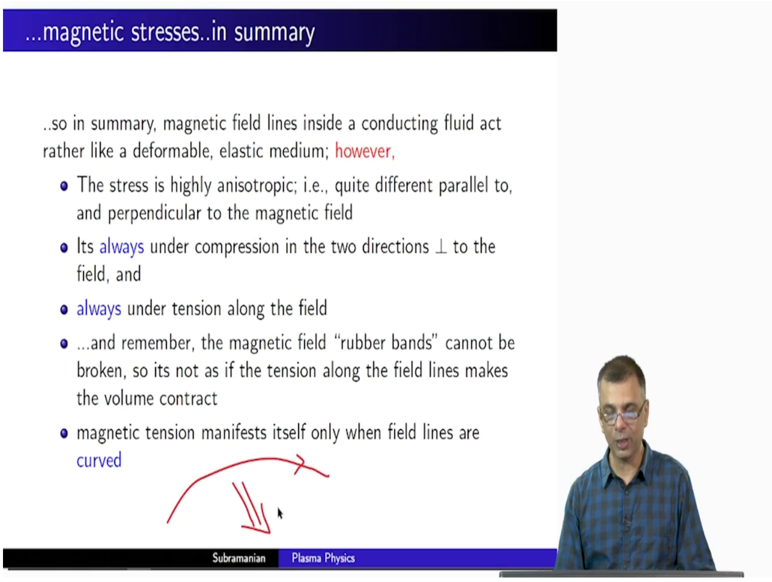

So, in summary, magnetic field lines inside a conducting fluid act rather like a deformable elastic medium. However, the stress is highly an isotropic. It is you know normally an elastic medium you know elasticity you normally do not think of elasticity as an isotropic thing, ok. Elasticity, the coefficient of elasticity or whatever it is the same in all directions. However, in this case it is highly anisotropic, ok. This stress is very different parallel to and perpendicular to the magnetic field as we saw.

Parallel to the magnetic field you know the stress tends to contract the magnetic field perpendicular to the magnetic field it tends to make the magnetic field expand, ok. So, it is highly anisotropic.

It is always under compression in the 2 directions perpendicular to the field. It is always under compression and is always under tension along the field, along the field in other words along the z-direction in this particular case. The example we took was the that the B you know was oriented along the z-direction.

Along the z-direction it is always under tension, ok, right. So, yeah. So, here is the other thing. The analogy of with rubber bands is good, it is all, it is all good. But, however, if you should not carry this analogy too far because rubber bands can be broken, you can cut rubber bands whereas, you cannot cut magnetic fields. Magnetic fields are bound to always be in loops. So, the magnetic field rubber bands cannot be broken. So, it is not as if the tension along the fields make the volume contract, no.

The volume as such remains the same, all right. The other thing is the magnetic tension really manifests itself only when field lines are curved. In other words, for straight field lines there is really no magnetic tension its only when you have curved field lines like this, ok. The tension along the field lines manifest itself. And what is more, it manifests itself and in the following way you get the equivalent of centrifugal force or is it centripetal, I think it is centripetal force, ok.

So, magnetic field lines tension manifests itself only when the you try to curve the field lines, it behaves like an elastic rubber band, it resist the curving, ok. It wants to snap back. It wants to become straight again when you try to curve it. So, we talk to little bit about magnetic tension.

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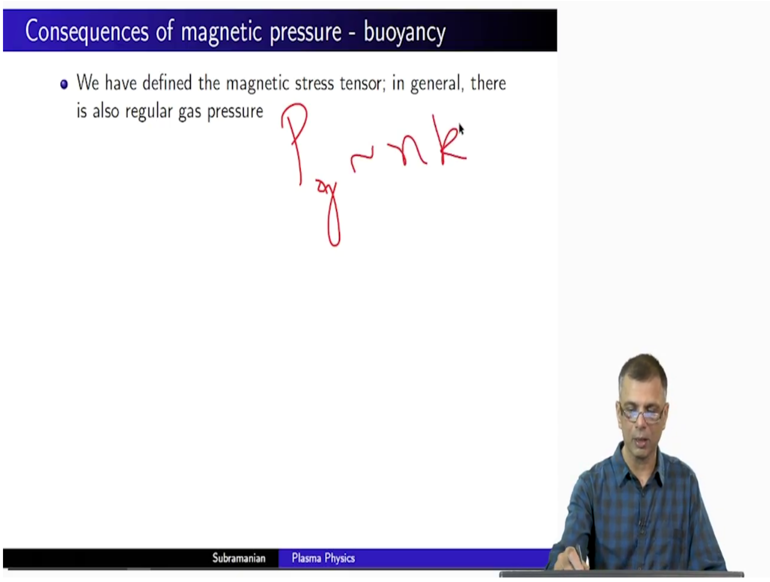

There is another interesting thing consequence of magnetic pressure and that is buoyancy. And this is very important in astrophysical situation, the concept of magnetic buoyancy. The fact that a magnetized fluid exhibits something called buoyancy. It is like this.

We have defined the magnetic stress tensor. In general there is also regular gas pressure, is not it. There is also regular gas pressure P gas, right. So, in addition to magnetic pressure 1 over 8 pi B square there is also regular gas pressure which is something like you know n k T, yeah.

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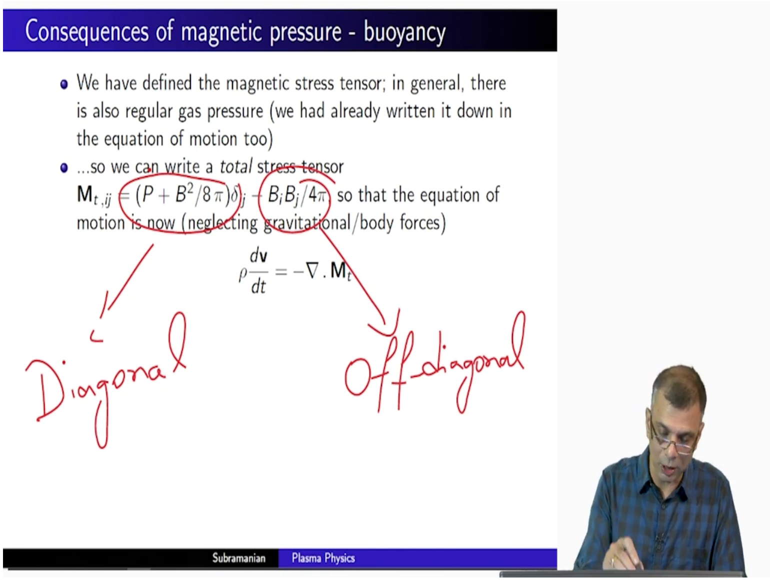

So, and we are already written down in the written it down in the equation of motion anyway as a gradient of P, yeah. So, now we can write a total stress tensor as this is the usual scalar gas pressure B and B squared over 8 pi delta ij also look like a gas pressure, regular gas pressure because these are also totally diagonal. So, you might as well club them both and write them, right here, ok. And these are the off diagonal terms. So, these would be the diagonal terms and these would be off diagonal terms, right.

So, now let us for a minute say body forces are not important. So, neglecting those you can write down the equation of motion as rho dv dt equals minus divergence of a total stress tensor which includes gas pressure as well as magnetic pressures; M t, right, where the M t is this.

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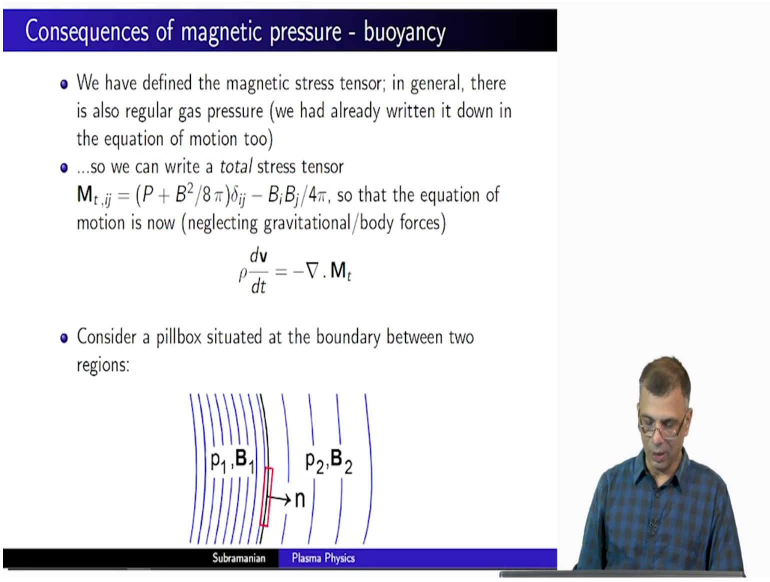

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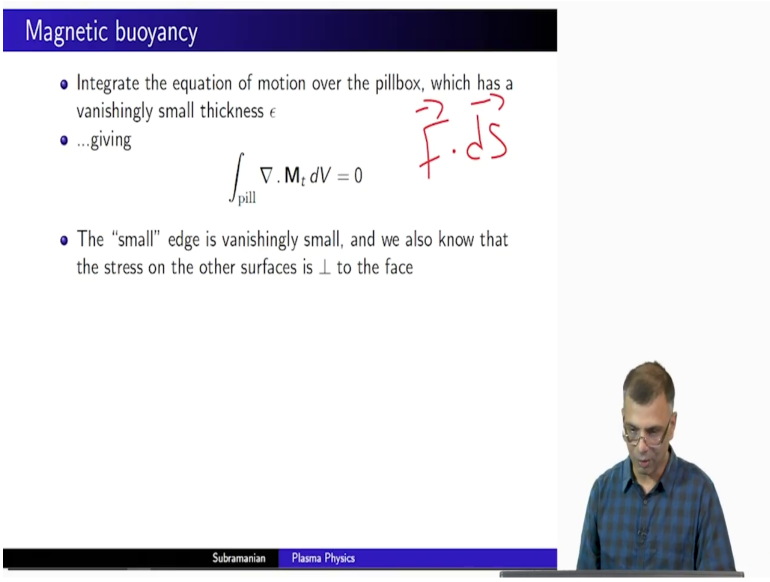

Now, essentially what we do is you consider region with 2 kinds of you know with a pressure p 1 with a gas pressure p 1 and a magnetic field B 1 on one side and a gas pressure p 2 and a magnetic field B 2 on the other side. And you consider this little pill box here, on the boundary with an outwardly directed normal like this, yeah. And so integrate the equation of motion over the little pillbox which has a vanishingly small thickness, ok.

And if you integrate because of the fact that you have a vanishingly small thickness you integrate this, yeah, which is essentially F dot d S by the divergence theorem, we integrate this over the pill box and that has to be 0. Because small edge is vanishingly small because you know this edge which is the small edge is vanishingly small and as far as this edge is concerned, the stress on the other surfaces is perpendicular to the phase.

So, therefore, the F dot, right; so, because the stress is perpendicular to the phase F dot d S has to be has to be 0, right. So, that is why this is 0.

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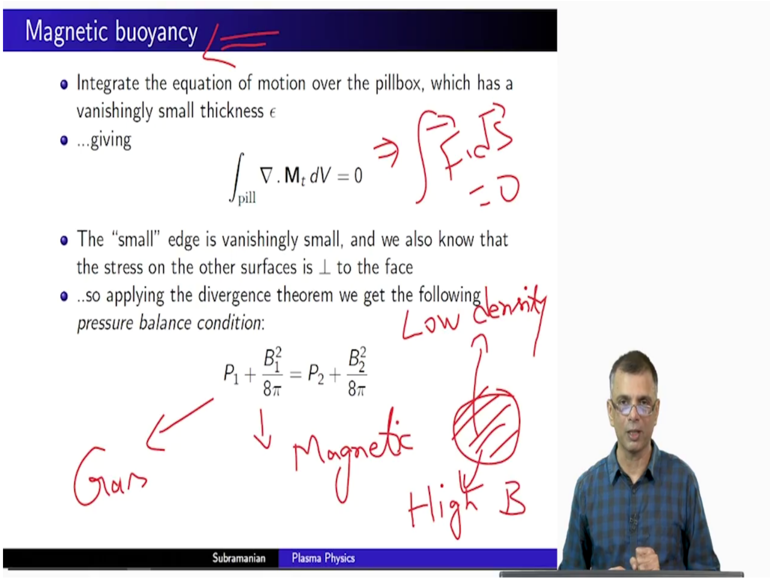

So, therefore, applying the divergence theorem which is which is essentially, essentially this is equal to saying that F dot d S equals 0 applying this. We get the pressure the so called pressure balance condition which is that this is the gas pressure and this is the magnetic pressure.

So, the sum of the gas and magnetic pressures on both sides has to be the same. So, if you have highly magnetized, if you have a volume; if for instance you have you know a bundle of magnetic field lines like this you have a volume where you have high in here you have high magnetic fields and outside you have low magnetic fields, ok. Because the gas pressure plus the magnetic pressure has to be the same inside and outside, ok.

Outside you have no magnetic field, so you have no magnetic pressure therefore, the gas pressure has to be higher. Inside you have high magnetic field fields therefore, you have to have low gas pressure, ok. What is one way of having low gas pressure? Having low density, ok. So, high B also means low density, ok.

So, in this volume which has somehow I say it is a flux tube, ok a flux tube with high magnetic fields; in here because of the high magnetic fields you have low density, low gas density. And what does element with low density do? It is acted upon by buoyant forces, right.

It floats to the surface because of buoyancy and that is what magnetic buoyancy is all about, ok. This flux tube is buoyant because you have high magnetic fields you have low density, low gas density, and because this low gas density this bundle of magnetic fields floats upwards. It is subject to buoyant forces, ok. And so, this is the central idea behind magnetic buoyancy. So, we will stop here for the time being.

Thank you.