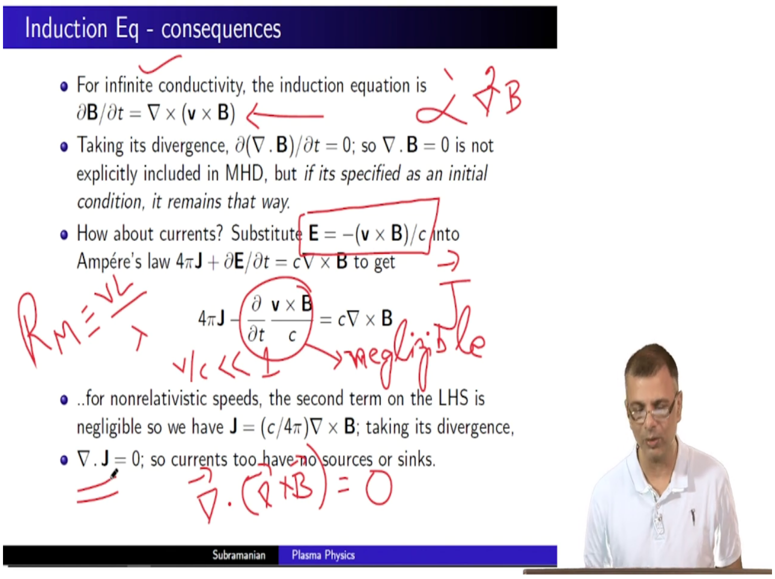
Fluid Dynamics for Astrophysics
Prof. Prasad Subramanian
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture - 49
Magnetohydrodynamics [MHD]: Currents in MHD, momentum equation and magnetic stress tensor

(Refer Slide Time: 00:16)



So, we will pick up from where we left off yesterday. We did a fair bit yesterday during the last session. In particular, we talked about this remarkable thing called the induction equation which you will see very often in MHD. If you remember, this induct, this form of the induction equation is valid only for infinite conductivity right.

If you have finite conductivity you will have an addition, an additional term which is proportional to del square B right you know, you will have an additional term here and the proportionality constant contains the conductivity right.

So, this term, the diffusive term assumes importance only when the conductivity is finite and not infinite ok and we also talked a little bit about the magnetic Reynolds number R sub M ok and the magnetic Reynolds number is essentially the ratio of it looks very much like the fluid Reynolds number except here, you have you know the appearance of the resistivity ok it is something like the resistivity appears in the bottom.

So, for an infinitely conducting plasma, the Reynolds number, the magnetic Reynolds number is you know formally infinite ok. So, whereas you know for plasmas with large with small conductivity rather large resistivity, the magnetic Reynolds number is small and in astrophysical situations typically, the Reynolds number is very high ok. So, we discussed all that, that was a very quick recap.

Now, what we will do and the we stopped last time was in starting to discuss some curious aspects about current density J in magnetohydrodynamics, in ideal magnetohydrodynamics for which the conductivity is assumed to be infinite. In ideal MHD, the conductivity is infinite so that the induction equation is just this with no additional terms ok.

So, now, you recall how we derive this E equals so, this is the electric field in the lab observers frame right. In the fluid frame, there is no electric field an if the plasma is infinitely conducting, the fluid frame by definition there cannot be an electric field. However, in the lab observers frame and it is only in the lab observers frame that this makes sense because you have the appearance of a v, it is only for the lab observer you have a finite velocity for the plasma right.

When you are sitting inside the plasma, there is no v right. So, it should be obvious to you that this refers only to the lab observer. So, in the lab observers frame, there is a finite electric field, it is another matter that since the v is non-relativistic, in other words, v over c is very

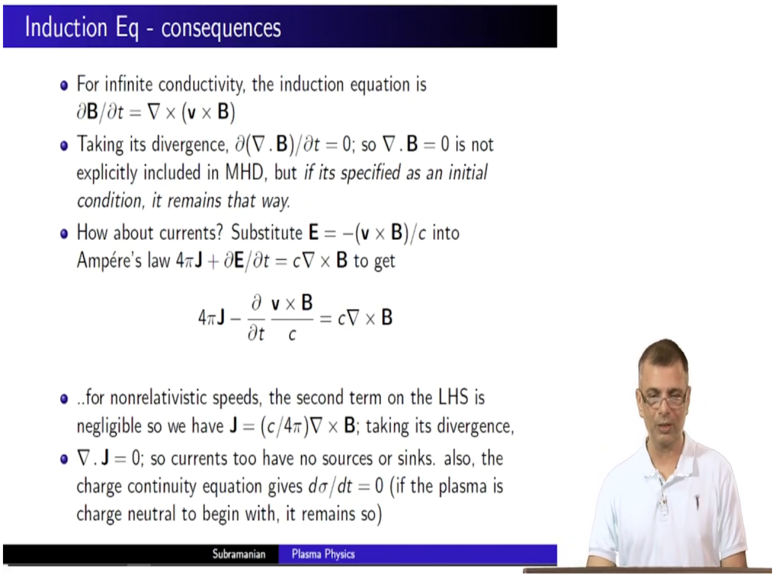
very small, the E, the magnitude of the E will also be small, but nonetheless it is small, but finite ok.

So, now, what we do is we substitute this in Ampere's law to get this kind of an equation ok and for non-relativistic speeds, when v over c is much much less than 1, this term is negligible right. So, we do not consider this term and so, you simply have J equals c times curl of B right.

And if you drawn analogy with the electrodynamics that you already know, you will immediately recognize that this is just the displacement current term ok, the regular current is not there at all ok, the non-displacement current part of it this is not there at all. Now, what about it?

You take the divergence of both sides, since the divergence of a curl is always 0 right so, this is always 0, it follows that the divergence of the current is equal to 0. So, the currents have no sources or sinks. This is something that we need to keep in mind very very clearly because this is one of the; one of the curious aspects of magnetohydrodynamics we will see why right.

(Refer Slide Time: 05:10)



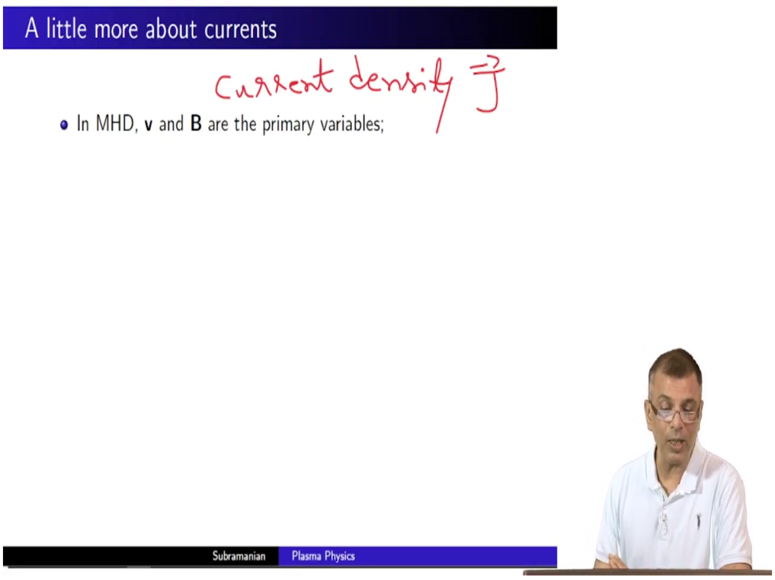
So, the divergence of J is equal to 0. So, the currents too have no sources or sinks, and the other thing is the charge continuity equation which essentially said that you know the divergence of J plus d sigma dt was equal to 0's, its since the divergence of J is equal to 0, d sigma dt is also equal to 0 right.

So, in other words, if the plasma is charged neutral to begin with, sigma is a net charge density you remember right. So, if the plasma is charged neutral to begin with, it remains that way for all time ok and you remember we talked about one of the basic assumptions of MHD and that has to do with the fact that the plasma although it is comprise of charge particles on large scales, it is charged neutral and it is only on this these large scales that where our theory is valid.

So, we are really talking about a bit of a curious beast here, you know a charged, but nonetheless charge neutral fluid which comprises of charge particles, but nonetheless on the whole is charged neutral and it carries a current density, very very strange, but this is a these are things that you will need to get used to.

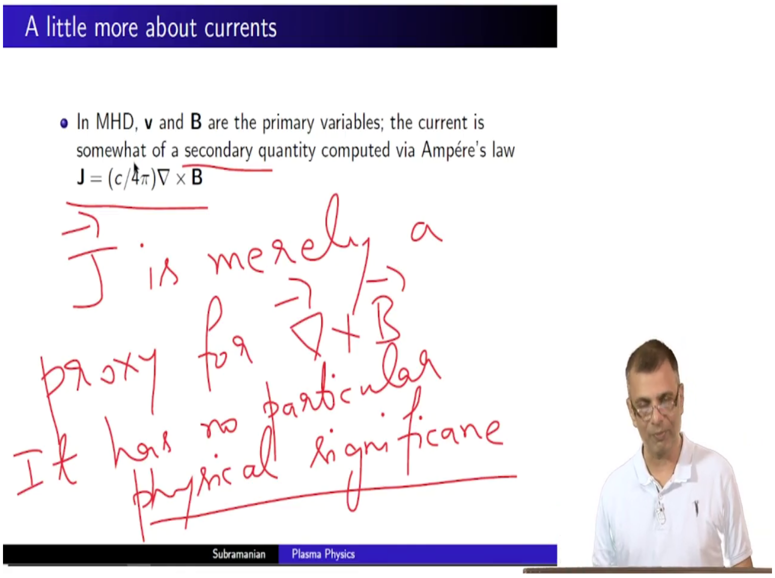
And what is this the fact that divergence of J is equal to 0, what is that telling you? Is telling you that if the plasma is charged neutral from the you know at an early time, it remains that way MHD ensures that it will never acquire any charge ok, it is somewhat like this divergence of B equal 0 is not guaranteed in MHD ok, what is guaranteed is simply that the time derivative of the divergence should B is 0. So, if it is, if you ensure the divergence of B equal 0 at the very beginning, then it remains that way, somewhat like that ok.

(Refer Slide Time: 07:04)



So, now let us talk a little bit more about currents because currents are in particular current density actually, I should say current density because it is bit of a strange thing or usual you know ideas about currents from electrical circuits do not quite apply here. So, let us spend a little bit of time trying to understand what the status of currents in MHD ok. In MHD, v and B are the primary variables ok.

(Refer Slide Time: 07:41)

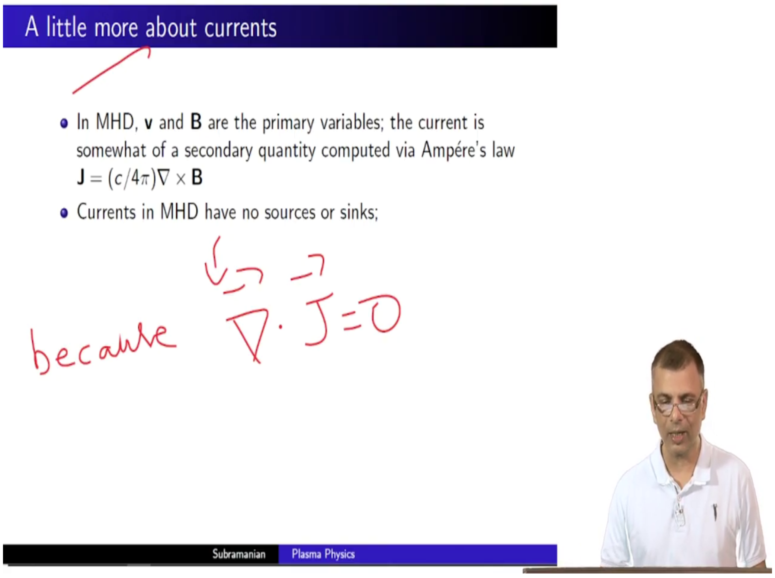


The current is somewhat of a secondary quantity ok computed via Ampere’s law. So, in other words, I would go so far as to say that J is merely proxy for curl B. It has I would even go so far as to say that it has no particular physical significance; physical significance ok. I insist on this, we will see why ok?

So, therefore, I would whenever someone asks you what is the actual meaning of current density you just point them to this equation, there is of course, in CGS ok. In CGS, you have the appearance of the c over 4 pi. In SI or MKS, this constant will be slightly different, but nonetheless its essentially curl of B. So, if someone asks you, what is the current density?

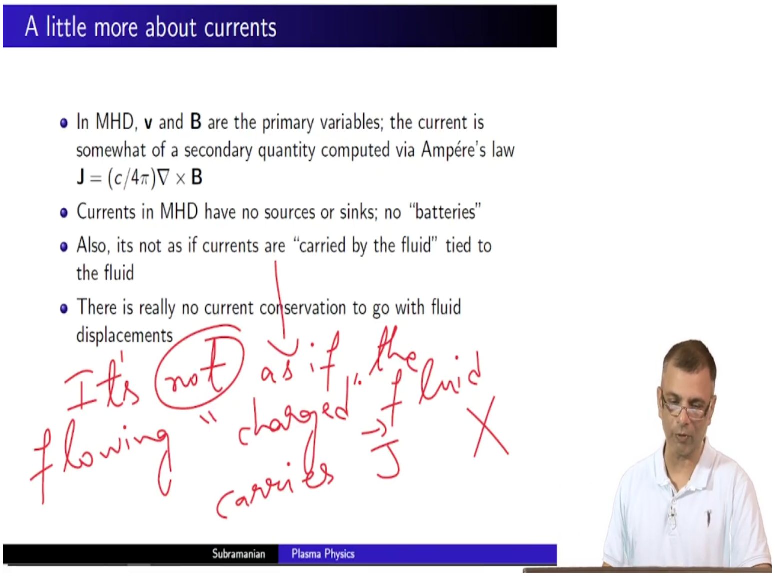
What is the meaning of the current density in MHD? You say that it is essentially do not worry about it, it is essentially telling you something about the curl of B that is it ok. Why? Because we will see this.

(Refer Slide Time: 09:15)



Why do I say that it has no real physical significance? First of all, we established in the last slide that currents in MHD have no sources or sinks right because this is because the divergence of J is equal to 0 right as we showed in the last slide divergence of J is equal to 0 and so, that means that there are no sources or sinks for current density in MHD right. So, this is a strange thing.

(Refer Slide Time: 09:53)

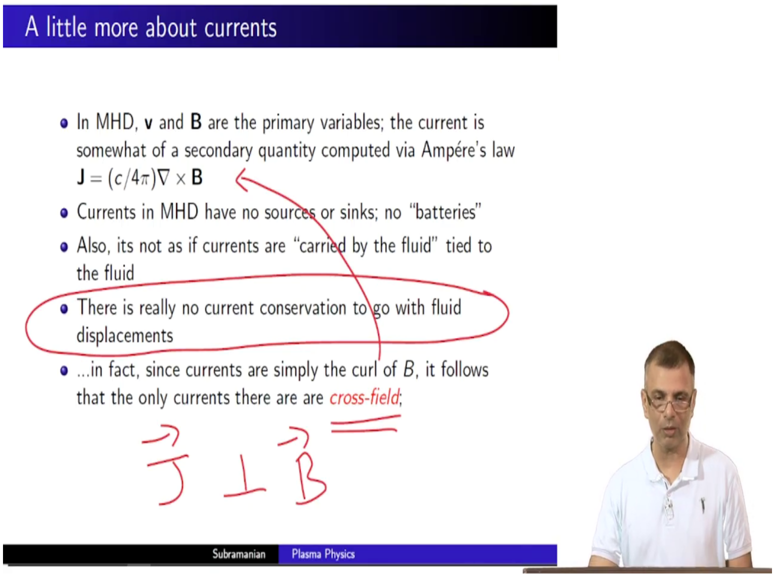


And in other words, there are no batteries ok. A battery is a source and a sink of current is not it? The positive terminal of the battery is a source of current and the negative terminal of the battery is a sink of current, there are no sources of sinks in MHD, there are no; there are no batteries very important. Normally or our notion of current in flowing in wires is precisely this you always think of a battery or a generator which generates the current in the other side you know taking the current back, but there are no such things in MHD.

Also, it is not as if currents are carried by the fluid as in it is tied to the fluid. One I mean you know this is the reason I am emphasizing this so much is because these are common misconceptions. You have a charge fluid; you have a fluid comprising charges and its flowing and it is natural to think that the flow of this charges comprises the current that is not so ok.

It is not as if; it is not as if the flowing charged well. First of all, the fluid is not charged as we said over large length scales, the fluid is electrically neutral, but since the fluid comprises of charged particles, many people think of it like this fluid carries a current, it is not so, this statement is not correct ok or as in the it is not as if the currents are tied to the fluid.

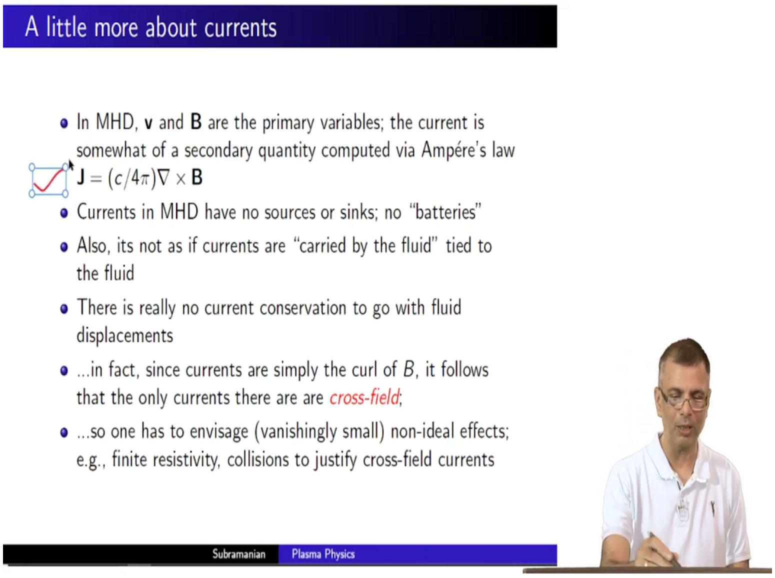
(Refer Slide Time: 11:45)



There is really no current conservation to go at fluid displacement, this is also very very important. There is charge conservation, but there is no; but there is no current conservation ok very important. In fact, since currents are simply the curl of B, you see here apart from a constant, they are just the curl of B, it follows at the only currents there are cross field ok because by definition, the curl of B is perpendicular to B.

So, the direction of the current density is actually perpendicular to B, the cross field meaning cross they are perpendicular to the magnetic field B ok. This is also something that bears you know looking at an and you know you should let this sink in a little bit.

(Refer Slide Time: 12:42)



So, one has to really envisage very vanishingly small non-ideal. If you insist on a physical picture for the current; for the current which I really would not pay much attention to I would simply say the current really has no explicit physical you know manifestation in magnetohydrodynamics, one has to stop thinking in terms of currents being carried by wires with a source and a sink that is not how it is in magnetohydrodynamics, the meaning of the current is just the current is simply a proxy for the curl of B.

But if you insist on a physical picture for the origin of current, you have to envisage vanishingly small non-ideal effects. We are talking about ideal MHD, but still if you insist on a physical picture, then you have to envisage non-ideal effects which are of course, vanishingly small because we are talking about ideal MHD.

For instance, what are the non-ideal effects we are talking about? Finite resistivity or collisions. Remember in MHD, the resistivity is technically 0 ok. So, the so, we have to envisage finite resistivity or collisions in order to justify these kinds of cross field currents ok, a some somewhat like this. So, what this means is that think of a magnetic field right and think of you know the orbits of protons.

And you know charged particles remember, there are no charged particles in MHD; there are no particles in MHD, but still since we are trying to ascribe a physical meaning to the current, let us for the for a moment think about protons and electrons, gyrating in opposite directions right, they obviously, I mean they will both gyrated around the magnetic field, but in the opposite directions right ok.

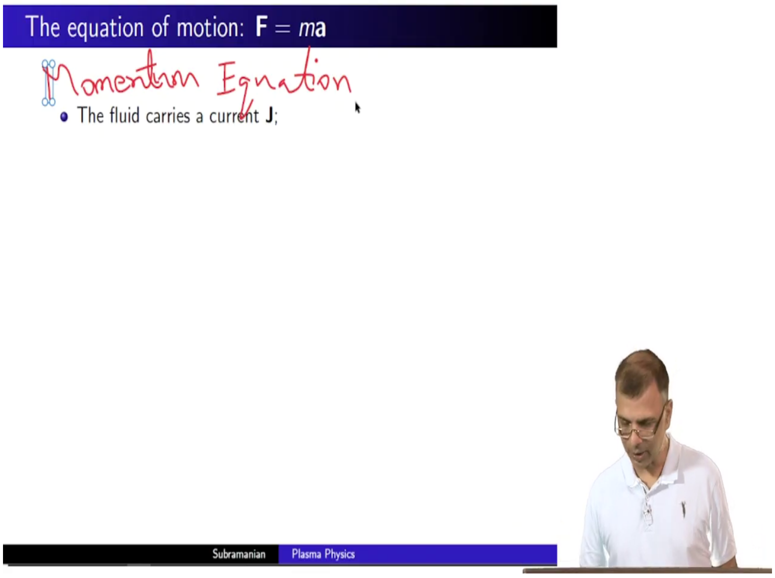
And now what is happening is there are; there are little collisions with each other which make the these you know gyro motions slightly displaced ok so, it is what happens is the proton motion does not exactly cancel out the electron motion. So, therefore, there is a slight charge displacement and that gives rise to the small current ok and since the gyration is perpendicular to the magnetic field.

This current that we are talking about, this small current which arises due to finite resistivity of collisions really which arises due to collisions which in turn give rise to resistivity, which I mean so, this small charge imbalances which arise due to collisions, they give rise to slight charge separations in a plane that is perpendicular to the magnetic field and that results in these cross-field currents.

But really, this is a lord hand waving frankly and I personally would I mean some people like to resort to physical pictures which is why I told you this, but I personally would not pay you know too much attention to these things, I would simply say that the current density is this J is really no point in getting in trying to ascribe to physical a meaning to the current density, I simply say that the current density is just this equation. So, it is simply a proxy for curl of B that is it and there is no need to discuss anymore ok alright.

So, these are some of the strange things about current density of currents in MHD. So, it is worth, I thought it was worth spending some time talking about it ok.

(Refer Slide Time: 16:13)

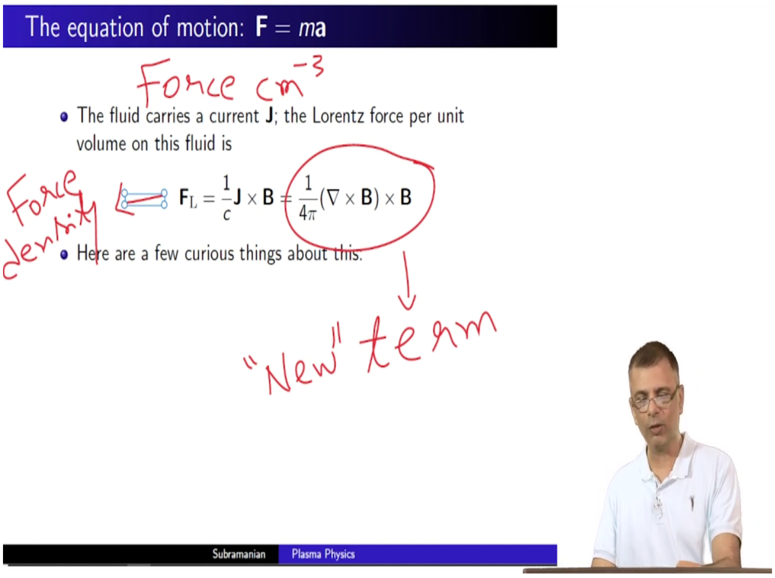


So, let us now get on to the momentum equation, the equation of motion. As we know, this is the momentum equation. We have spent a fair amount of time talking about the momentum equation in fluid dynamics. You remember the Euler equation and then, Navier-Stokes equation all of them are the momentum equation.

Now, how is it altered that is the whole point, why are we talking about MHD? Because in a magnetized fluid, you know the fact that there is a magnetic field you know in the fluid means that you know there will be additional terms to the equation of motion and what are what is that?

You already have a fair idea of what it would be, it has to be something like the Lorentz force and that is exactly what it is we will see what. So, the fluid carries a current J, mind you what it simply means is that the fluid has a curl of B ok because a current J is simply the curl B ok.

(Refer Slide Time: 17:15)

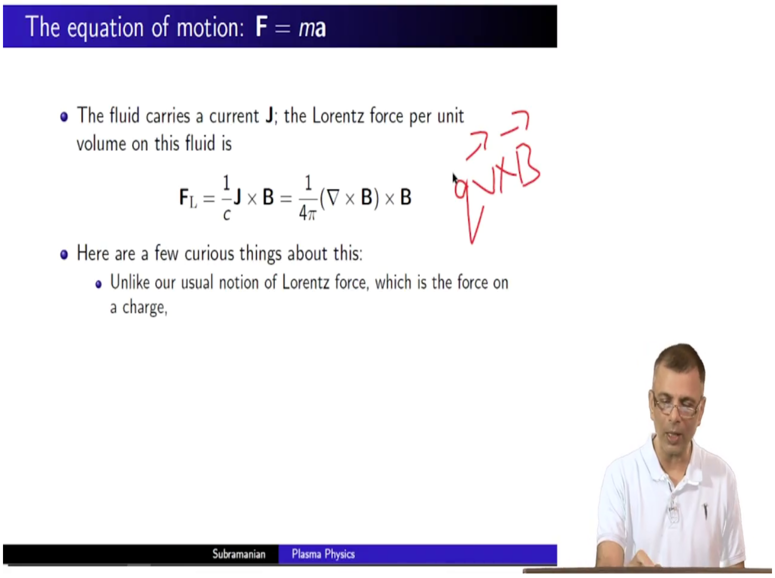


So, and the Lorentz force per unit volume on this fluid is simply J cross B ok. So, this is the force density; force density in other words, force per centimeter cube and that is those are the dimensions of this and so, that is simply J cross B, a this simply arises from q v cross b the kind of thing that you are already familiar with and so, qv would be the J, but since there are no discrete charges in MHD, I would not like to you know appeal to discrete charges, it is simply J cross B right and we know that J is just curl of B and so that is what it is.

So, this is the Lorentz force per unit volume on this fluid. So, this is the force that needs to be added to the momentum equation to the F equals MA. We already know the way to write down MA, we already know the way to write down different kinds of F right, the pressure gradient, the body force and all that and this is a new piece, this is the new term that needs to be added ok.

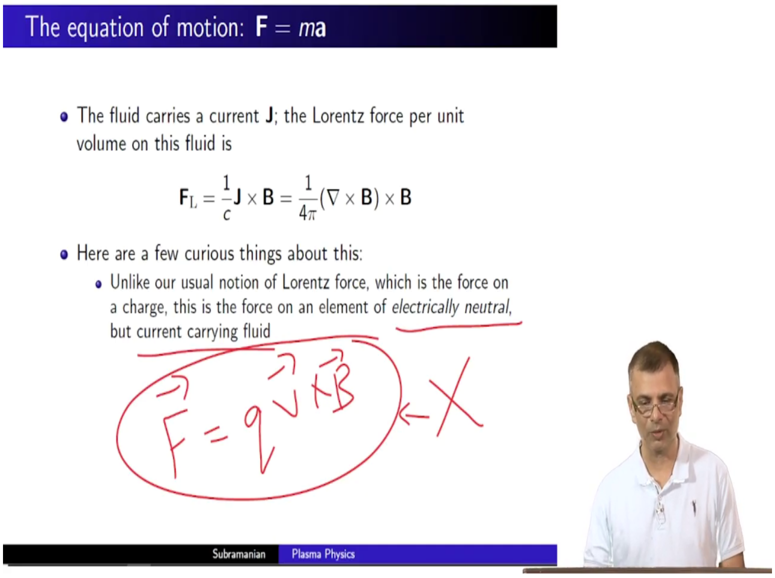
Now, here are few curious things. Again, curious things MHD is all about I mean the point is you know many of our intuitive ideas do not quite hold in MHD and so, it is important to pause and look at these things, there are few curious things about this force.

(Refer Slide Time: 19:01)



Unlike our usual notion of Lorentz force which is force on a charge right qv cross B, this is how we normally think of this Lorentz force right, it is a force on a charge q.

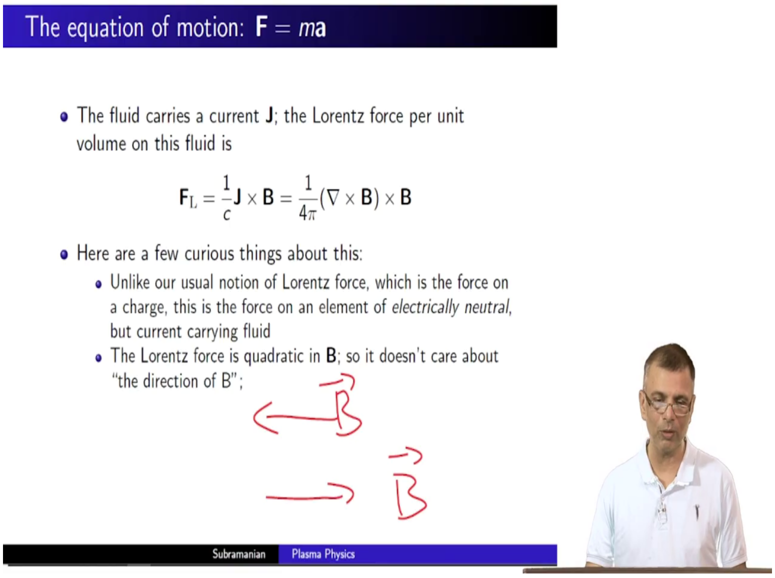
(Refer Slide Time: 19:20)



However, what is this? This fellow is really this is the force on an element of electrically neutral, there are no charges, the element of fluid is actually electrically neutral, but it still carries currents very important. So, the curious thing about this Lorentz force is that unlike our usual notion of Lorentz force, which is force on a charge, you know we normally write the Lorentz force as F equals qv cross B.

This is not how; this is not how we should think of the Lorentz force in MHD. Why? Because there are no charges at all. This is actually the force on an element of electrically neutral, there are no charges, but the fluent is electrically neutral, but nonetheless it still carries current ok. So, these are a couple of curious things about this Lorentz force and so, this is important to you know keep in mind.

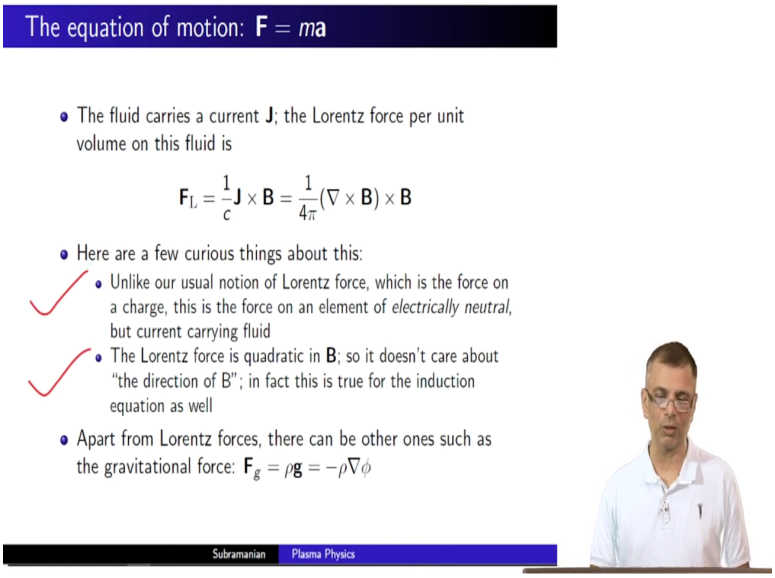
(Refer Slide Time: 20:29)



The other thing is that the Lorentz forces quadratic in B, it goes as B squared you see is B; B here and B here. So, it goes as B squared. In other words, what and so, what is the implication of that? What about it? It does not care about the direction of B. Whether B is; whether B is this way or this way, it does not matter, the Lorentz force will still be in the same direction, this is another strange thing ok.

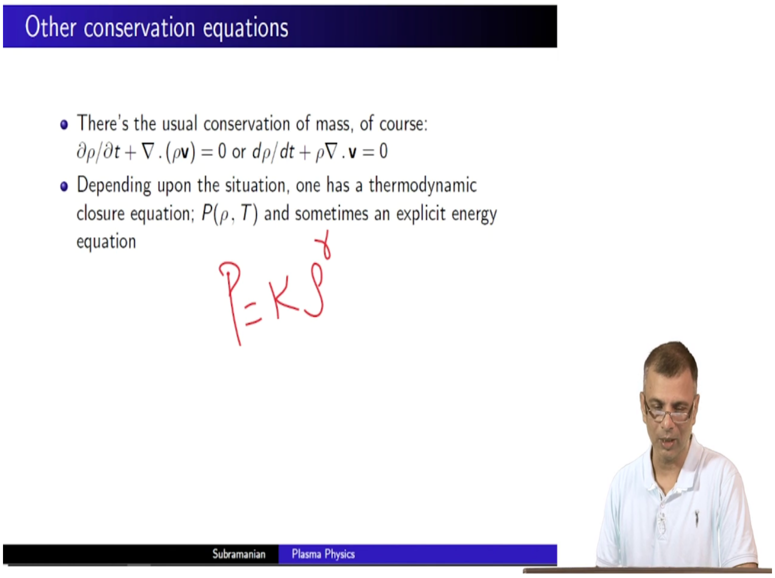
So, important to keep this; keep these things in mind and it is true that it is that this is the only additional term that appears in the momentum equation so, it is almost as if we can you know take this term and dump it in into the momentum equation that we already, we are already you know so familiar with and we can go ahead yes, but it is important to understand the features of this term ok and that is what we are doing now.

(Refer Slide Time: 21:26)



And in fact, this is true for the induction equation as well. The induction equation that we wrote that was also quadratic in B ok alright and of course, as we know apart from Lorentz forces, there can also be other forces such as the gravitation force, the body force right.

(Refer Slide Time: 21:45)

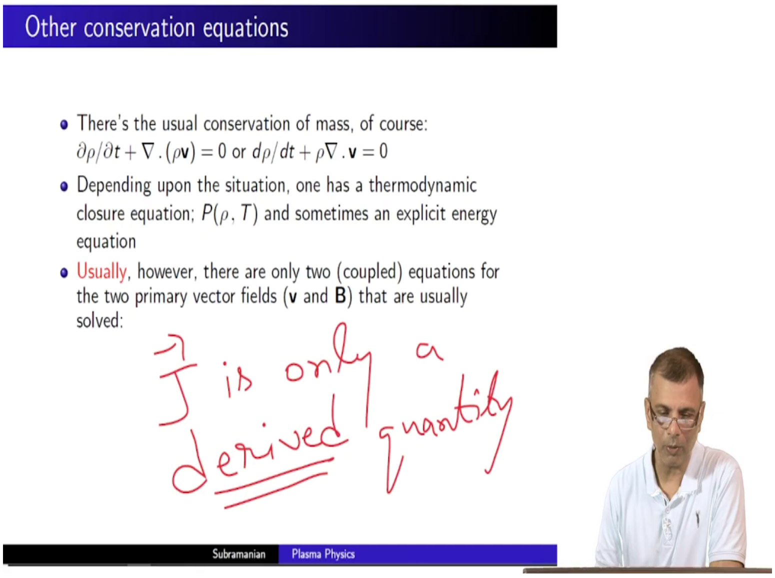


And so, we know this anyway from regular fluid dynamics, there is no need to go too much into it only real additional term is this and it is important to keep these two curious points in mind about the Lorentz force ok. The other a as far as other conservation equations right, there is a mass conservation equation of course, which we are very familiar with. So, we will go straight ahead.

So, that that is Eulerian form, and this is the Lagrangian form of the mass conservation equation, we have seen this. So, we will go right ahead pass this and depending upon the situation, one has a thermodynamic closure equation something like P equals p which is a function of rho and T or maybe a polytrophic in equation like P equals k rho raised to gamma.

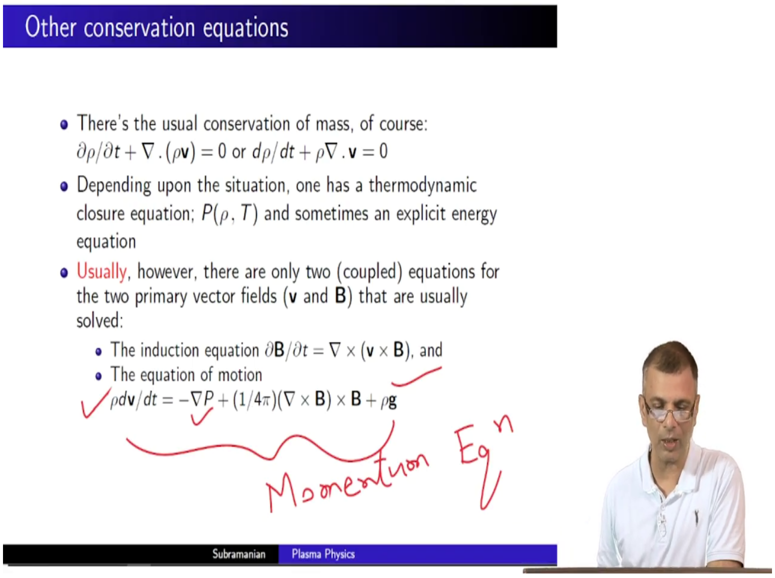
So, you might have this or sometimes an explicit instead of this, you might have an explicit energy equation alright so, that depends on the situation.

(Refer Slide Time: 22:49)



Usually, however, there are only two coupled equations for the two primary vector fields v and B that are usually solved ok. Remember this with regards to MHD, we might talk a lot about current density and so on so forth, but it is only a derived quantity ok. J is only a derived quantity. The primary quantities are just the fluid velocity and the magnetic field that is it ok.

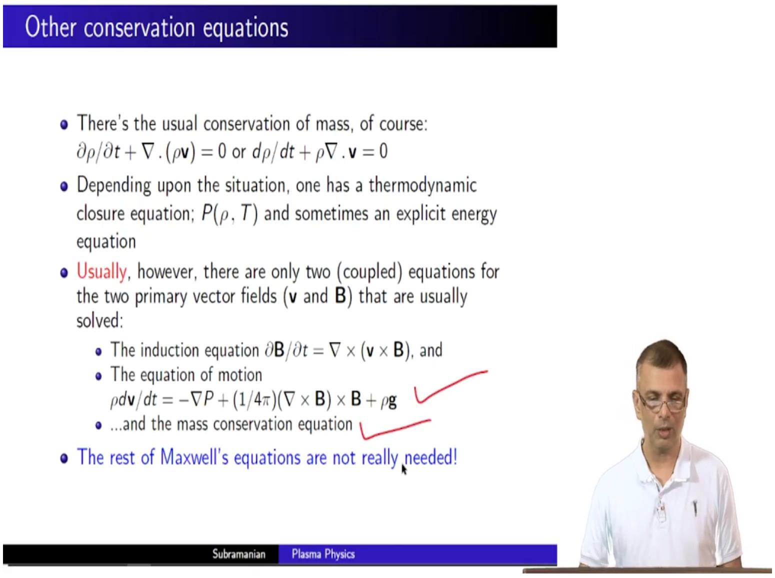
(Refer Slide Time: 23:34)



And these the coupled equations that you know you solve the induction equation and the equation of motion or the momentum equation. These are the only two equations that typically need to be solved. Of course, you would have an energy equation, sometimes in the guy's offer thermodynamic closure equation or an explicit energy equation ok.

So, and you remember how the equation of motion looks like. We are already familiar with this piece, with this piece and with this piece. This is only new piece here, the Lorentz force right and so that is that. So, the induction equation and the equation of motion, these are the only two equations that normally need to be solved in for in magnetohydrodynamics and of course, the mass conservation equation which is this ok.

(Refer Slide Time: 24:29)



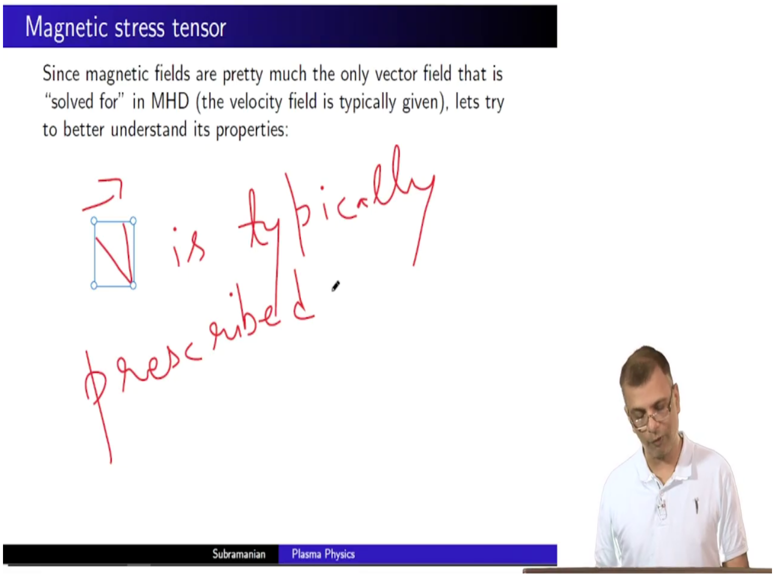
The rest of the, the point I want to make is that although we are talking about a magnetized fluid, you know the rest of Maxwell's equations which is where we got the induction equation from, we can might so forget about them, they are not really needed. Once we derive the induction equation, we can just keep it, I mean we can just forget about the rest of the Maxwell's questions ok.

We can rely, we are already; we are already familiar with the equation of motion, from fluids, we are already familiar with the mass conservation equation, we already know this, and we are also in fact, familiar with the energy equation either in this guys or in that guys ok.

And so, the only new equation that we need to be worried about is just the induction equation, the other bunch of Maxwell's equations not important anymore because we have already derived the induction equation from those and that is it, we can forget about the rest of the Maxwell's equations.

And of course, it is also you know there is an important new term in the in the equation of motion and that is the that has to do with the Lorentz force of course, you know that is there. But the, but it is curious that we do not have to worry about Maxwell's equation anymore alright. So that is how it goes.

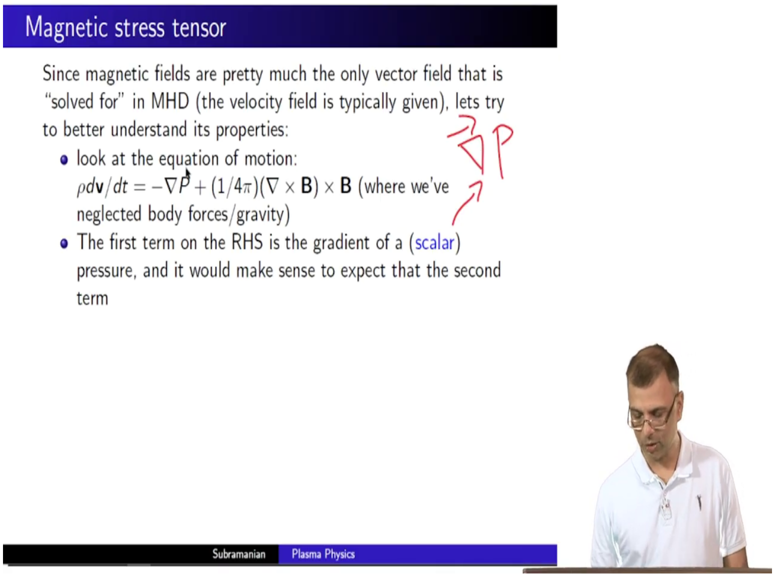
(Refer Slide Time: 25:51)



And the next thing that we will worry about a little more is the nature of the magnetic fields now ok. What the magnetic fields do to the equation of motion and like this line says since; since magnetic fields are pretty much only vector field that is quote unquote solved for in MHD in that v you see the velocity field is typically given or typically prescribed ok.

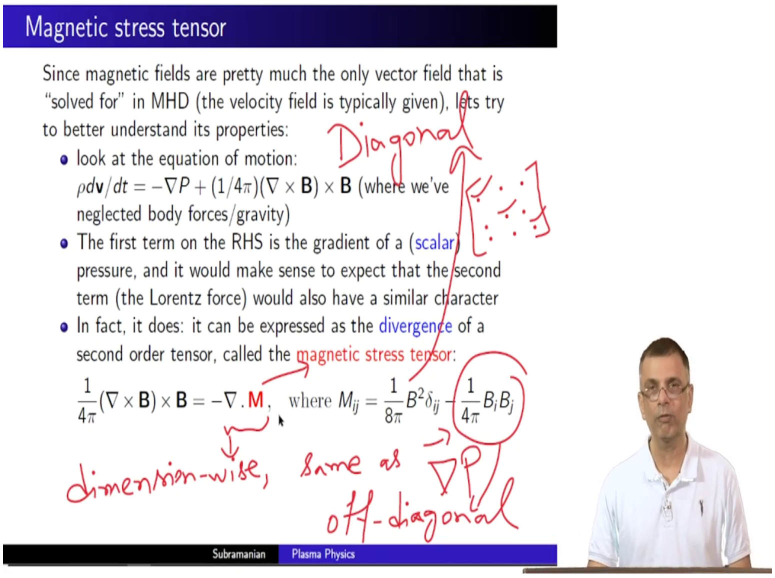
I give you a certain velocity field, I give you fluid that is flowing in a certain manner and MHD typically likes to answer ok, given this velocity field, how does the magnetic field that is present in the fluid, how does it evolve, how does it evolve with time ok? So, magnetic fields pretty much the only vector field that is solved for an MHD. So, it is important to try and understand its properties a little more.

(Refer Slide Time: 27:02)



So, you look at you know the equation motion where we have I mean this contains only the pressure term, the gradient of pressure term and the Lorentz force term, we have neglected body forces of gravity and the first term on the RHS which is the; which is the you know essentially the gradient of a scalar pressure, the as it is just the gradient for a scalar pressure.

(Refer Slide Time: 27:33)



And therefore, it would make sense to expect that the second term, the Lorentz force should also have a similar character. At the end of the day, this should also look like the gradient of a scalar pressure, this entire term this curl B cross B. In fact, it does, it can be expressed as a divergence of a second order tensor which is called the magnetic stress tensor and.

So, the magnetic stress tensor, which is M, this thing in red ok, this thing can be expressed as the divergence of a magnetic stress tensor and dimension wise, it is the same as gradient, so, the gradient of a scalar pressure that is what this is ok. And the elements of this tensor the ij-th element of this tensor is given by this 1 over B square 1 over 8 pi B square delta ij so, these would be the diagonal terms, and these would be the off-diagonal terms.

So, this is the diagonal term of the stress tensor and these are the off-diagonal terms, this B i B j ok. So, the stress tensor would essentially be a matrix like that, like that, like that, this,

this, this, this, this, this and the diagonal terms this, this and this are given by 1; 1 over 8 pi B square and off-diagonal terms would be given by B i B j where suppose this is x; x x, x y so, the x y would B x B B y and this would be x z so, there would be B x B z so on so forth ok.

So, we will discuss, given this there are some curious properties of this magnetic stress tensor that need to be understood before we go forward in MHD and we will discuss it when we take it up next. So, that is all for the time being.

Thank you.