Fluid Dynamics for Astrophysics
Prof. Prasad Subramanian
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture - 48
Magnetohydrodynamics [MHD]: The induction equation

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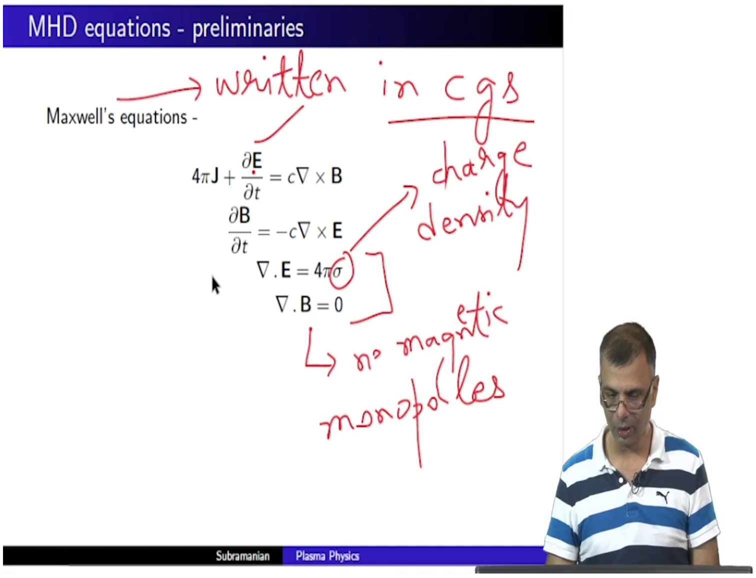

So, let us now establish the basic equations of Magnetohydrodynamics. And since, we are going to be dealing with electric and magnetic fields. Yes, we did say that electric fields are technically 0 inside of a fluid. But, nonetheless you know we have to start with them and then show that they reduce to 0 right.

And so, since we are talking about electric and magnetic fields, we have to deal with Maxwell's equations right. And so, these are the four Maxwell's equations, these are the two divergence equations. This is essentially saying that the divergence of B is 0 in I E, there are no magnetic monopoles that is what this equation is saying, no magnetic monopoles or magnetic field lines are always closed.

This is essentially saying establishing the relation between ok, here sigma is not conductivity, sigma is this is charge density. I beg your pardon for the change in notation, but normally we will use sigma for conductivity, but in this particular equation it is charge density. So, this is telling you how the electric field is related to charges right. And so, these are the two divergence equations. And this is telling you how a changing magnetic field produces an electric field, and this is telling you how a changing electric field produces a magnetic field.

And so, this would be this is essentially the displacement current term and this would be the physical non displacement current part of the current, if you will ok. So, that is the displacement current and this is the non displacement current.

Taken together this at the left hand side here would comprise the charge density, the current density and the right hand side is the curl of B. And I must emphasize that the all of these equations are written in cgs ok. T hat is why you see the appearance of c like the speed of light ok.

If you are writing this you might, you can equivalently write these things in SI units, it is just that these constants will appear slightly different. If you are more comfortable with SI units that is fine no problem, the basic physics is still the same. But, I wanted to you know emphasize that these four equations the Maxwell’s equations are written in cgs units ok right.

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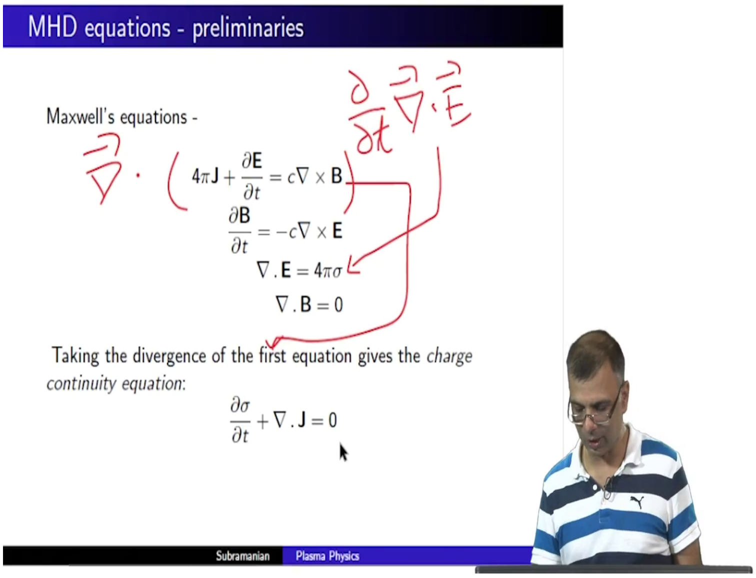

So, now, let us take the divergence of the first equation of this equation right. So, let us take the divergence of this equation, in other words I operate this entire equation by like that right. So, I have the divergence of J which is that right and the divergence of well, I this would essentially become d over dt of the divergence of E. I am assuming that the time and space derivatives can be interchanged. And for the divergence of E, I substitute from here right, for the for this one I substitute from here ok.

And the divergence of a curl is always 0. So, therefore, this term taking the divergence of the right hand side just gives you 0. So, taking together taking the divergence of the first equation gives you what is called the charge continuity equation ok, which you must which you know you are familiar with from basic electrodynamics. This is essentially saying that the divergence of the current density is equal to the time derivative of the charge density ok.

If you know there is no current flowing inside or out of volume through the surface. There is no way the charge density inside that volume will be changing with time that is what this equation is saying. You can also think of this as the equation for conservation of charge. Charge is always electric charge is always conserved that is what this equation is saying ok so, that is one thing.

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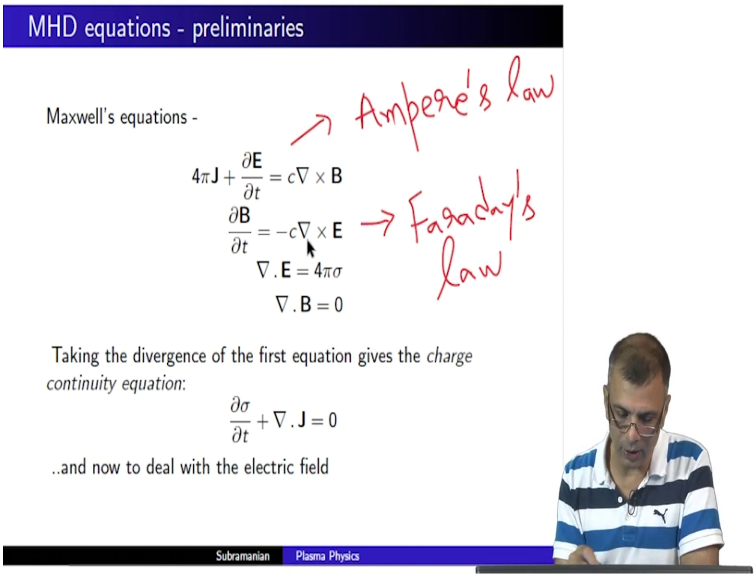

And now, to deal with an all important electric field, we made a big fuss about the electric field remember, we kept saying that you know the electric field in the fluid frame is equal to 0. So, let us deal with it ok alright.

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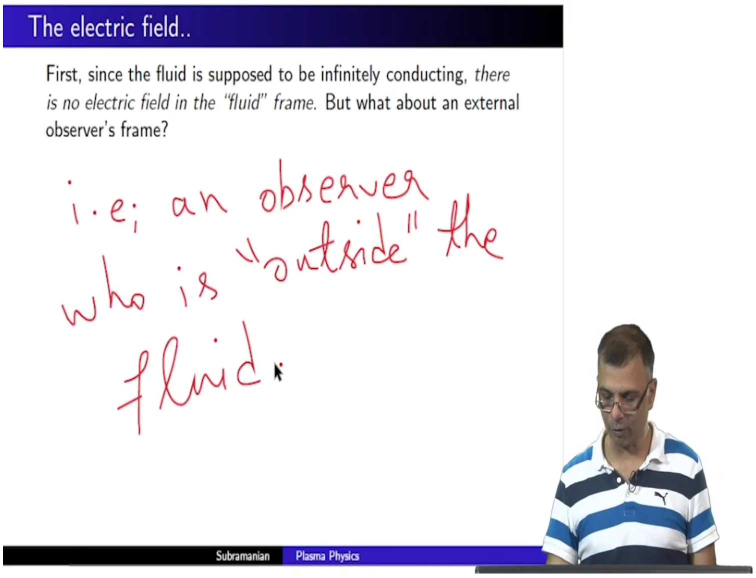

So, firstly, the fluid is supposed to be infinitely conducting right. The conductivity is infinite therefore, by you know there is no way you can have an electric field inside of the fluid, there is no way you can have an electric field in the quote unquote fluid frame, that is not possible ok, right.

So, ok so but what about an external observers frame, what about an observer i.e; an observer who is outside the fluid, what about this kind of observer ok? So, it is clearly the way we are talking about this, we are implying that the electric field need not always be 0, for someone who is not immersed in the fluid. The electric field has to be 0 for someone who is immersed in the fluid simply, because the fluid is infinitely conducting.

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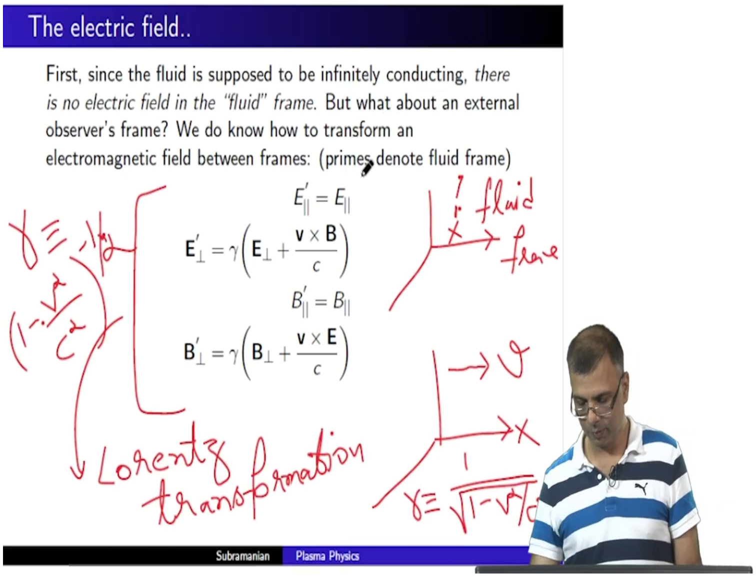

But, in order to in order to answer this question, in order to answer this question; we have to resort to a standard Lorentz frame transformation and that is this ok. So, the prime so, the primes would denote the fluid frame ok. So, and we have a parallel and and perpendicular in other words what we are going to do, is you have suppose you know you have two frames. One a fluid frame and one is an observer's frame ok.

And the fluid frame has primes in it. So, wherever you see primes E prime, B prime and so on, so forth you are in the fluid frame. Primes denote the fluid frame ok and say for just for concreteness, this is the X axis. So, this would be the X prime ok, and this would be the X axis here.

So, this would be the observer frame, the external observer frame. And the main thing is the external observer is moving with a velocity v with respect to the fluid frame, and that is how you make the transformation. So, the parallel refers to you know parallel to v and perpendicular refers to perpendicular to v ok. And, for simplicity we have taken the you know in this particular way of doing things, we have taken v to be parallel to the X axis ok.

This is no loss of generality, because even if you wanted to treat v, some other v you would just split up the components that is all ok. The basic results are exactly the same right. So, now, this is the Lorentz transformation of electromagnetic fields, Lorentz transformation ok. Which is relating the fields in the fluid frame in other words; the frame and all quantities in the fluid frame have a prime on top of those like this.

And all quantities in the non fluid frame, in the external observer frame. Which can potentially be moving with respect to the fluid frame ok, those do not have primes ok and the parallel denotes parallel to the velocity of movement and perpendicular denotes perpendicular to that direction right.

And this essentially how you transform between these two frames and this is called Lorentz transformation. The E parallel prime is the same as E parallel; however, that is not the case with E perpendicular E.

And this gamma is essentially, you see this v here the gamma is essentially the Lorentz factor. And the gamma is defined as 1 minus v squared over c squared raised to minus 1 half like that ok, maybe I should erase this and write it again. Well it is a little difficult this is essentially 2 ok. So, gamma is 1 over square root of 1 minus v squared over c squared ok.

So, that is what this gamma is it is essentially v it is a function of v ok, c is the velocity of light ok. And so, the E perpendicular prime is not the same as E parallel E perpendicular that is this v cross B business sitting in here. And, however so, that the B parallel prime is the same as B parallel, so, as far as the parallel components are concerned that both the electric field and the magnetic field, they are unaffected. It is only the perpendicular components which are affected.

So, this is how you transform between the prime frame and the unprimed frame ok right.

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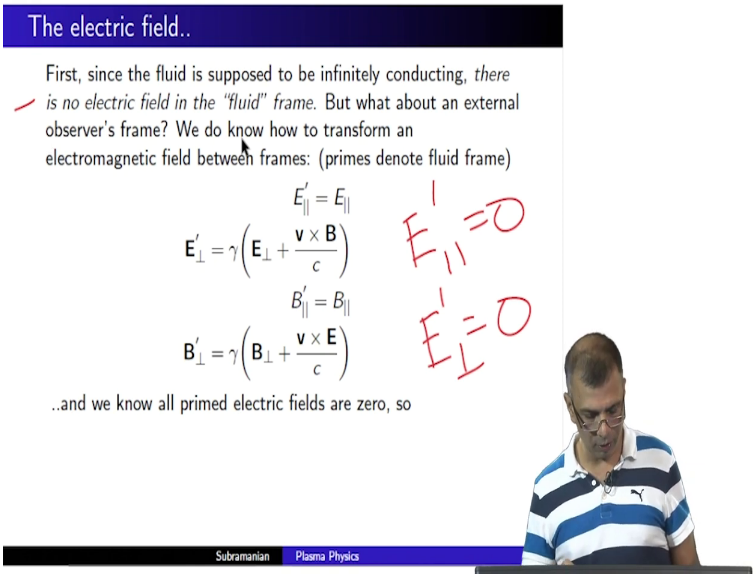

So, let us keep this in mind. Now, we know that all primed electric fields are 0 right, there is no electric field in the fluid frame in the prime frame, there is no electric field. So, E parallel prime is 0. So, therefore, E parallel is also 0 for that matter E perpendicular prime is also 0 ok. So, in other words E parallel prime is equal to 0 and E perpendicular prime is also equal to 0 by definition.

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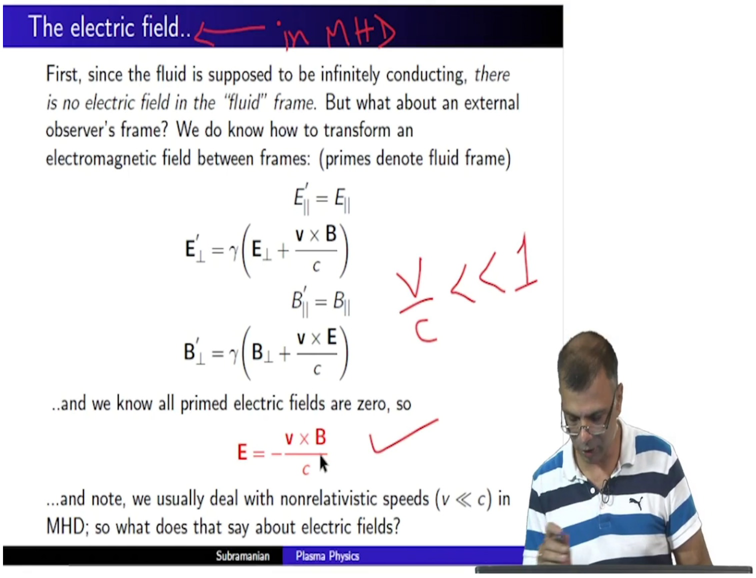

Therefore, you get so E parallel this is already 0 just. So, directly from here you get this, it is very interesting ok. So, the quantity in the brackets is equal to 0 and that just gives you this. So, now, pay close attention this is the electric field not in the fluid frame that is always zero, but this is the electric field in the external observers frame and that is non zero that is related to the magnetic field that the external observer sees in this manner ok.

And note we usually deal with non MHD, there is such a thing as relativistic MHD, but you know generally speaking, we deal with non relativistic speeds in MHD. And therefore, v over c is much much smaller than 1. So, even in the non prime frame even in the external observer frame, because v over c is much much less than 1. The bare magnitude of the electric field is very very small, in comparison to what well the magnitude of the magnetic field for instance.

So, what does that say about electric fields, electric fields are technically 0 in the fluid frame and even in the external observers frame. Because, we are restricting ourselves to non relativistic speeds, even in the external observer frame the electric fields are pretty small ok. So, this is something to keep in mind very very firmly right.

So, this was a brief detour into the role of the electric field in MHD ok. And this is about it we will not be talking too much about electric fields anymore, as we said we will mostly be encountering magnetic fields from now on, not mostly exclusively ok. And this is why ok? Before going ahead it is important to be very clear about the details and this is why, you know you do not worry about electric fields at all in MHD ok right.

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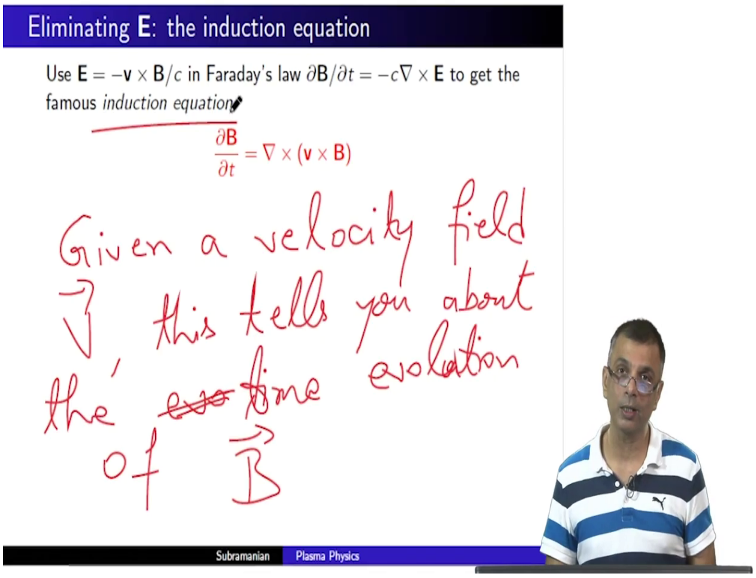

So, now, using this fact using this fact that we just derived E equals minus v cross B over c. What we do? Is we use this in faradays law in the so, you have the curl terms were Faraday's law and Ampere's law right. So, in the basic Maxwell's equations that we saw, this one this was Faraday's law and this is Ampere's law right.

So, what we are going to do now is substitute for the E that we have found inside Faraday's law ok. So, we are going to use E equals minus v cross B over c in Faraday's law right. So, to eliminate this E in order to get what is called the induction equation, which is which gives you the time evolution of B, there is only B you see in this equation there is no E at all ok.

So, this given a certain velocity field, this tells you how the magnetic field evolves ok, that is what this equation is telling you. Given a velocity field V, this tells you about the evolution of the magnetic field the time evolution, the time evolution of the magnetic field B, you see the time evolution.

All you need is v right and this gives you this is a complete dynamical equation for B ok. And this is what is called the induction equation it is very basic in MHD many times, when you see the equations of MHD written down. They just start from the induction equation assuming that you know right. So, it is important to know where the induction equation came from and this is where it comes from ok.

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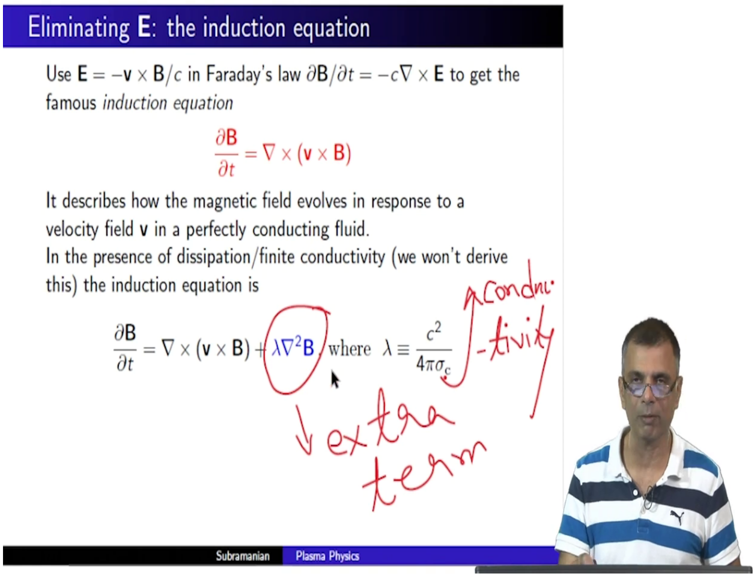

It describes how a magnetic field evolves in response to a magnetic field to a velocity field v. Of course, in a perfectly conducting fluid, why a perfectly conducting fluid, because this whole thing this thing applies only to a perfectly conducting fluid, E equals minus v cross B over c. If the fluid is not perfectly conducting this does not apply ok.

So, as such the equation as it stands applies only to a perfectly conducting fluid. However, we will not derive this, but we will say this in the presence of a finite conductivity in other words in the presence of dissipation. And we will not derive this the induction equation has this extra term, this is an extra term. Where this lambda is related to the conductivity sigma c, this is the conductivity ok.

So, you have the conductivity in the denominator here ok. So, this term in blue gives you the effects of finite conductivity or finite resistivity however, you choose to look at it ok. If you did not have this term if the conductivity is infinite, then this lambda would tend to 0. And you would have the plain vanilla induction equation. And we derived this induction equation, we will not we did not make any attempt to derive this extra piece, I am simply stating this ok right.

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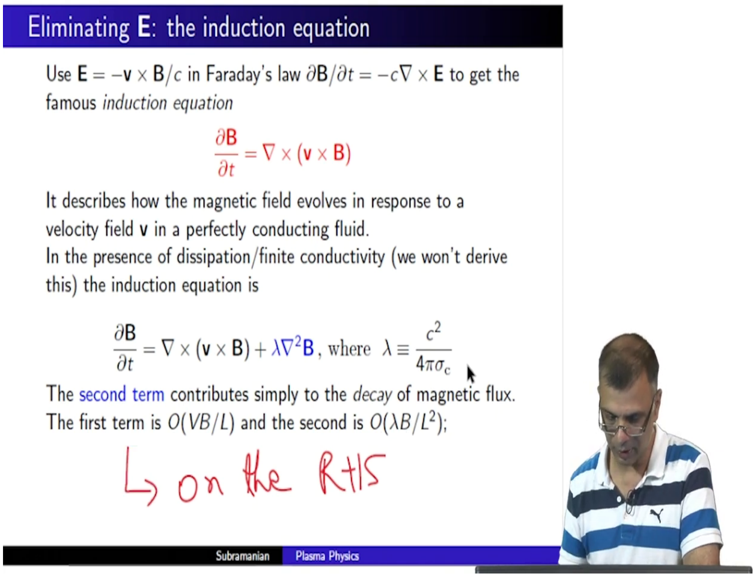

The second term contributes simply to the decay of magnetic flux, the second term in blue, we will see how. Now, just like we did dimensional dimensionless numbers in fluid mechanics, you know and we did make some mention of the dimensional dimensionless numbers in magnetohydrodynamics.

But, here is the first time we are considering them explicitly. And so, let us see here the first term you see, this guy the first term on the right hand side I must say. First term on the RHS: this guy is some kind of v times B and this curl is a differentiation right. So, it is a d over dx kind of thing, so you have and so, the d over dx can be some I mean you know approximated by 1 over L right. So, this first term is order VB over L.

The second term is of order what we do is we do not bother about the dimensions of lambda, we keep lambda as it is right. And, the B as it is and since there is a del squared this would be something like a d square over dx square. And therefore, that is represented by L square here ok. So, the first term is order VB over L, the second term is of order lambda B L square lambda B over L square.

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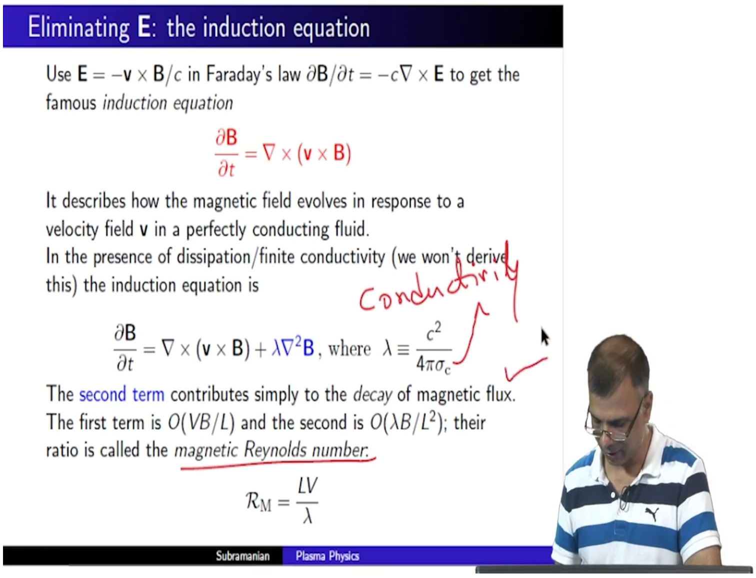

And the ratio of these two numbers, the ratio of the first term to the second term is called the magnetic Reynolds number, very similar to the fluid Reynolds number that we talked about earlier. And the magnetic Reynolds number is given by L times V, this divided by this order of magnitude LV, where L is some macroscopic length, V is some macroscopic velocity that we are talking about say the fluid flow velocity which of course, has to be non relativistic ok, divided by lambda.

So, for an infinitely conducting fluid lambda is technically 0, you see the sigma c appears in the denominator so, and this is the conductivity right. So, lambda is technically 0. So, for an infinitely conducting fluid since this is 0, the magnetic Reynolds number shoots up to infinity right. And for a highly resistive fluid, where the conductivity is very finite, where it is large and therefore, lambda is small.

The magnetic Reynolds number conversely is small. So, infinite conductivity infinite magnetic Reynolds number that is how it goes ok; it is somewhat like you know 0 viscosity fluids Reynolds numbers went to infinity you remember that right.

Fluid Reynolds number was LV over nu, where nu was the coefficient of viscosity. If the fluid was in viscid, then the fluid Reynolds number went to infinity. Here if the fluid is infinitely conducting, then the magnetic Reynolds number goes to infinity.

We made this comment here, saying that the second term, this term contributes to the decay of magnetic flux let us justify this.

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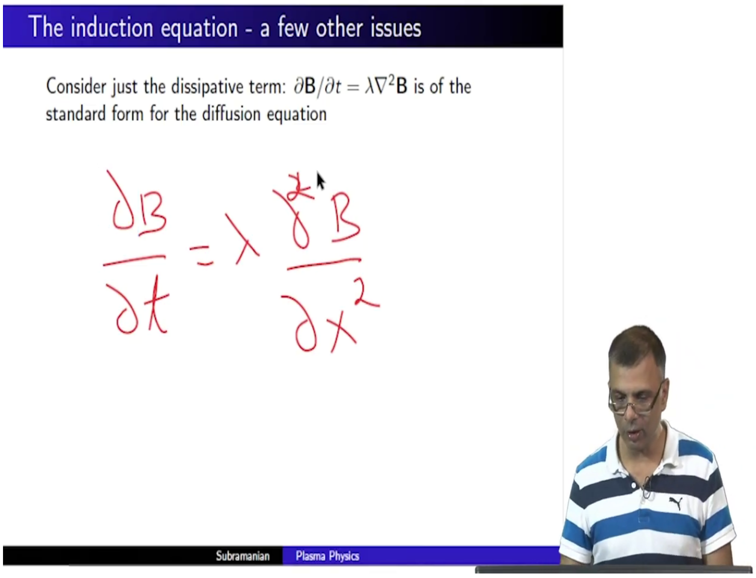

Consider for instance, just the dissipative term in other words, we will not bother about this term, we will only bother about an equation that has this term and this term ok. Which is like this d B d t equals lambda del square B. This is a standard form for the diffusion equation.

This is a this is a diffusion equation, this is like you know in one dimension for instance, this would be something like d B d t equals lambda d square B d x square right this is a diffusion equation is not it.

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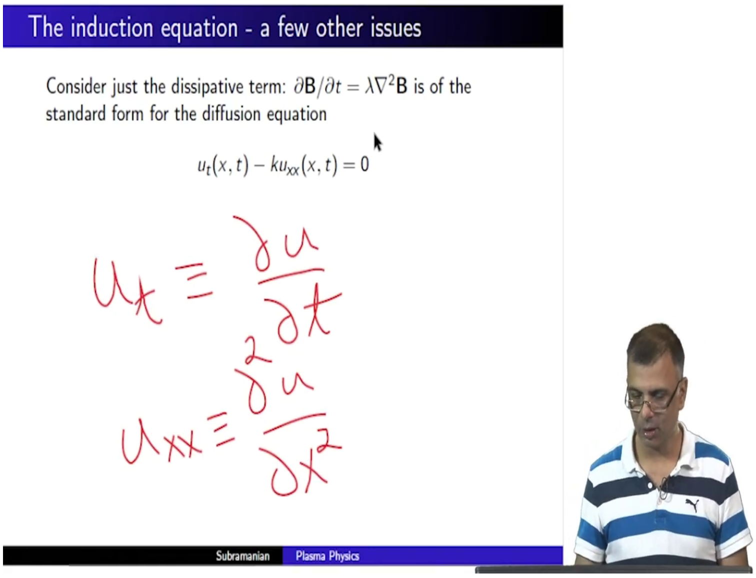

And the solution to a diffusion equation is well known the so, like that ok u xx is d square u d, where I say u t is d u d t, and u x is d square u d x square that is a notation ok. So, this is a standard diffusion equation.

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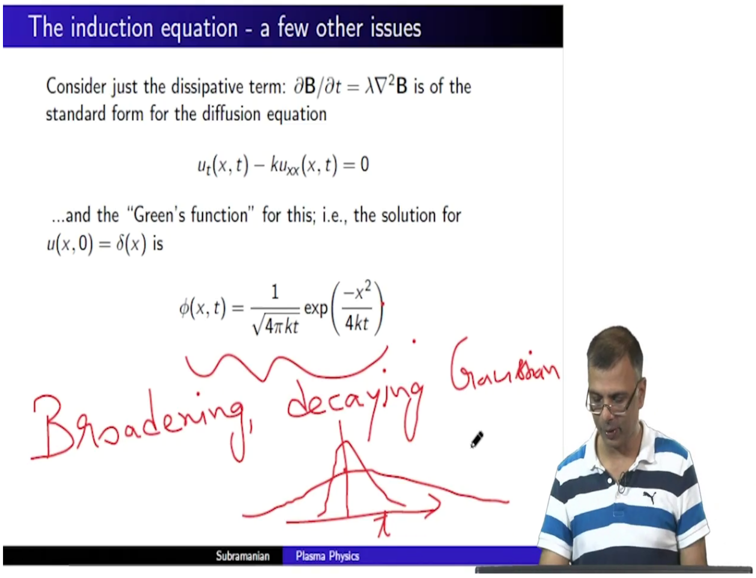

And the Green's function for this the Green's function is just, you can think of the Green's function as a response to a delta function excitation ok. In other words the solution for at time t equals 0 you have a delta function injected into this. And you want to see how this excitation proceeds with time that is what the diffusion equation is going to tell you and the solution is this.

So, you have a broadening decaying exponential. So, this would be decaying Gaussian actually it is something like this. You have a time on the x axis right you started out with a delta function like this. And with increasing time what is what you are saying is the variance increases and the height also, the height decreases, with increasing time the height is decreasing and the variance is increasing right.

So, you started out with the delta function at time t equals 0, at a later time it would look like this. And what does this represent? This represents the magnetic field at a still later time it looks like this. In other words you started out with a point like magnetic field, as time progressed what happens was the magnitude of the magnetic field decreased. And it broadened and as time progressed even more the magnitude decreased even more, and it broadened even more. And what is this?

This is essentially saying that the magnetic field is spreading out its diffusing ok.

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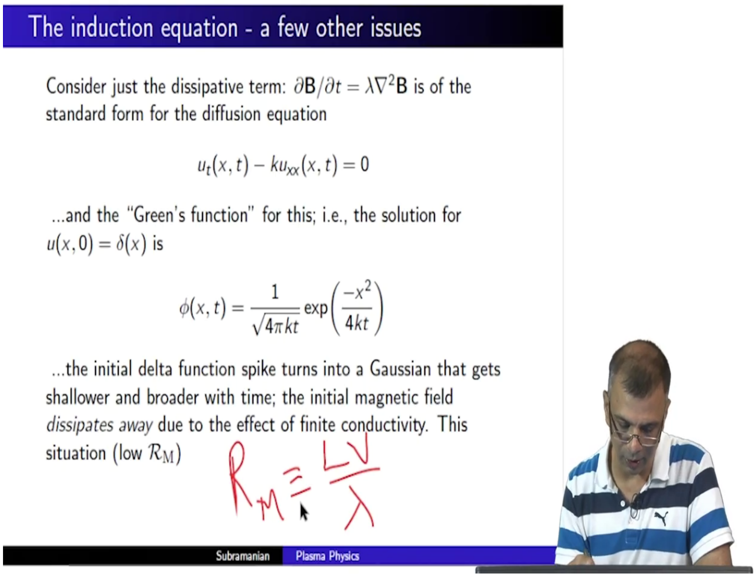

So, an initial data function turns into a Gaussian that gets shallower and broader with time. So, the initial magnetic field dissipates away due to the effect of the finite conductivity. And the effect of the finite conductivity is all embodied, in this lambda in this lambda term right.

So, if there was no if the conductivity is infinite, then this term would not be there and the magnetic field would not diffuse away, it would not dissipate away. So, that is what I that is why we made that statement in the previous slide. So, and so, this would be a low Reynolds number situation, you remember Reynolds number, the magnetic Reynolds number was defined as LV over lambda.

When the when the conductivity is finite and that would mean that lambda is essentially small, and sorry lambda is large. And therefore, the magnetic Reynolds number is low ok.

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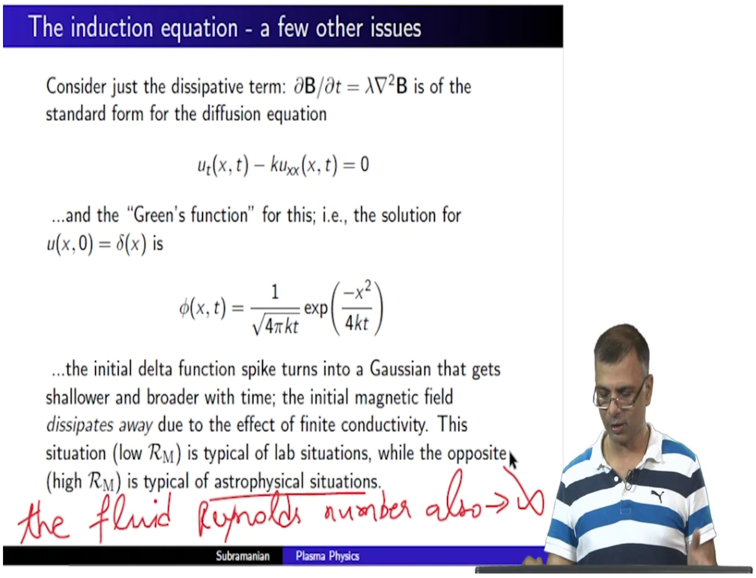

So, this situation low is typical of lab situations, low magnetic Reynolds numbers are typically found in lab plasmas. Whereas, a high Reynolds number situations are typical of astrophysical situations. In astrophysics, we typically deal with high magnetic Reynolds numbers ok.

And, if you remember the several examples that we discussed regarding fluids in astrophysics, you know accretion even shock propagation supernovae so on, so forth. In astrophysics we were most we were also mostly considering high fluids Reynolds numbers ok. And as we are saying here; the magnetic Reynolds numbers are also very high in astrophysical situations.

So, in other words in astrophysical situations the fluid Reynolds number also tends to infinity ok. So, both the magnetic Reynolds number and the fluid Reynolds number in astrophysical situations are very high.

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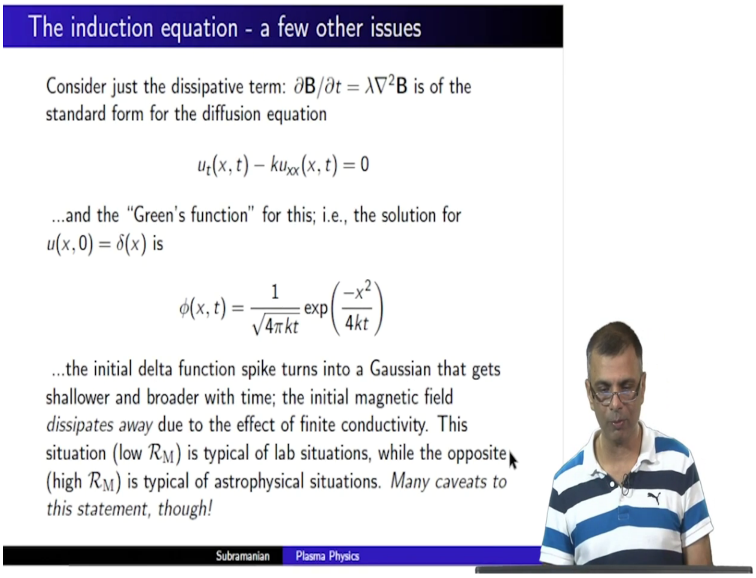

So, this is something to keep in mind there are many caveats to this statement, though I just I although the statement is generally taken to be true, one should accept it only with a grain of salt, I just wanted to you know make that clear.

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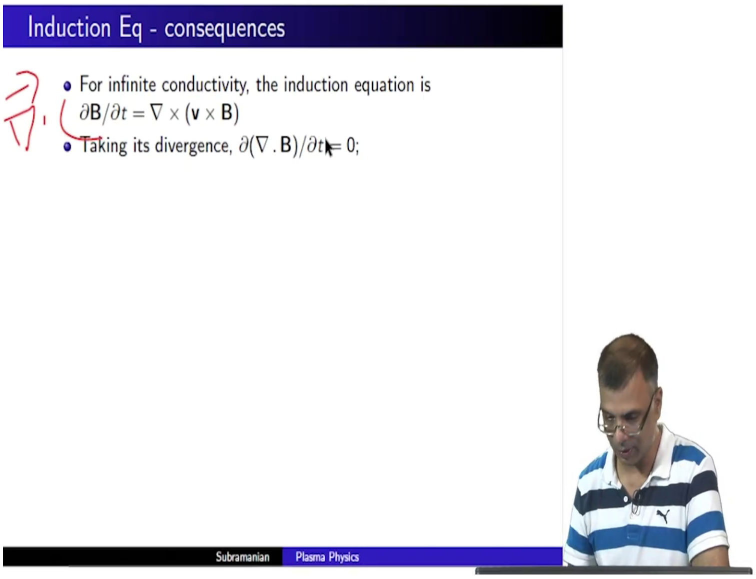

So, there are some consequences for the from the induction equation. So, for as if we take the conductivity to be infinite in other words, if we take this term to be essentially 0, in other words if we take the magnetic Reynolds number to be technically infinite.

You only have this, the induction equation is just this right. What you do is you suppose, you take its divergence, you take the divergence of both on both sides right. So, you take just this of this whole thing right. And you allow you know you can take the divergence inside the time derivative ok, then you know that the divergence of a curl is always 0 right. Therefore, what it is saying is that the time derivative of the divergence of B is 0 ok.

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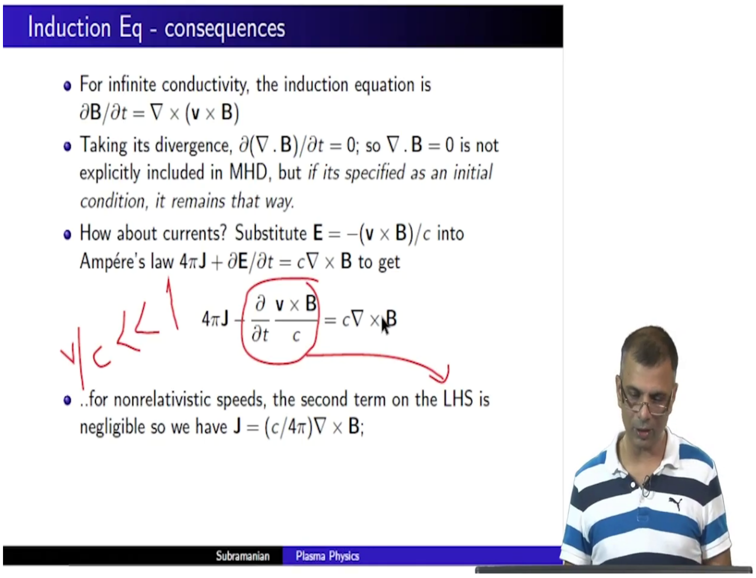

So, therefore, this is not explicitly saying that the divergence of B is 0 mind you; this is how we have normally learned our electrodynamics right. The divergence of B is always 0 this is something that is sacred. That is not what MHD is saying it is simply saying the time derivative of the divergence of B is 0 ok.

But, if it is specified as an initial condition in other words, if divergence of B is 0 at the beginning of your simulation it will remain that way ok. It will that condition will not change with time, on the other hand, if you had a non zero divergence of B due to some reason ok. Due to artificial resistivity or some problem in your simulation, then that will remain frozen as time progresses that is what this equation is saying ok.

Now, how about currents this is a very interesting. So, what we do is? We substitute E equals v cross B over c into Ampere's law here right. So, we this d E d t we substitute this here and we get that ok. There is there is no mystery I just substitute this in here and I get this. Now, for non relativistic speeds in other words for v over c much less than 1, this term is negligible right. So, this is what I mean here therefore, we have J is equal to essentially curl of B ok.

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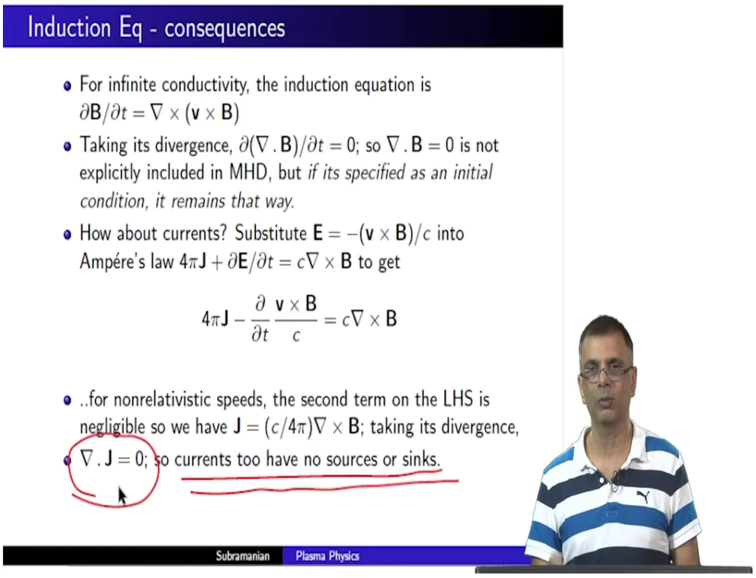

So, J is essentially curl of B how about we take its divergence, we take divergence of J. And then you know what's going to happen I am going to take this is just a constant. So, I am going to take a divergence of the curl of B, which is saying that the divergence of J is equal to 0. So, currents also have no sources or sinks and this is a very very important statement, which needs to be thought about in some detail ok.

And this is very unlike our normal notion of currents in electrical circuits. And this is so important that we will consider it, when we come back the next time, this statement the fact that the divergence of current density E is equal to 0, technically equal to 0 in magnetohydrodynamics. So, this is something that needs some thought. And so, we will consider this when we meet next for now, we will close.

Thank you.