Fluid Dynamics for Astrophysics
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Lecture - 40
Disk accretion: Removal of angular momentum, Shakura-Sunyaev viscosity parameter

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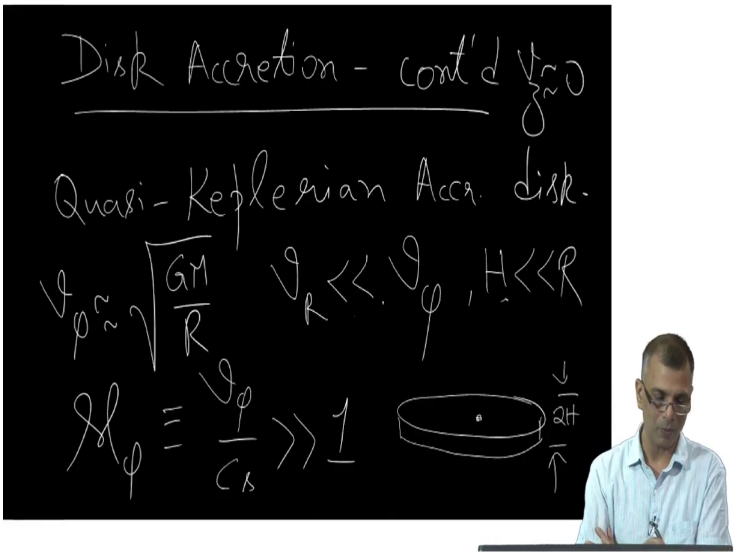

So, we will continue our discussion of Accretion Disks. What we had done until now is consider the mass conservation equation and the momentum conservation equation. But remember we were looking at the momentum conservation equation component by component, right. So, we first looked at the z component of the moment, and also the other thing is we are working in cylindrical coordinates because that is the you know most natural thing to do when dealing with disk accretion.

So, the first thing we did was we looked at the vertical component of the momentum conservation equation and then we looked at the radial component of the momentum conservation equation, right. And in both cases, we essentially derive the following conditions for a Quasi-Keplerian accretion disk, a quasi-Keplerian accretion disk which is to say that the V phi is nearly the Keplerian velocity GM over R.

Not quite because exactly the Keplerian velocity then there would be no accretion at all, right. This is a an exactly Keplerian accretion disk is the perfectly stable one. And you know I mean a parcel of gas rotating at exactly the Keplerian velocity around a central object remains in equilibrium.

There is no motivation for it to actually sink in. There is no motivation for it to have a radial velocity. Hence, I think quasi-Keplerian accretion, this is it is almost Keplerian, ok. So, hence this nearly equal sign. So, this and then the fact that the radial velocity is much much smaller than V phi, ok and the fact that the height the local height of the accretion disk is much much smaller than R.

And the fact that the Azimuthal Mach number which is defined by V phi over C s is much much larger than 1, all these conditions are all coupled, ok. You break one the rest of them break down, ok. If the disk becomes puffy, ok, if the disk is not thin, if here is a central object and here are the streamlines of the accretion disk and the thickness of the disk would be 2H like this, ok.

So, the condition here is that the this thickness is much smaller than the local radius. If this is not true, then this breaks down, this breaks down, all of these breakdown simultaneously, ok. And of course, the other thing is that you know V z, V z is nearly equal to 0. So, this, this, this, this, and this these are all simultaneously satisfied, ok. You break one the rest of them break down, ok.

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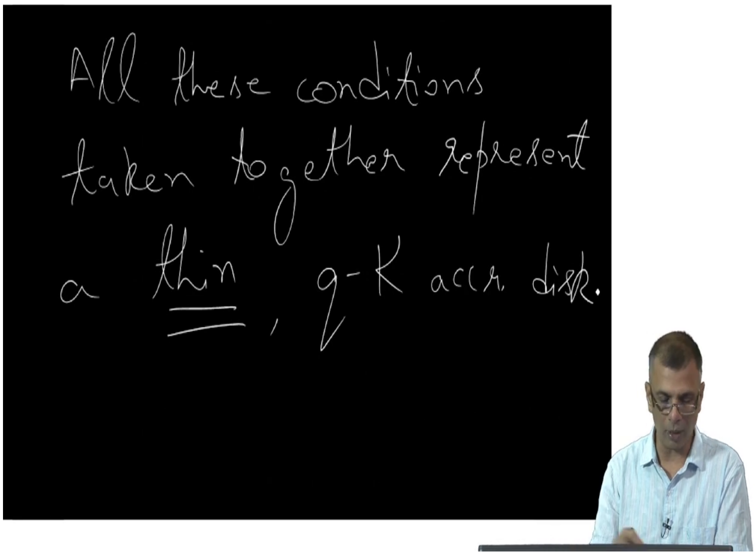

So, taken together all these conditions, taken together represent a thin quasi-Keplerian accretion disk. And we have derived each of these conditions in what we have done so far, ok. So, the only thing now remaining is to consider the azimuthal component of the steady state momentum equation, ok.

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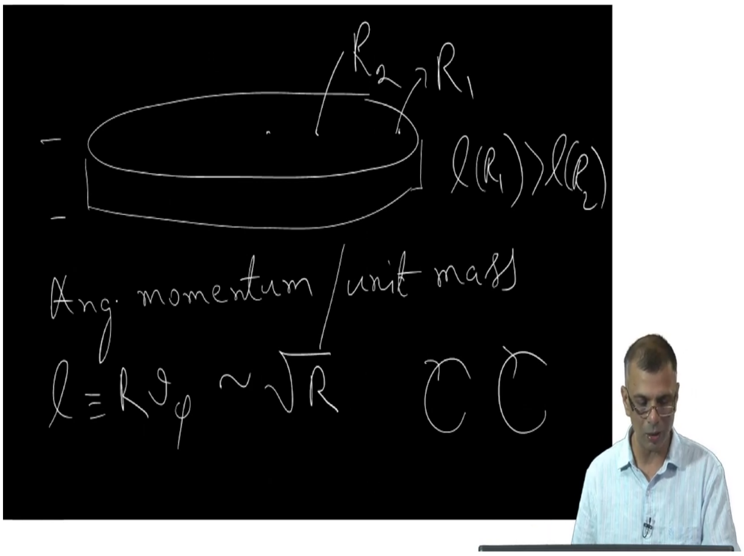

And now even before that let us we drew this cartoon, right, ok. So, matter is being supplied from a companion star, and it already has some angular momentum to begin with and that is why it has a preferred rotation axis and that is why it cannot simply you know directly be sucked in by the central object because it already has angular momentum.

So, angular momentum has to be lost, ok in order for it to settle down to a central object. Why do we say this? We say this because if you know the angular momentum per unit mass, I say all this is a prelude to discussing the azimuthal component of the steady state momentum equation which is the only; well that is the last component of the momentum equation that needs to be discussed. And then we will discuss the energy equation. So, this this is our basic brief, ok.

So, even before that we should, and we have discussed what were going to say we mentioned this very briefly when we met last, but it is worth going over it a little bit, ok. So, the angular momentum of a parcel of gas per unit mass per unit mass is important, it is simply r V phi, r cross V phi actually.

So, I denote this by l and l is equal to r V phi, right and V phi goes a square root I sometimes I write this as small r sometimes I write this as large R, I should I should be consistent, and they are the same, ok. So, R times V phi and you remember the V phi goes as square root of GM over R and so, R times V phi this goes as square root of R and this is the important thing, right.

So, a larger radius say R 1 and R 2. So, the l at R 1 is greater than l at R 2. So, the angular momentum of a parcel of gas here is larger than the angular momentum of a parcel of gas here. That is what this formula is telling you.

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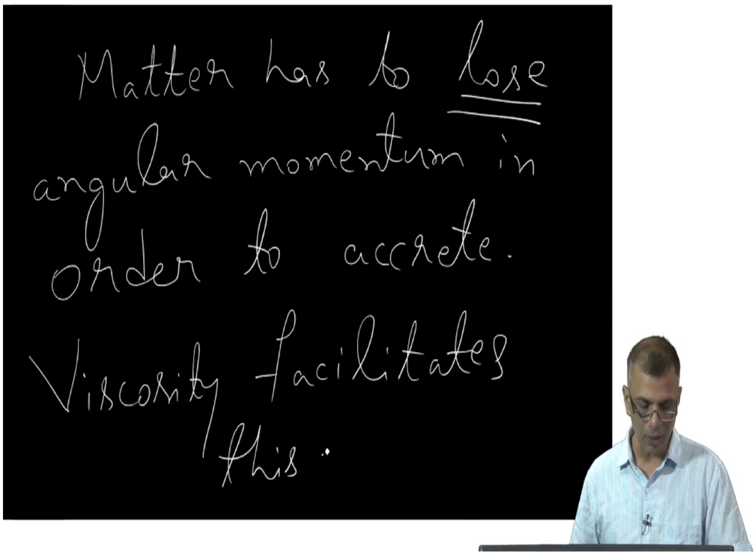

In other words, matter has to lose angular momentum in order to accrete, right. So, what does accretion mean? Matter moves in from radius R 1 to radius R 2 with a very small, but finite radial velocity, ok. This is fine. But you see since matter here has larger angular momentum than matter here the azimuthal velocity here might well be larger than the azimuthal velocity here, but the angular momentum is R times azimuth velocity, ok.

Therefore, the angular momentum here is larger than angular momentum here. And therefore, angular momentum has to be lost in order for matter to accrete, ok. And viscosity, the process of viscosity facilitates this. What do I mean by this the loss of angular momentum? So, viscosity facilitates accretion, disk accretion, ok. Disk accretion is closely tied with viscosity technically speaking shear viscosity, ok.

Before actually discussing the azimuthal component of the you know momentum equation, can we say a little more about viscosity? Sure we can. You see this accretion disk out here this is a side view, and if I was looking from the top I would see a picture like this.

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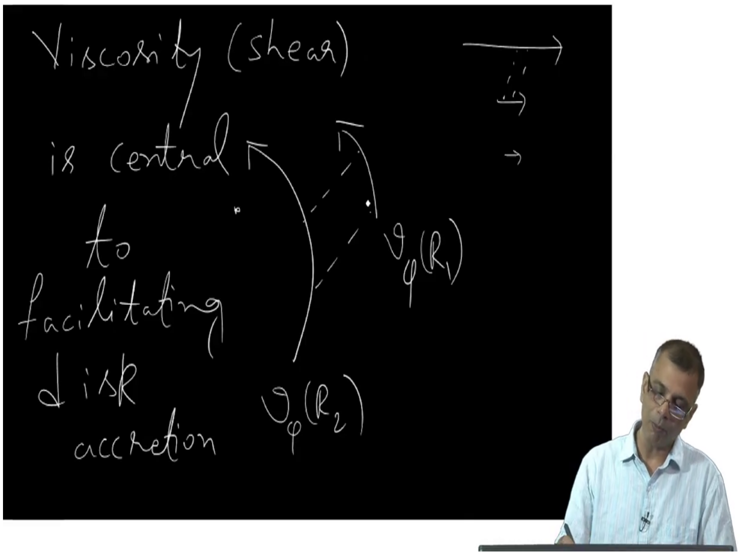

I would see the central object and I would see like we remarked the azimuthal velocity here is larger than azimuthal velocity here and therefore, you know if I was to draw velocity vectors this would be something like this, ok. And this would be V phi R 1 and V phi R 2, right. So, the length of this velocity vector is smaller than the length of this velocity vector and I am looking at it from the top.

So, this is a shear situation, is not it? This is exactly, this situation is exactly like the ones we used to draw so many times before where you have the you know where a velocity vector here is larger than the velocity. This this is essentially the same situation, except now we are talking about a rotating frame of reference that is all.

So, just like we used to talk about you know rubber bands linking these two layers similarly we would have little rubber bands linking these two layers preventing the velocity shear, is not it? So, this so, it is the same concept, it is the same viscosity concept that we were discussing for plane parallel flows is exactly the same viscosity concept here. And viscosity is you know a viscosity, shear viscosity in particular is central to facilitating disk accretion for this matter I mean.

So, this is the physical picture of viscosity that you can think about. Here is a shear kind of flow because you know the velocity here is larger than the velocity here. So, it is a by the definition of a shear flow. And you know viscosity is that process is that property of the liquid that prevents this shear from happening, ok.

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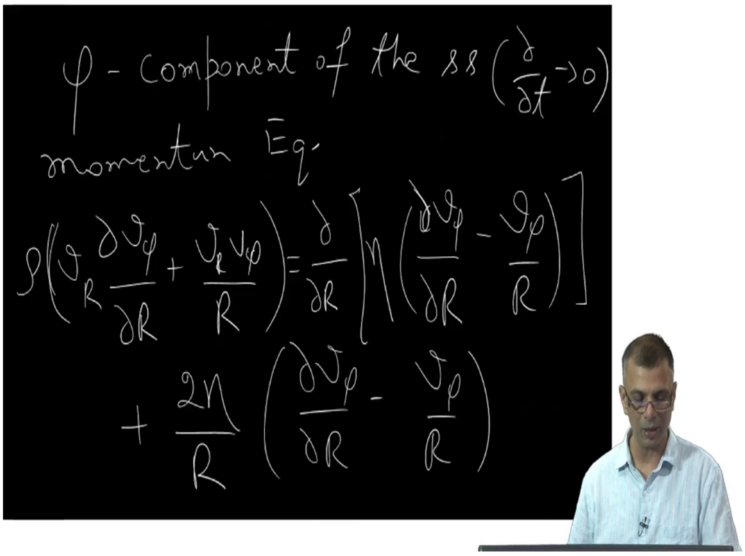

So, enough of all this. Let us now write down the azimuthal component, phi component of the steady state. In other words, d over dt this kind of thing momentum equation, ok. And this would look like this slightly large, but that is ok. So, what we are going to write is this would be rho and I urge you to there are several books which give you; so, far we have we have been looking at the momentum equation purely in you know vector language.

And so, this is just breaking it down is exactly the same momentum equation, ok. It is essentially this Navier-Stokes equation if you will. And now we are including viscosity, so you know it is the Navier-Stokes equation we are writing down, but it is simply the azimuthal component that is all. It is no different, ok.

It might look more elaborate and laborious, but it is important to write out the entire equation, but there is nothing new in this, nothing new is simply writing it down in cylindrical coordinates that is all, ok. This this would be M a, ok. And the right hand side would be the f, eta is a viscosity coefficient d V phi d R minus V phi over R, ok, yeah, that is it ok.

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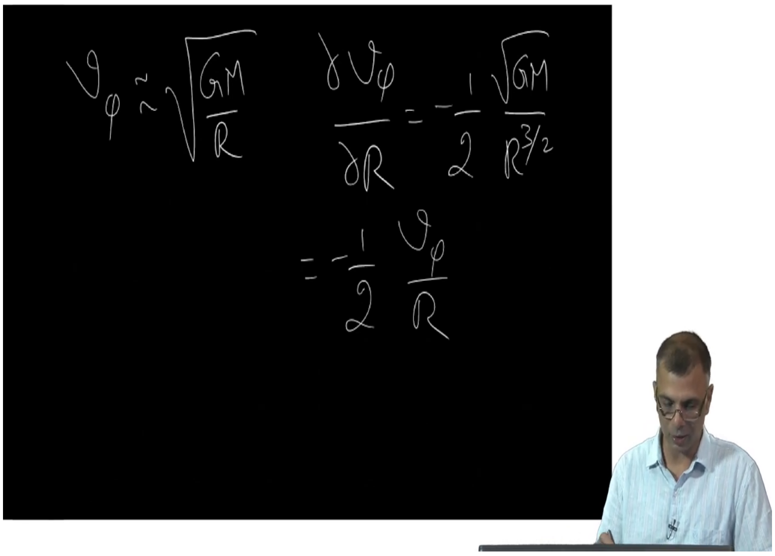

Now, we use the fact that V phi is equal to, we use the fact that V phi is almost equal to the Keplerian value. And therefore, you know things like d V phi d R, ok this would be something like this would be something like one-half, well minus one-half, right.

So, this is essentially equal to minus one-half, right is not. It you see you just divide this by R and you essentially get this. So, we use anytime we see a d V phi d R, we simply write it as V phi over R like that, right. So, using things like this the R phi component of the viscosity tensor defining the R phi component of the viscosity tensor. Let me take a separate slide to write this.

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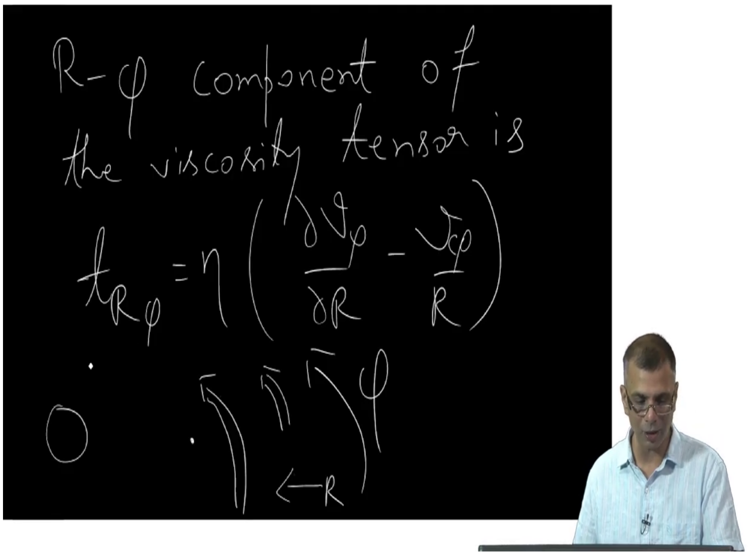

The R phi and I will discuss this a little bit more why am I considering only the R phi component, other components are not so important. I will say this in a moment let me write this down of the this is by definition d V phi d R minus V phi over. And before I say this I just look at this definition it goes as d V phi d R minus V phi over R and you can see that it is exactly in that combination that all these terms appear eta times d V phi d R minus this and eta times this.

So, both of these can be immediately you know written very compactly in terms of this t R phi. Now, why is only t R phi important? What we are interested in if you recall looking down at the top view, we are interested in the central object here and a velocity flow that is something like this, is not it? So, what does t R phi mean? t R phi means the phi directed and this would be the phi direction, right. This would be the phi direction.

And so, I mean just to distinguish this these would be the velocity vectors and this is simply denoting the phi direction. So, this would be the phi direction, right. And so, this would this t R phi would mean the phi directed stress, right on a surface whose outward normal is in the R direction. This would be the R direction.

And what is a surface whose output normal is in the R direction? If I draw a cylinder whose top view is like this and so, the cylinder is essentially going down into the plane of the screen, right. So, the side surface of the cylinder, that is the surface we are talking about. The side surface of the cylinder is the one the normal to which is pointing in the R direction.

And that is exactly that is the only since the viscosity we are talking about the shear viscosity we are talking about is the one that prevents you know two of these azimuthal azimuthally directed velocity vectors or stream lines from sliding over each other, right. So, therefore, the only relevant component of the viscosity tensor is this t R phi, the one which represents a viscous stress in the phi direction on a surface whose outward normal is in the R direction.

In other words, the surface the side surface of the cylinder which extends into the plane of this screen, ok. Hence this is the definition of t R phi and using this definition this entire this large expression essentially boils down using this expression for t R phi it just boils down.

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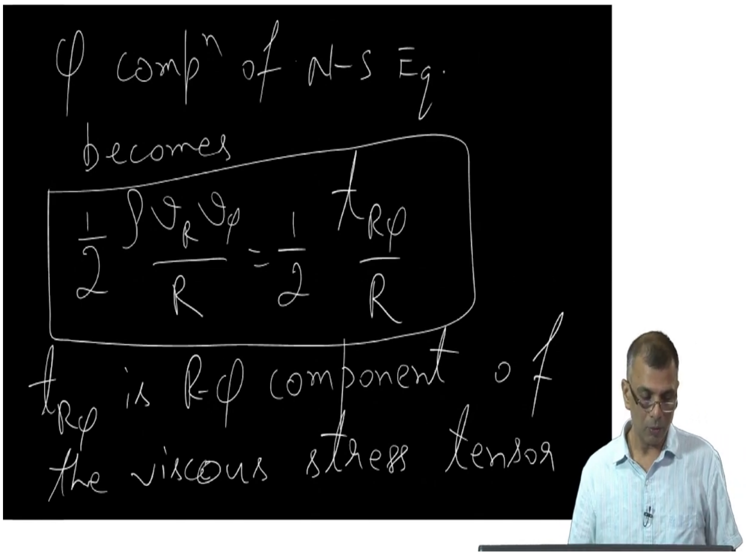

So, the azimuthal phi component of the Navier-Stokes equation, steady state Navier-Stokes equation essentially becomes one-half rho V R, V phi over R is equal to one-half t R phi over R, ok. So, this makes it very simple. And that entire large equation just boils down to this using the t R phi, ok.

Now so, t R phi let me write this down once again just to emphasize it, t R phi is the R phi component of the viscous stress tensor. Now, let us discuss and then; so, let me just highlight this a little bit because we will return to this in a second. Now, there was one breakthrough that happened in 1973 due to Shakura and Sunyaev, 1973, Astronomy and Astrophysics.

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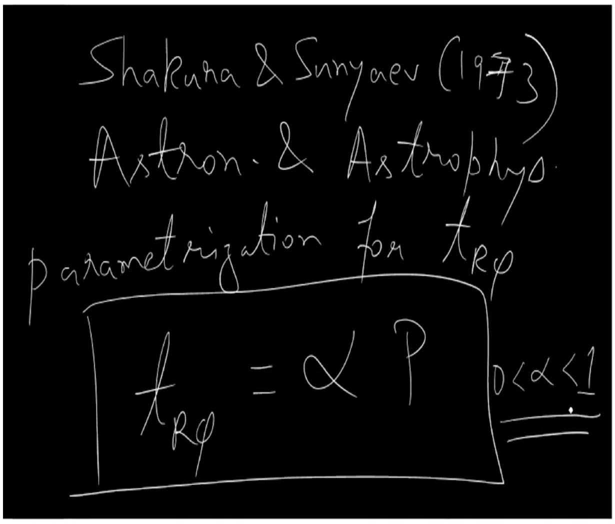

You should go to this journal and look up this paper. And so, what they said was that they adopted the Shakura and Sunyaev parametrization for t R phi. They said is equal to some fudge parameter alpha times P. This is a very important parametrization which allowed you know discussion of accretion disks to proceed very fast.

Now, what is the logic in this? Really, the logic is very very simple, ok. It is like this. You remember the viscous test stress tensor is on an equal footing with the pressure, right.

They have the same dimensions, force per unit area, force per unit area, ok, same thing. In fact, what we know as a scalar pressure they are simply the diagonal elements of the viscous stress tensor, the off diagonal elements are things like these. And of the several off diagonal elements in this case this is the only off diagonal element that is of any interest to us.

So, what we are really, what Shakura and Sunyaev's prescription merely meant is that the R phi component of the viscous stress tensor is some fraction alpha of the ambient pressure. And they said by you know the common sense thing is that alpha is less than 1, and of course, greater than 0. This is the other important thing.

In other words, the viscous stress tensor is only a perturbation on the ambient pressure, the viscous stress. The viscous stress, the shear viscous stress the only component of the shear viscous stress that is relevant is only a fraction of the ambient pressure, ok. It is a fraction that is less than 1, well you know.

Therefore, alpha is less than 1. It is only a fraction. And of course, it has to be greater than 0 you do not want a negative you know a negative viscous stress that makes no sense. So, this in some sense was a very simple you know prescription.

Now, what happens is this entire thing using this this entire equation this equation quickly collapses if you write down the you know the quasi-Keplerian prescription for V phi, it is simply you know it using this prescription wherever we see t R phi, I simply write alpha P, ok using this prescription this guy which is essentially you remember where we got this from. We got this from here, it is essentially this.

This entire phi component of the steady state momentum equation nicely boiled down to this and using the Shakura-Sunyaev prescription for the viscous stress tensor it boils down to an even simpler form it becomes alpha P is equal to rho V R V phi, ok.

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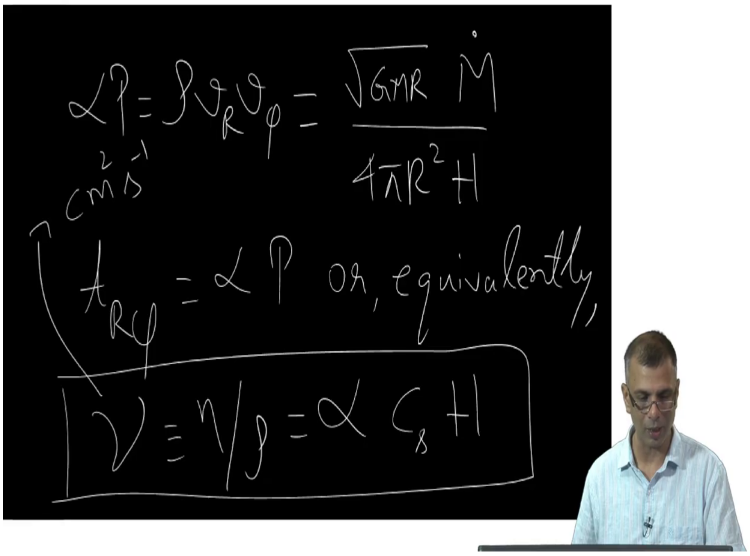

Because you remember this is simply saying that rho V R V phi is equal to t R phi and therefore, and t R phi is alpha P, right. So, alpha P is equal to rho times V R V phi and when I substitute the Keplerian value for V phi, this becomes this simply becomes square root of GM R, M dot over 4 pi R squared H.

It is very nice what we have done here is of course, substitute for the V R in terms of the accretion rate. You remember we are talking about, we are talking about accretion and so, to the extent possible we should always go back to the accretion rate, the accretion rate is the one important thing in this entire exercise, ok; so, this is it. Alpha P is equal to this expression on the right and this is the azimuthal momentum equation that, is it, ok. So, we are pretty much done.

I also wanted to point out one important thing the Shakura-Sunyaev prescription as such was this that, this Shakura-Sunyaev parameterization was such this. It can be shown that this is equivalent to this the viscosity coefficient new which we know is essentially eta over rho this is equal to alpha C sub s H, ok. Many times the Shakura-Sunyaev prescription is written in this form.

Now, what is the logic in writing this? We discussed. We simply said that well this is the this is the viscous stress the shear stress and this would be the scalar pressure, all this is saying is that the viscous stress is a fraction of the scalar pressure, ok. It is a fraction is an alpha necessarily has to be is (Refer Time: 25:20) of course, has to be positive, but it has to be less than 1, it is a fraction of the scalar pressure.

It can be shown that this is equivalent to this. And what is the logic in writing this? Well, you recall the dimensions of this are something like centimeters square per second. In other words, it is some kind of velocity times some kind of length scale, ok. So, what is the velocity here? The velocity would be the speed of sound. And what is the length scale here? The length scale is the height of the disk. And remember let us look at, yeah. So, let us look at this.

So, what you really want what this is prescription is really saying is that if you think that the viscosity is arising due to turbulent Eddies and we will discuss why this is so, ok. You want the size of these turbulent eddies to be smaller than the height of the disk. You do not want these eddies to be larger than the height of the disk.

And so, the natural length scale to choose would be the height of the disk which is H here and you do not want these Eddies to be supersonic because as we know supersonic flows and tail things like inconvenient things like you know shocks and everything, so you want the for a reasonable velocity you simply want to write C sub s.

And of course, these are the upper limits, right. The velocity of these eddies would be the speed of sound at most and the length scale of these Eddies would be the height at most. Note the word use of the word at most, ok. So, that would be the upper limit and therefore, the alpha would be a fudge parameter. Alpha less than 1 ensures that the velocities of these Eddies is less than C sub s and or the height the length scale of these eddies is less than the disk height.

Now, you might wonder why am I suddenly starting to talk about fluid Eddies, ok. We thought viscosity was due to you know molecular collisions of molecules with each other and so on so forth. And so, why are we suddenly starting to talk about Eddies now; why it is almost as if we are holding these Eddies responsible for the viscosity. And indeed that is exactly what this is saying; then that is indeed true.

We will indeed be appealing to a turbulent source, a turbulent viscosity, not the regular molecular viscosity and we will discuss this in some detail as we go along. But at the moment we will stop here by noting that this is essentially the azimuthal component of the momentum equation, ok. So, this finishes our discussion of the momentum equation and what we will take up next is the Energy Equation. So, we will stop here for the time being.

Thank you.