Fluid Dynamics for Astrophysics
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Lecture - 39
Disk accretion: Mass conservation and vertical hydrostatic equilibrium

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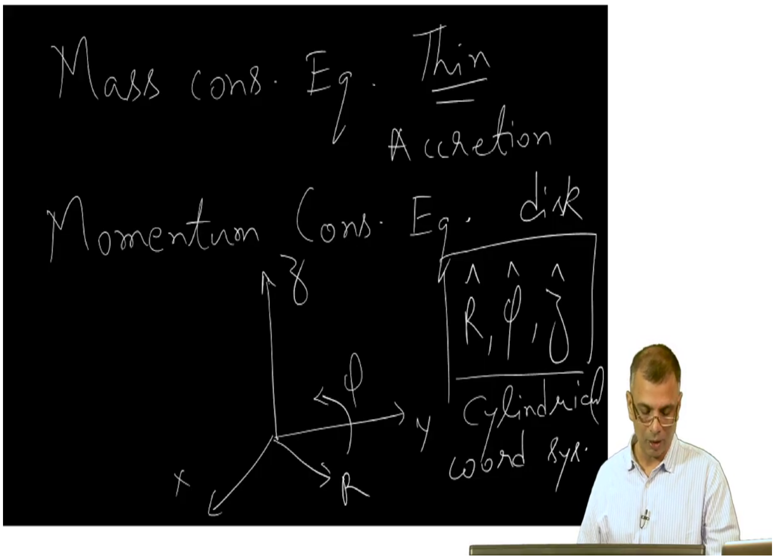

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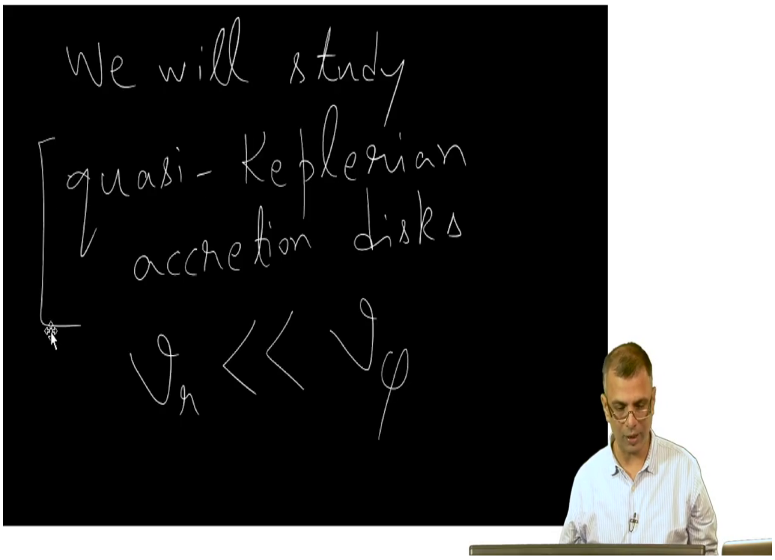

So, what we will be doing now is consider a quasi-Keplerian accretion disk of this kind, right and consider the mass and momentum conservation equation; what else, right. I mean there are only these two things. And try to figure out what the structure of such an accretion disk will look like, ok. Now, before doing that we have to do a little bit of geometry and that is we are going to be talking about a thin accretion disk, ok, a thin accretion disk.

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In other words one which looks like this, ok. What does the word thin mean to you? Well, it looks thin, ok. What do you mean by that? Really the vertical dimensions, it looks like a thin plate, right. So, the vertical dimensions are very small in comparison to say you know the radial dimensions, right. So, this is what we are talking about.

And the appropriate geometry, I mean you one should always when one was talking about spherical accretion the you know the appropriate geometry to the appropriate coordinate system to adopt was a spherical coordinate system. In this case, turns out that the cylindrical coordinate system is much more convenient.

A cylindrical coordinate system would be one where for instance you had the standard Cartesian axes x, y and z like this, right. And a cylindrical coordinate system would be one where you would have an r coordinate in the x y plane, ok. It is lying in the x y plane, and a phi coordinate which would be like this, right, a phi coordinate which would be like this and a z coordinate which is just along the z axis.

So, essentially you would have an r hat, phi hat, and z hat. So, this is the cylindrical coordinate system within which cylindrical coordinate system; this is the coordinate system in which we are working, ok. And we will first consider the mass conservation equation, ok.

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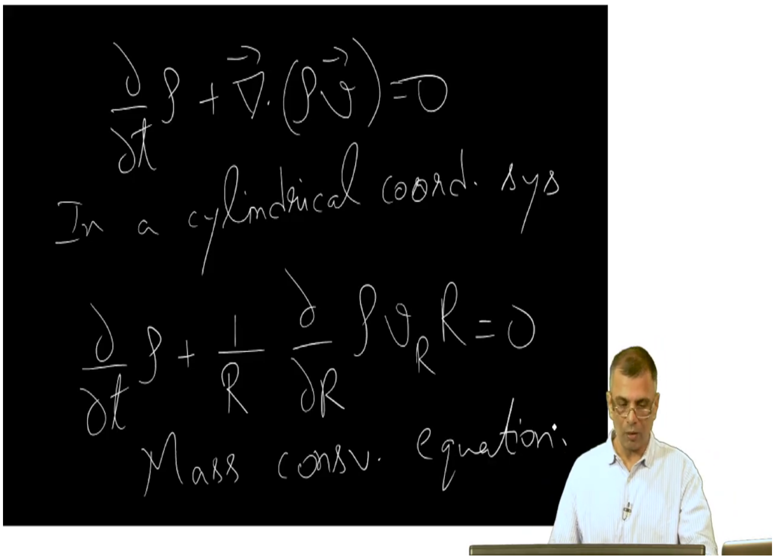

So, you remember what the mass conservation equation look like in general. The mass conservation equation looked like d over dt of rho plus gradient of rho v was equal to 0 this is how it is written in you know general in a coordinate system free manner, where the divergence operator assumes an appropriate form in the appropriate coordinate system, right.

You have for instance in a Cartesian coordinate system this would be just x hat d over dx, y hat d over dy so on so forth. For a cylindrical coordinate system, in a cylindrical coordinate system, which looks like this, this essentially would look like d over dt plus 1 over; and now, I will be writing capital R, in place of there is a capital R essentially means this, ok.

So, maybe I should actually erase this here and write a capital R here just to be consistent, ok. It is also good to write it this way because you know I distinguish between the spherical R and the cylindrical R that I am using. Remember the spherical R would be pointing somewhat like this. Whereas, the cylindrical R this pointing outwards and it is always in the x y plane, ok.

And so, in a cylindrical coordinate system the mass conservation equation would look like this 1 over R. This and this are the same, alright. Now, let me integrate this equation let me write this down once again, ok. This is the mass conservation equation I want to, ok, right. So, now, let me write this, let me rewrite, let me write this down because I am going to integrate it vertically.

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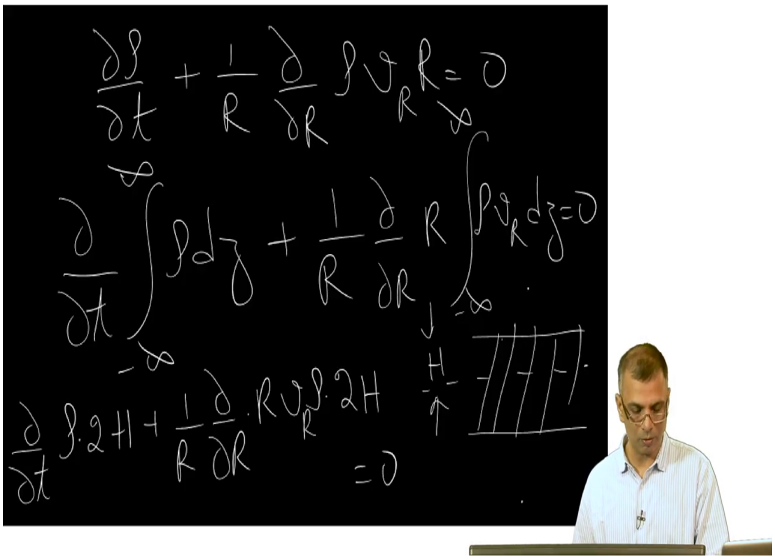

So, let me write it down once again just to be a little safe, so that we do not make any unnecessary mistakes. What is this V sub R? This is all important radial velocity, ok. The velocity at which material is accreting in a radial manner. In a perfectly Keplerian accretion disk this term simply would not be there V sub R would be 0.

There would be no accretion, ok. Mass conservation would simply be this, that is it, ok. Whereas, in this case there is a we assume a finite V sub R and we will see what the conditions of V sub R would be. So, I integrate this vertically, right. So, I integrate this vertically to write like this. In the vertical direction would simply mean in the z direction, ok.

I am assuming that the accretion disk is so thin that there is no appreciable variation in the z direction, ok. We will come to that, but for the time being I integrate this vertically to get what is called the surface density. So, I am assuming that I can interchange the integration and differentiation, so I am keeping the d over dt outside and I am integrating like this.

I am formally writing from minus infinity to infinity, the z extends from minus infinity to infinity, but of course it extends only over a finite thickness. So, the only contribution comes from that finite thickness the rest of it just goes to 0. And now, what I am going to do here is 1 over R, d over dR and R comes outside this R I come I take outside and I allow the rho and the V sub R to be functions of z. So, I can write rho. So, this is essentially a vertically integrated version of this, ok.

Now, if I the assumption is that there is no vertical structure, in other words this rho dz essentially becomes rho times H, where H is the half thickness of the disk, ok. So, it actually becomes rho times 2H, where H the disk looks like this. So, this would be the vertical plane and the half thickness would be H, ok.

So, the vertically integrated version of this essentially becomes. So, I have a d over dt, right rho times 2H. So, this integration, so because there is no vertical structure I can pull the rho out, right and integration of dz from minus infinity to infinity essentially becomes 2H. Because this is nothing else, this is a accretion disk, right and there is nothing here or there, right.

So, this essentially becomes rho times 2H plus 1 over R d over dR R. And I pull both of these the rho and the V R here formally they are allowed to be functions of z, ok. Whereas, if I assume that there is no disk structure I can pull both of these out and this becomes R times V R times rho times, ok. I forgot to write a minus infinity to infinity here. So, this also becomes 2H that is equal to 0.

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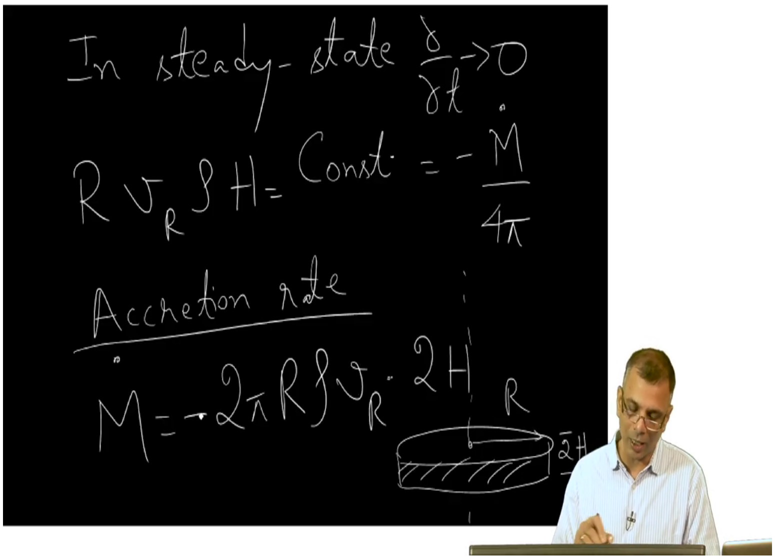

Now, we are almost there, right. So, and now what happens is I get; now I consider in steady state; and you remember what we mean by steady state, right I mean all across steady state simply means that according to the observer in the lab, there is no time variation. In other words the partial, d partial t goes to 0.

So, in this equation I simply throw away this entire term and I only worry about this term, right. So, if I do this and I integrate over R, right, so this essentially yields R equals some constant, which I the constant can be anything. I choose to write this as minus M dot over 4 pi, ok.

So, this is simply an integration of this equation, an integration of just this term of the equation. I integrate this, and so, I just solve this differential equation and this is of course, the all important accretion rate, right. And so, essentially I can write the accretion rate as you will see in accretion rate our entire discussion of accretion disks. The accretion rate for plays a very very important role, ok.

So, the accretion rate is essentially, this is essentially the same equation as this M dot is equal to minus, and the minus is important I will tell you why; 2 pi R rho V R times 2H. Now, so this is a very important equation, the accretion rate equation, right. And so, what does this really represent? Consider a disk a sort of a you know like this, like that, and there is a central object. And so, that would be the you know axis.

And so, what this is essentially saying is that you see what is the surface area of you know the side of the disk that would be 2 pi R, where this would be R of course, ok. So, it would be 2 pi R times 2H 2 pi R times 2H, right. So, that is an area and the area times the density the mass density times you know the accretion velocity, obviously, gives you grams per second. So, that is essentially what this equation is saying, ok.

So, you are essentially accreting in this kind of a thin disk geometry, and that is what this is saying and the minus simply arises from the fact that the V sub R, R as such is pointing outwards whereas, the V sub R is inwards we are talking about accretion. So, the V sub R intrinsically has a negative sign and that cancels this negative sign and you get a positive accretion rate, ok.

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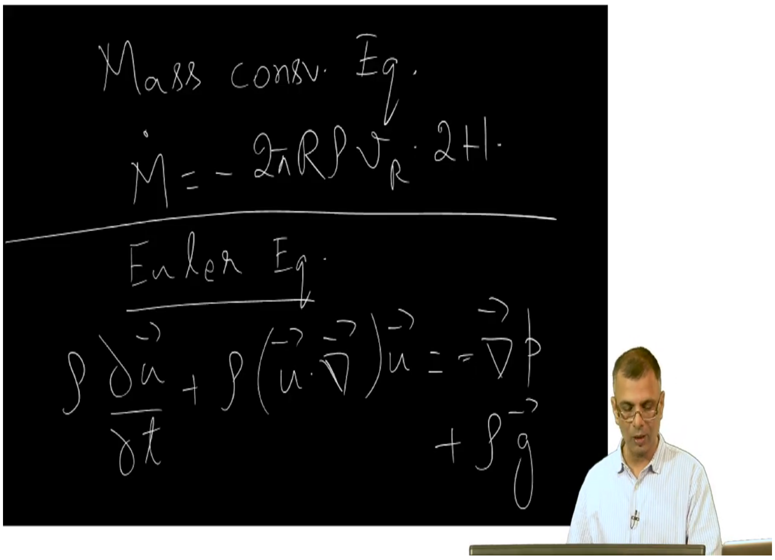

So, the mass conservation equation, just to emphasize, the mass conservation equation is just is just M dot equals minus 2, this is the same thing that we wrote down on the previous slide, no different, ok. So, we are done with our discussion of you know mass conservation, right. So, we are done with that.

And now, let us look at the momentum conservation equation, right. So, the momentum conservation equation in general we look at the Euler equation, in other words we neglect viscosity, except of course, for yes and no for the time being let us just proceed with this.

The Euler equation generally is given by rho like that this is a nabla minus gradient of pressure plus are the all important body force system. This is the version of the Euler equation that we have been seeing so far, right. And in this case what will happen is we will have 3 different versions, I mean 3 different components of the Euler equation.

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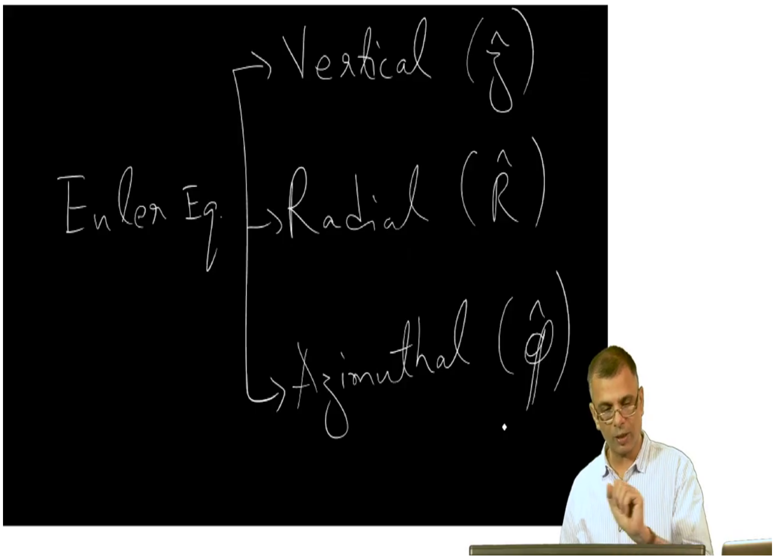

We will consider the, of the Euler equation we will consider, we will consider the vertical component which is essentially the z hat component, right. And we will consider the radial component which is the R hat component, and we will consider the azimuthal component. We will consider these three components of the Euler equation separately. It is important, ok, right. So, let us now consider first the vertical component of the Euler equation. Let us see how that looks like.

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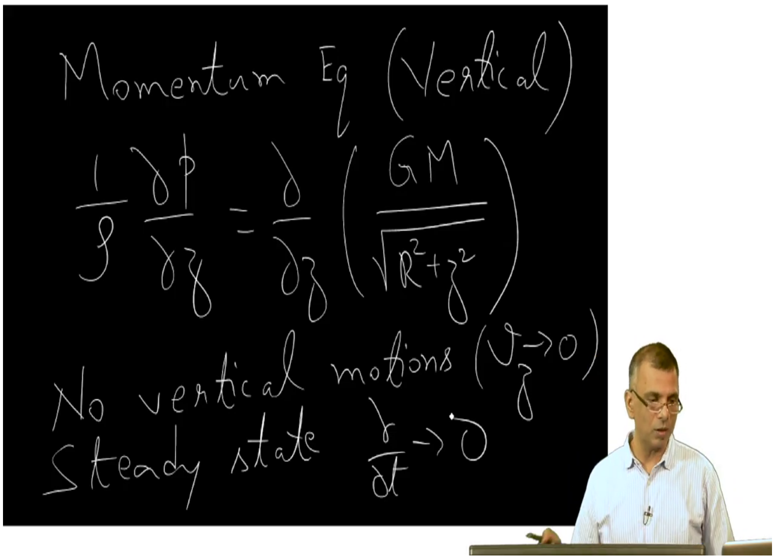

And the vertical component of the Euler equation of the momentum equation. The momentum equation the vertical component looks like 1 over rho dp dz. So, you only have the pressure gradient term and this is equal to d over dz, I will explain what these terms are in a minute.

But before that I just want to say that this assumes that you see you look here I am considering; well first of all its steady state therefore, this term is completely gone, right. And the other thing is I am assuming that there are no velocities at all, there are no vertically directed velocities, ok.

In the vertical direction, you only have a competition between the pressure gradient and the gravity, that is what this is, you have the pressure gradient and gravity that is all you know. So, this entire thing is essentially the minus rho times small g, ok. So, what this is assuming is that no vertical motions, i.e, V z 0 and it also of course, steady state these are the two important assumptions that go in d over dt goes to 0. So, these are the two important assumptions, ok.

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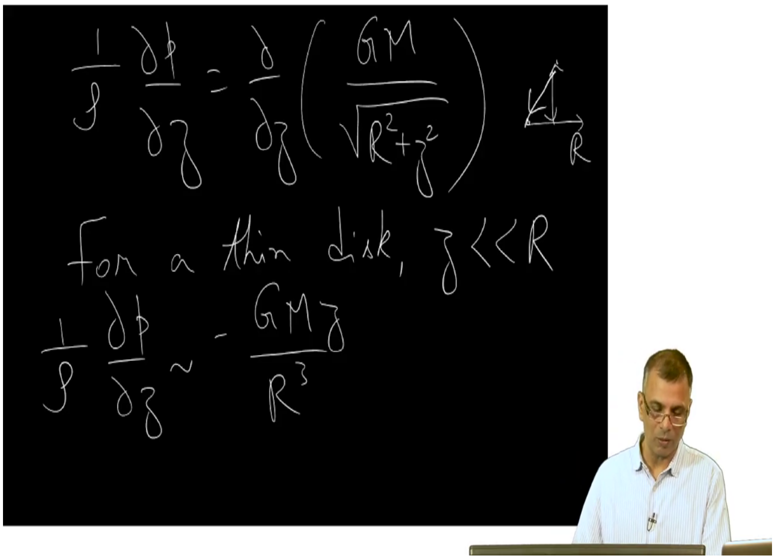

So, you see we had written down, you know the vertical structure the vertical component of the momentum equation as this. And on the right hand side is essentially the gravity term. This is not just G M over R, G M over R squared plus z squared it takes into account the fact that you have the central object and you are at a certain distance R from here and at a certain height z.

So, you are really considering this, you really this takes into account the fact that the gravitational attraction is not simply along the radial direction is actually along this direction that is what this is all about. But at the same time having done that we then recognize that for a thin disk z is much much smaller than R, right.

So, this essentially becomes 1 over rho approximately, ok is equal to minus G M z over R cubed. Where does that come from? You pull out, you pull out an R, ok you write 1 over z squared over R squared and then you do a binomial expansion, and the fact that the z over R is a small parameter, right and so, this simplifies to this, ok, alright.

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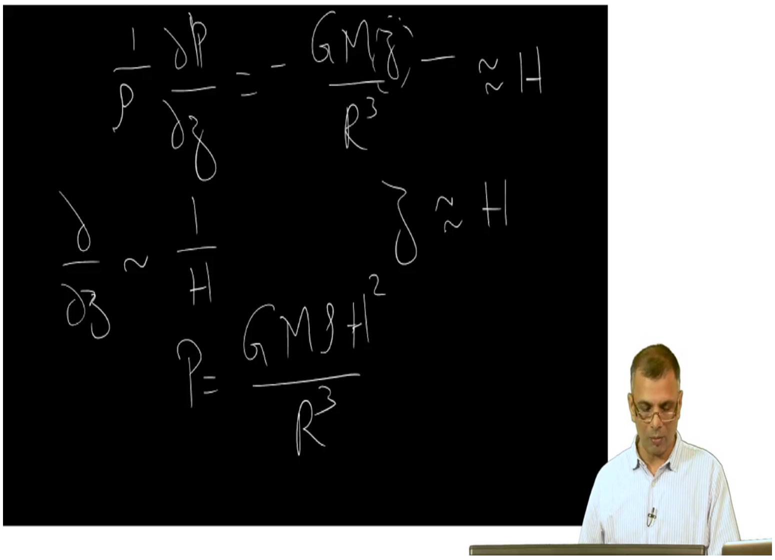

And next what we do is we say, so what we have done is we have come to a stage where we have written 1 over rho, sometimes I write it as a capital P sometimes I write it as small p, its all the same there is no difference, write this. Now, we do a horrible thing which is also a very practical thing, ok.

We have seen the spirit of doing things earlier in the course. So, what we do is we say whenever we see a d over dz we replace this as 1 over H, right that is because we really have no vertical structure, ok. So, d over dz can be replaced by a 1 over H, ok. And we also replace this z here as approximately H.

So, and we also say z is approximately H. So, in this spirit of approximation we can write the vertical momentum equation essentially becomes P equals GM rho H squared over R cubed. And not to despair this will all become much simpler as we go along, right.

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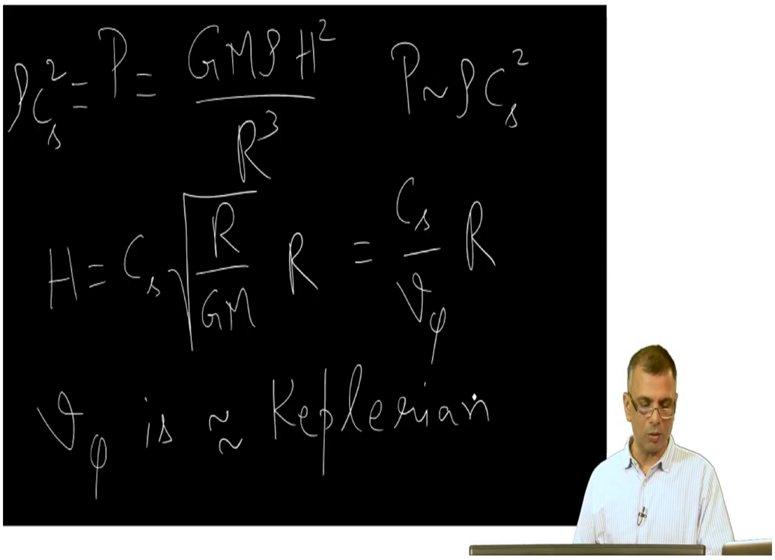

So, let me go to the next slide and reproduce this what we have written is that we have simplified the vertical component of the momentum equation to look like this. And just to emphasize how did we get here? We took the vertical component of the momentum equation, right. Actually, it is, strictly speaking it is this one.

We made the assumption that we have a thin disk, so z is much much less than R. So, I do a binomial expansion in the small parameter z over R and that gives me this. And I am still not satisfied, I say I really do not want this partial differentiations I would just write you know d over dz is something like 1 over H and here in place of the z I just replace it with H. And that gives me this, that gives me this, ok.

Now, what I am going to do is I am going to say we have a well-known relationship between P and rho, right. We have the sound speed, right. So, I am going to write P is something like rho C s squared you agree with this. And now we are back to writing the sound speed as C sub s and not a, as we used to do in spherical accretion; it is always. I hope you are you are able to keep up you know with these changes of notation and they are fairly transparent, ok.

Now, if we substitute this as if we write this as equal to rho C sub s squared, then I get a ready expression for the height, ok. The height of the thin disk is essentially equal to C sub s square root of R over GM times R. And we have seen this before we have seen this quantity square root of R over GM, you remember right. This is essentially one over the Keplerian velocity. We have seen this before V phi, right. So, V phi is equal to square root of GM over R, we have seen this.

So, using that here what we find is that this is equal to, this is essentially equal to C s is already there, C s times V phi is a Keplerian velocity times R. In other words I would write, in other words the H over R is nothing but C s over V phi. And this is such an important relation that I will write it on a separate slide.

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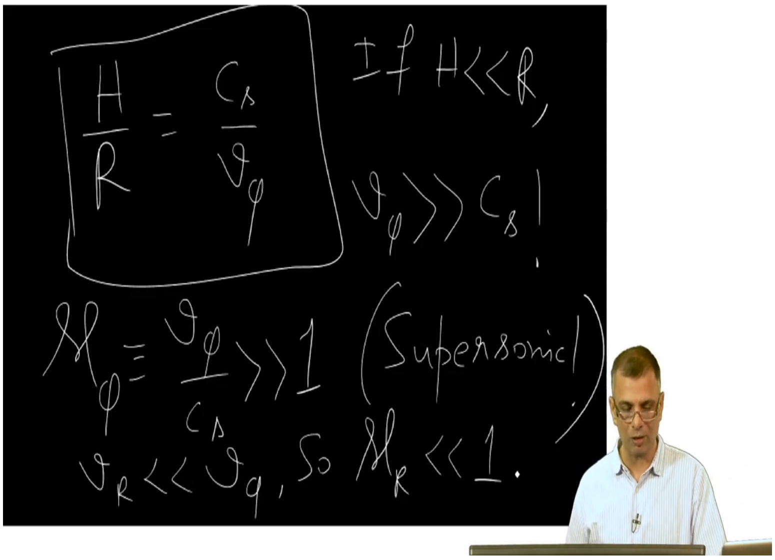

In other words, H over R is equal to C s over V phi. So, this is a; this is a measure of how thin did the disk is, the quantity H over R, ok. If H over R is small; that means, the disk is small and that has a certain demand on, ok. So, the other thing of course is we are assuming that this is indeed V phi. In other words the disk is indeed quasi-Keplerian. So, here what we are implicitly saying is that V phi is very nearly Keplerian.

So, this is where we make the assumption of a quasi-Keplerian accretion disk, ok, right. So, here what does it mean? So, what would the H over R? If you had if you foresaw the fact that the disk is thin, in other words, if H is it means that V phi has to be, right. In other words, the azimuthal Mach number which is defined as V phi or C s very supersonic, right.

So, if you have a thin disk it automatically assumes that you know the azimuthal Mach number is much much larger than 1, ok. And in a quasi-Keplerian disk of course, the whole assumption of quasi-Keplerianity is that V R is much less than V phi. So, the radial Mach number is much much less than 1. In fact, V R is almost 0 in comparison to V phi. In fact, we have actually assumed that V phi is exactly equal to Keplerian. If that is the case, if V phi is exactly Keplerian we have seen earlier that V R is technically 0.

So, V phi is not we, although we call this V phi you know it is not exactly Keplerian there can be some departures from Keplerian value, which gives rise to a small, but finite radial velocity. But any rate the radial velocity is much much smaller than the azimuthal velocity. And azimuthal velocity is highly supersonic if the disk is thin and you can see that right from here.

A thin disk automatically implies a supersonic azimuthal velocity, ok, alright. So, and of course we have, in writing this as a momentum equation we have you know we have assumed that there are no vertical motions, so V z is equal to 0. So, this is you know one important part. The next thing, So, we what we have done is we have derived a very important thing from the vertical component of the momentum equation.

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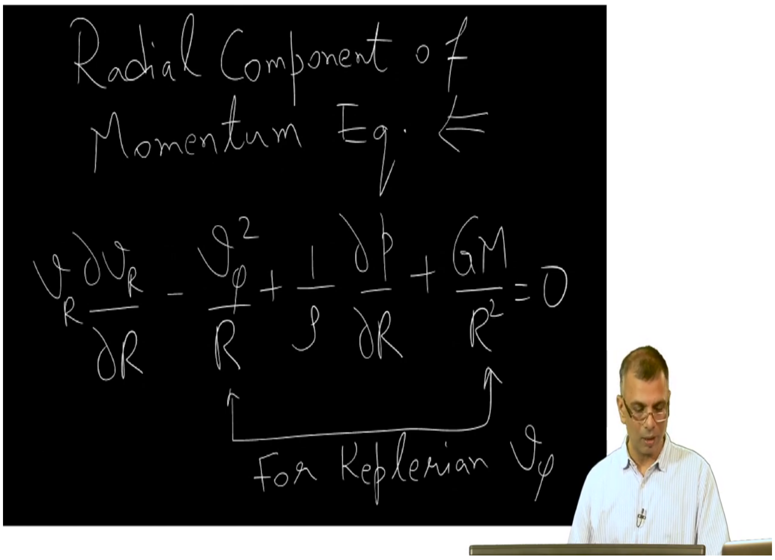

Now, let us turn our attention to the radial component of the momentum equation. Remember we started out by saying that we will split up the momentum equation, we will consider the vertical part the radial part and azimuthal part all separately. So, now, we consider the radial component of the momentum equation, which in cylindrical coordinates looks like this, ok.

This would be the u, u dot grad u term, right both of these taken together. And then you have the pressure gradient term of course, except now the pressure gradient we are talking about the pressure gradient in the radial direction, right and then of course, the gravity term. And we do not have to worry about the square root of R squared plus z squared, we are staying on the equatorial plane therefore, we only have to worry about this.

Now, you immediately realize that we have written a V R here, ok. This is assuming that you know there is a small, but finite V R, ok. Now, if the azimuthal velocity is Keplerian, right, we know that the second and the last terms in the equation these two cancel each other. These two are exactly the same, this is equal to this, if the azimuthal velocity is Keplerian, right. So, this term and this term exactly cancel each other for Keplerian V phi, ok, right.

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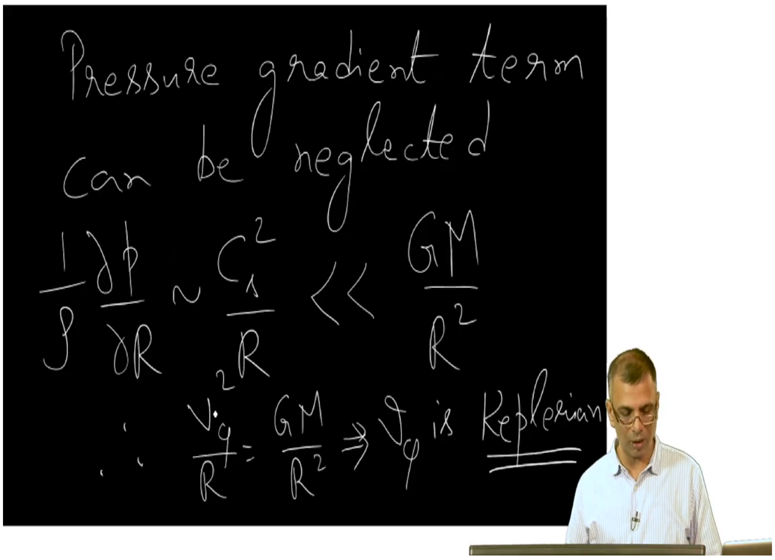

And the pressure gradient term can be neglected. Why? You see the 1 over rho dp dR is something like C s squared over R, right. And because you know the p over rho dp d rho p over rho roughly equal to C s squared, and then 1 over the d over dR is roughly 1 over R, ok. And this we know is much much smaller than GM over R squared because GM over R would be V phi squared, so this would be essentially V phi.

So, this becomes GM over R squared. This is simply this follows from here, this follows from this fact, ok. So, therefore, the pressure gradient term can also be neglected, right. Therefore, so essentially the pressure gradient term is neglected, right and you know the V R is small, but finite, ok. But since it is very small in comparison with the other terms it can essentially be neglected, right.

And so, we are left with the inescapable conclusion that minus V phi squared over R is equal to GM over R squared, ok, and so not minus V phi square over R V phi squared over R is equal to GM over R squared. In other words, the V phi is Keplerian. In fairness, this was something that we already knew, ok and. So, essentially, so in other words we know that, we are already assumed that the rotation was quasi-Keplerian.

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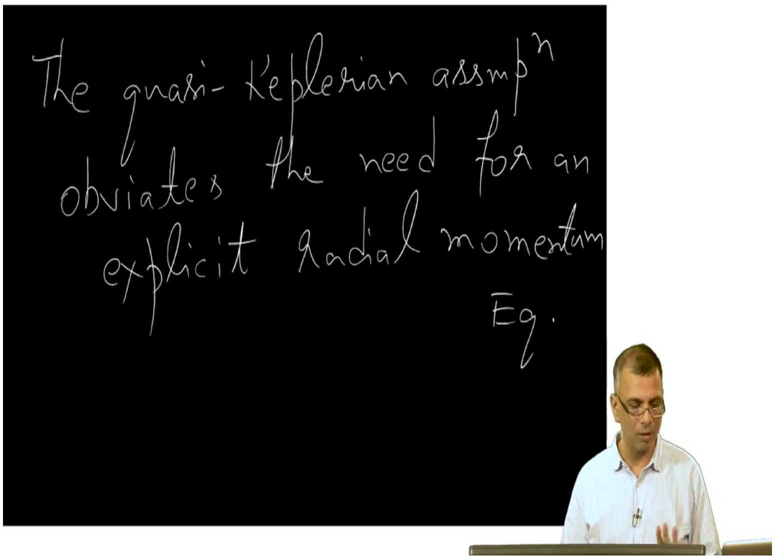

So, therefore, the quasi-Keplerian assumption, assumption essentially obviates the need for a radial for an explicit radial momentum equation. If you assume quasi-Keplerianity, then you immediately say that this and that balance each other, but of course, you have to realize that the V R is very small and you have to justify the neglect of the 1 over rho dp dR.

And the neglect of the 1 over rho dp dR comes from the fact that you know the C s is much much larger than V phi and that came from the vertical momentum equation, ok. So, at any rate we are already considering quasi-Keplerian equation disks to begin with and therefore, we find that that is what the radial momentum equation is telling you in any case. So, we can forget about that. No need to worry anymore about the radial momentum equation.

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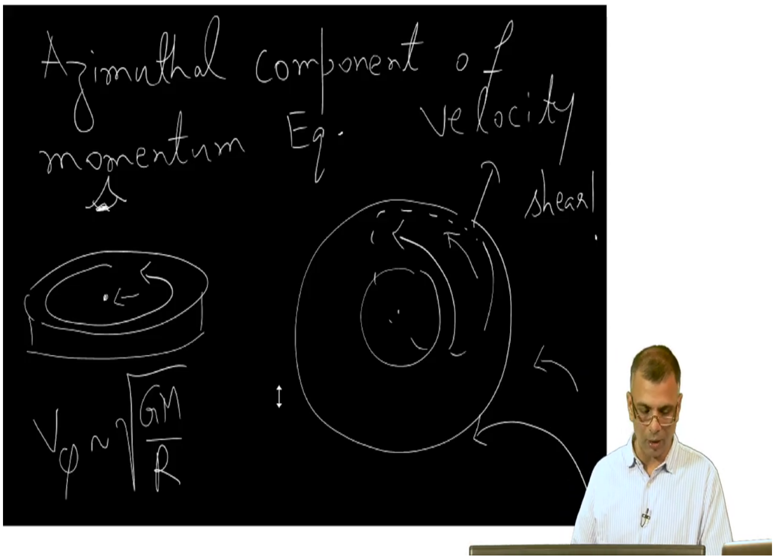

Now, the azimuthal component, this is a very important component. And the azimuthal component of the momentum equation this involves viscosity, ok. Why is that? Well, you see suppose you were looking top down and we will not write down the equation, right at the moment, but suppose we were looking top down on the accretion disk, right.

So you have in one picture you have you know the central object here and accretion disk here, which looks somewhat like this with a finite thickness, right. And matter is swirling round and round in a highly supersonic manner and at the same time there is also very slow inward drift very very slow inward drift, ok.

Now, suppose I looked top down, I looked at this accretion disk from the top, ok. What would I see? I would see orbits like this, like this, yeah. And what does the V phi look like? The V phi goes as square root of GM over R, is not it. So, the essentially matter is that the V phi here at larger radii is smaller and the V phi at smaller radii is larger, you agree.

So, the length of the vectors, maybe I should draw this a little better. So, the length of the vectors are such that at smaller radii because the R is in the denominator, at smaller radii matter is swirling around faster than it does at larger radii. Now, what this is what is this telling us? This is telling us that this entire thing points to the fact that there is a velocity shear. The fact that you have a smaller velocity there and a larger velocity here, ok. This is nothing, but shear. You can always transform to a frame to a rotating frame where I can draw this like this.

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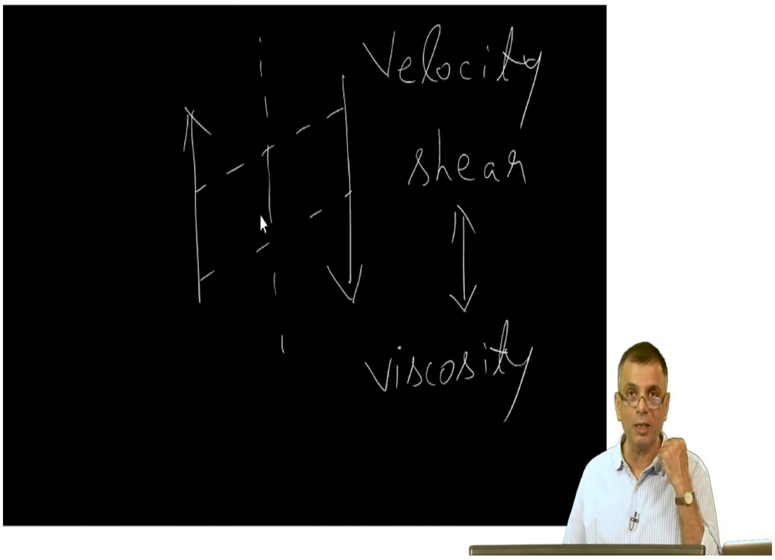

I have you know, I have an in between you know layer at rest and on one side of the layer the velocity is going up that way and on another side the layer the velocity is coming this way, so this would be the V phi, ok. And this is exactly what a shear flow is, ok. So, what happens now? Now, the reason I said viscosity is important is because it is exactly in this kind of situation that you think about is exactly in this kind of a situation where you encounter shear that you start thinking about viscosity, right.

So, you start thinking about little rubber bands linking this layer to this layer, preventing the sliding, preventing one layer from sliding with relative to the other layer and these little rubber bands are essentially a way of thinking about viscous forces, ok. So, this is one thing.

So, the presence of velocity shear is intimately linked with viscosity, ok. If there was no viscosity there would be no shear. In fact, because you know the layers would freely slide with respect to each other is the fact that there is viscosity in the fluid, in other words there are these little rubber bands connecting the two shear layers that is what gives rise to the phenomenon of shear itself. And here if you are assuming a quasi-Keplerian disk, it has to have velocity shear.

There is no way around it. A quasi-Keplerian accretion disk there is azimuthal velocities look like this. So, this is since the R is in the denominator at smaller r, you have a larger V phi as compared to a larger R and this is essentially velocity shear like this and you can transform it to something like this, ok.

Now, why is viscosity important for us? Because we know that apart from appearing in the momentum equation viscosity also appears in the energy equation, right because viscosity is a dissipative process.

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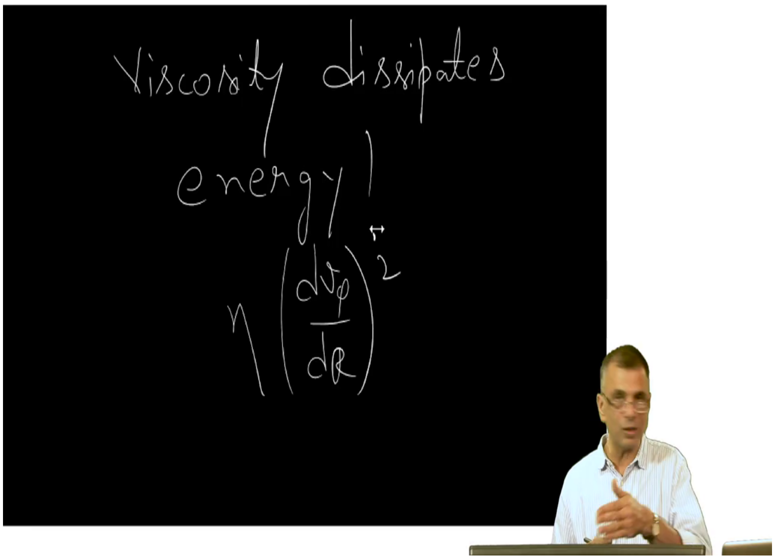

Viscosity dissipates energy, the rate of dissipation of energy is something like, something like this that is the rate at which viscosity dissipates energy where this is the coefficient of viscosity, ok. Now, why is energy dissipation important because that is the whole point, is not it.

You wanted to transform; and what energy is dissipated? It is gravitational potential energy that is dissipated. You want it as large an M dot as possible. And where is M dot coming from? It is coming from the gravitational potential of the central object that is fine, but the ultimate observable is not the accretion disk itself, it is the photons that are emitted from the accretion disk.

And why are the photons emitted? That is because accretion disk gets hot. And why is accretion disk get hot? Because there is viscosity in the accreting fluid which dissipates energy that heats up the fluid and a heated fluid essentially emits photons, assuming there is a black body, ok.

So, viscosity is central to not only the physics of the accretion process; if we had no viscosity you would not have accretion at all, ok. And it is also central to explaining the fact that the disk is luminous. Why is the disk luminous? Because it is hot. Why is it hot? Because viscosity dissipates energy, ok.

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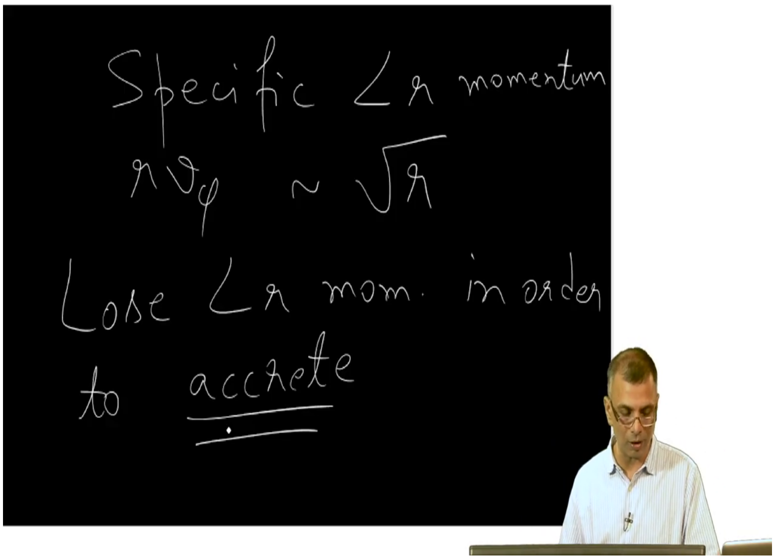

You can also look at you know the specific angular momentum, the specific angular momentum which is something like r V phi, ok. This goes as square root of r. In other words, larger radii have more angular momentum as compared to smaller radii. The larger radii rotate slowly, but they have more angular momentum as compared to smaller radii, that is what this is telling you.

So, matter has to lose angular momentum in order to, matter has to lose angular momentum in order to accrete. And what causes it to lose this angular momentum? Viscosity; and viscosity also does another very important thing. It heats up the disk and makes it radiate which is what ultimately makes the disk observable. So, we will stop here and we will consider the azimuthal component of the momentum equation when we start next.

Thank you.