Fluid Dynamics for Astrophysics
Prof. Prasad Subramanian
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture - 33
Spherically symmetric transonic flows (contd)

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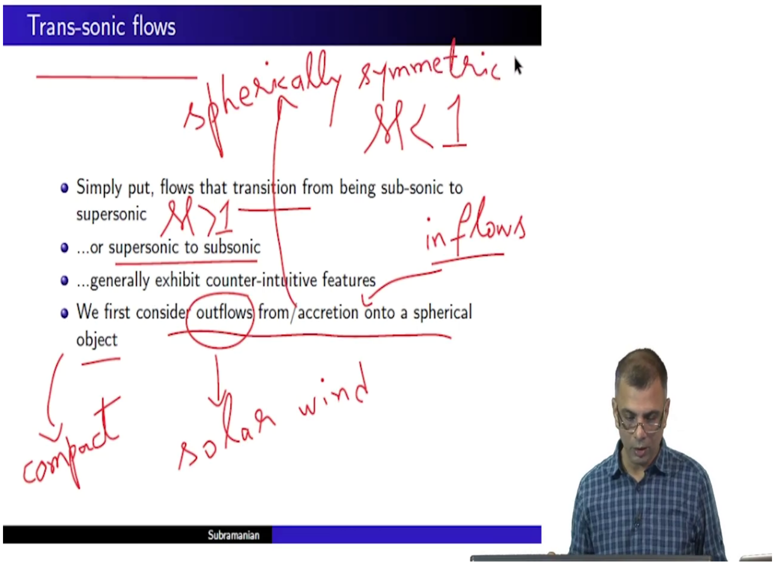

So, we will continue with our discussion of Transonic Flows. So flows that transition from being subsonic which is to say M Mach number less than 1, to supersonic which is a say mach number greater than 1 ok; or supersonic to subsonic either way; either way there is a transition from subsonic to supersonic, right. And we did this very briefly the last time. And we will do it a little in some more detail inflows and outflows accretion onto a spherical object.

And as we remarked the last time, the mathematics is exactly the same. When we are talking about outflows essentially we are talking I mean you know the physical situation, we will be considering as the solar wind. And when we are talking about inflows, inflows are spherically symmetric inflows.

This problem, the physical basis of this problem is accretion onto a spherical object, I should say a spherical compact object. Again I really should not say spherical object, a compact object spherically symmetric accretion. I really should say spherically symmetric, this is really what we are you know talking about, right.

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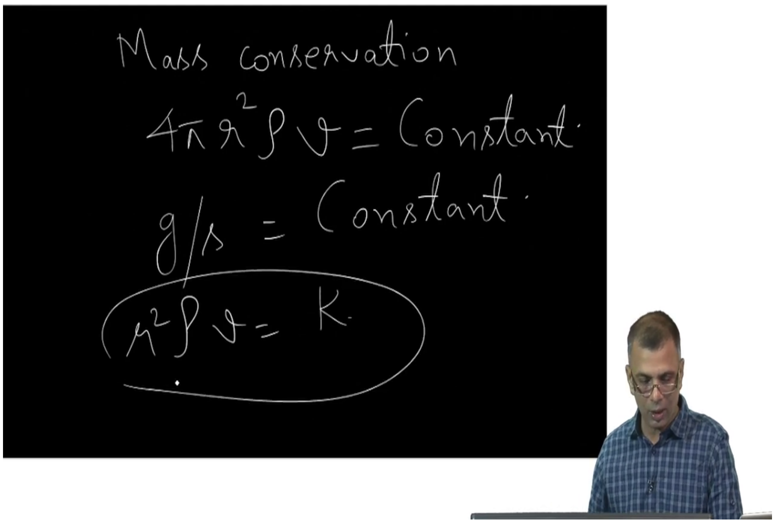

So, let us start off like this we introduce mass conservation. In the following guys, we said, which is the area of a sphere right, 4 pi r squared is the area of a sphere of radius r. So, the amount of you know that is the area, and this is the mass density, right, and this would be the velocity.

The velocity can have either sign either it is accretion, which would be in fall or it is you know solar wind in which case it is an outflow either way this is constant. And as we remark the last time, this would mean you know this is the area, this is the mass per centimeter cubed, and this is centimeter per second.

So, what this is essentially saying is that the grams per second is equal to constant. Equivalently I really do not need this 4 pi, this can be absorbed in the constant. Therefore, I can write this as r squared rho v equals some constant. Equivalently if I you know so remember this equation, and I choose not to. So, let me write this again.

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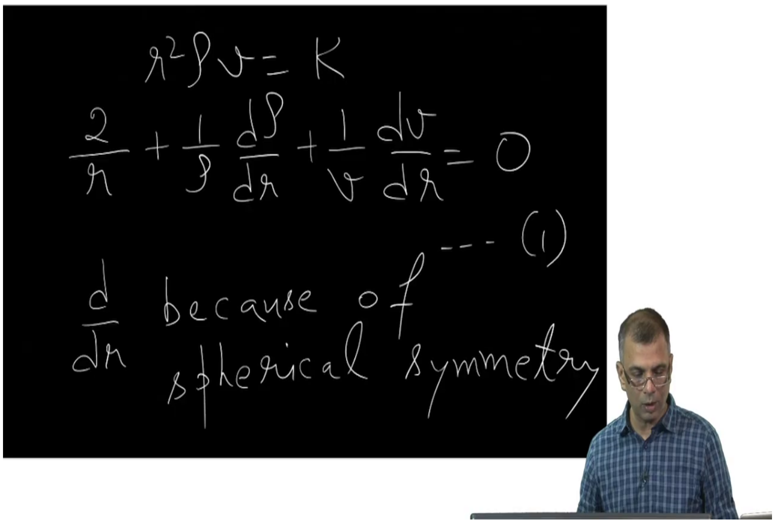

And I choose not to use this form of the equation. I differentiate both sides with respect to r, ok. And recognizing that rho is a function of r, v is also a function of r, but K is not a function of r.

When I differentiate both sides with respect to r, the mass conservation equation essentially becomes, remember these are straight derivatives, not partial derivatives because of the assumption of spherical symmetry. 0 because you know d K d r is 0, right. So, this would be equation 1 and alright.

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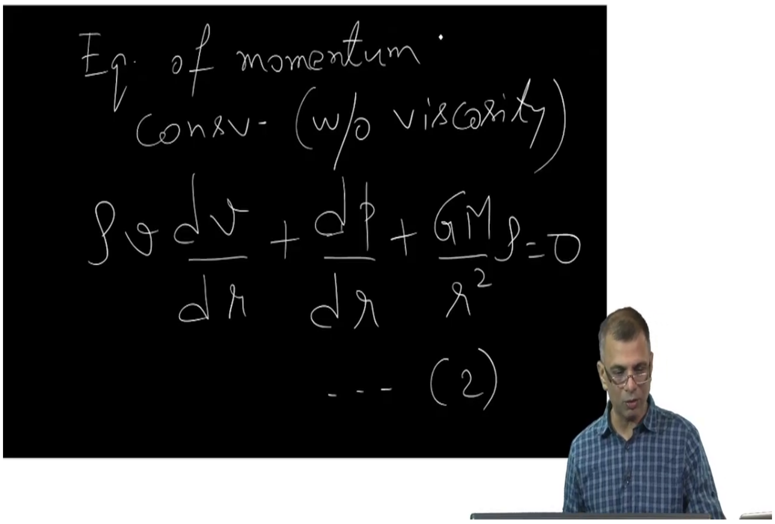

So, the next one similarly the momentum conservation equation, equation of momentum conservation without viscosity in viscid fluid; in other words Euler equation is written as I realize I was using u for the velocity in my slides, and now I am using v, I hope you know it is you understand what I am writing it is not a big deal.

Now, what is the new thing? We have been writing the Euler equation over and over again many, many times in this course. Most of the time, we have been neglecting gravity, but now of course gravity is vitally important, ok. So, no viscosity, but gravity is included, the body forces are included, ok.

In lab situations, gravity does not really matter. It is good enough to ignore gravity especially when you are for inferences dealing with horizontal flows, or you are dealing with flows whose vertical extent is very limited. It is to ignore gravity, but not here, ok. Now, what you do? You try to combine these two equations.

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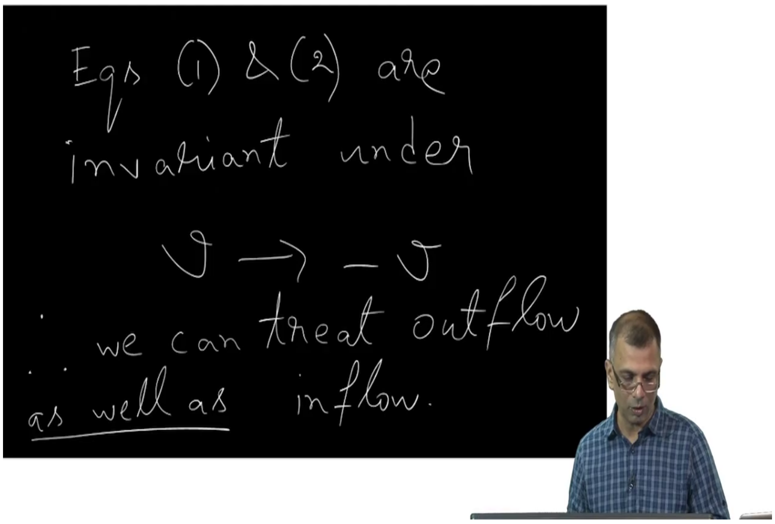

So, the first thing is that to note is that equations 1 and 2 are invariant under, ok. You can see this. It is evident from here. We replace v by minus v here, and you replace v by minus v here, it is the same equation. It does not matter. So, which is why, therefore, can treat outflow as well as inflow with the same two equations.

In other words, we can treat the problem of a spherically symmetric outflow which would be the solar wind as well as a spherically symmetric inflow, which would be spherically symmetric accretion, spherically symmetric accretion onto a compact object. Both of these can be treated you know uniformly using the same formalism.

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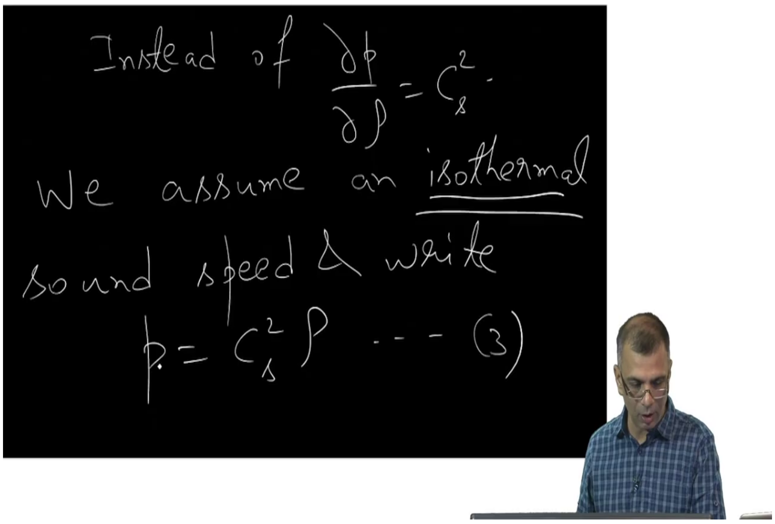

Now, what we do is we write instead of we write we assume an isothermal sound speed, and write p equals C s squared rho. Now, this is equation 3, ok. So, this is another very important step. We are assuming an isothermal sound speed. So, in order to eliminate the p here, you see this p ok, I eliminate this p. So, I can write. So, essentially this rho v essentially what happens is this thing this.

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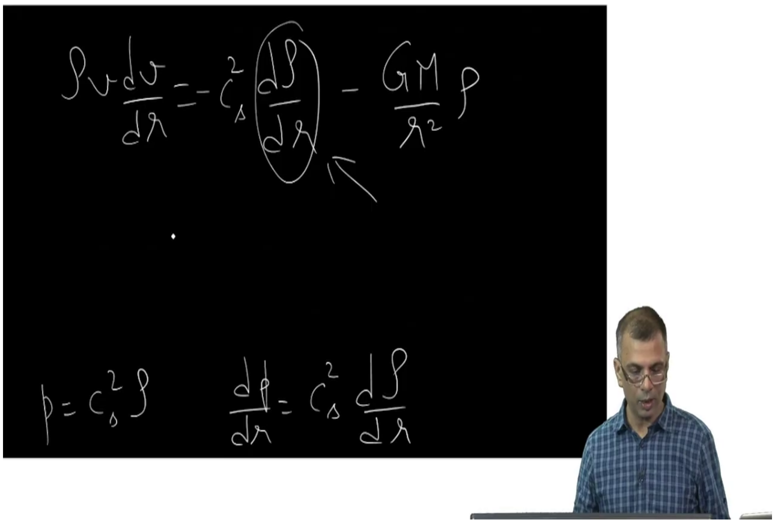

Rho v equals instead of writing minus you know dp dr, I can write minus remember we had written p equals C s squared rho, right. So, dp dr is equal to C s squared d rho dr. You agree? So, I had a dp dr here. And instead of that, I can write C s squared d rho dr right, minus ok.

And now, I have this d rho dr. And I substitute for this d rho dr from not here, sorry from here. You see I have an expression for d rho dr here. I substitute for d rho dr from the mass conservation equation into this equation here. When I do that, I get the following equation, ok.

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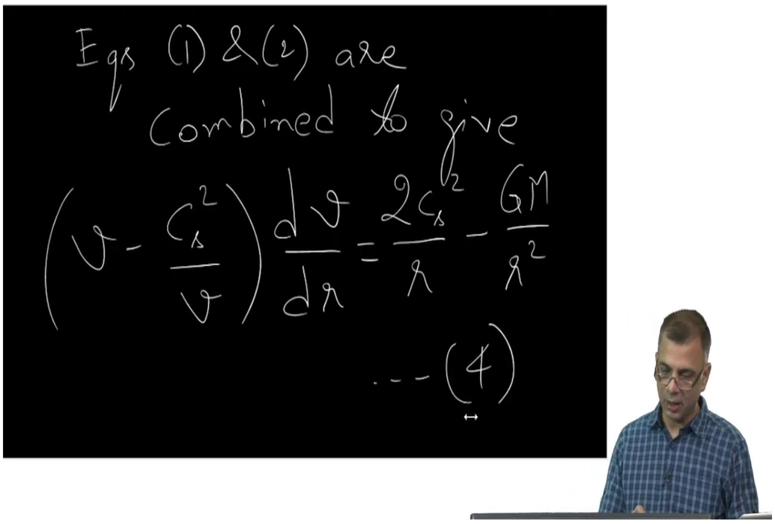

So, equations 1 and 2 are combined to give v minus C s squared over v d v dr is equal to I have told you exactly what the steps are, and I urge you to work them out in any case 2 over r minus, and I believe I call this equation 3. So, this should be equation 4. And equation 4 is the most important equation we need to worry about. Now, one is I mean the main thing about this whole transonic business you can write this.

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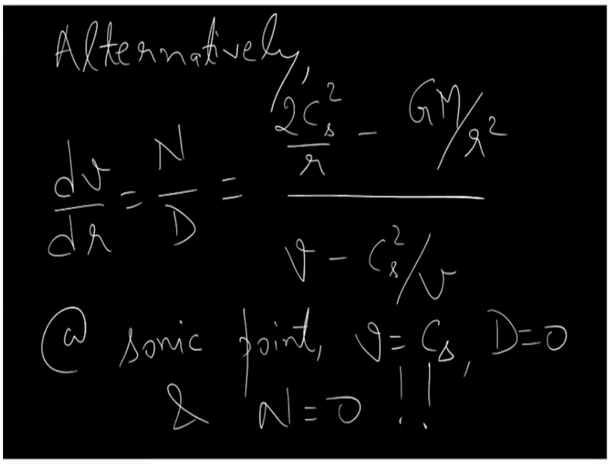

You can alternatively write this as, and this is a slightly friendlier way of writing it I would say, alternatively is some numerator over some denominator where the numerator is 2 C s squared over r minus over r squared and the denominator is v minus, ok. Now, what happens here? You see we are dealing with transonic flows remember.

And in other words, flows that transition from subsonic to supersonic or supersonic to subsonic is not it? Ok, but the main thing is either way the transition happens they pass through the sonic point, ok.

So, at the sonic point, in other words they pass through Mach number equal to 1. So, this you see this equation becomes something like v square minus C s squared, and the v goes upstairs yeah. So, when this becomes v square over C s square, the denominator becomes 0, and the numerator also becomes 0, ok.

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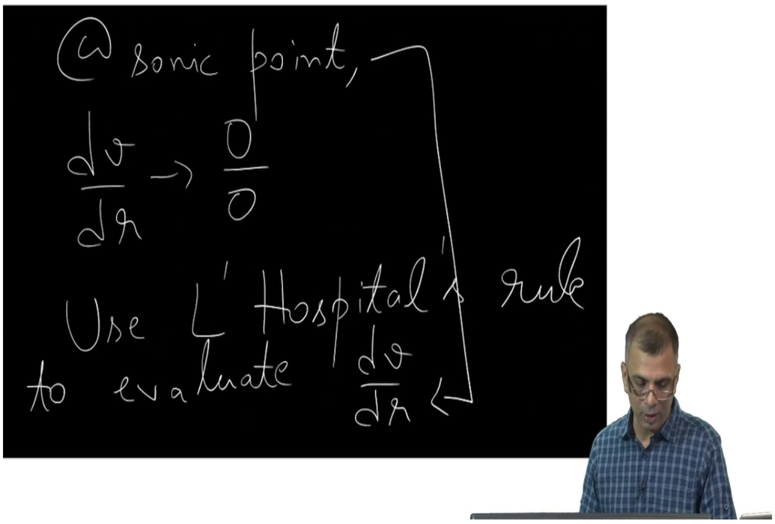

So, in other words, at the sonic point the (Refer Time: 13:36) assumes 0 over 0 kind of character. Now, what do you do in such a situation? In order to you know deal with this kind of a problem, you do not just throw up your hands. What you do?

You would remember from your, you know 12 standard calculus. You use at the sonic point of course, you use the L Hospital's rule, alright. Now, let us now. So, this is one thing you should remember, right. Now, let us now pay a little more attention to ok.

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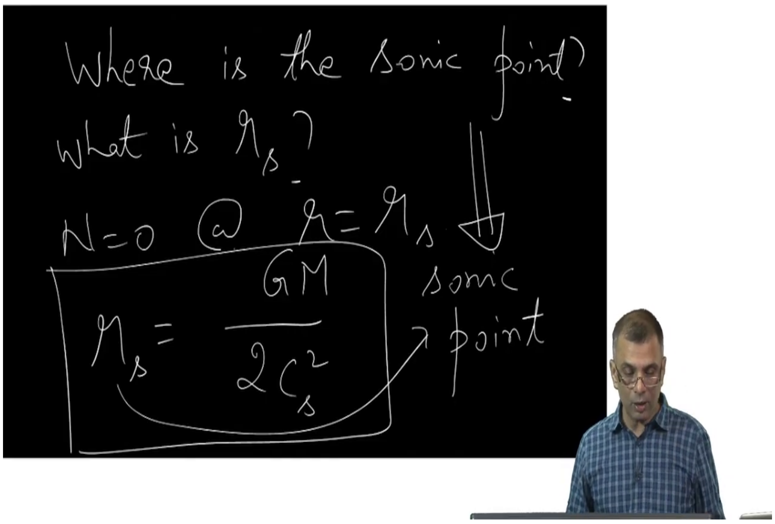

So, we talked about the sonic point, but let us now pay a little more attention to where is the sonic point ok. I probably jumped the gun in writing this; we will come back to this. You might be it is evident how the denominator becomes 0. You see at the sonic point v equals C s, and v squared equals C s squared. So, it is evident how the denominator becomes 0 that is evident.

Now, if you do not want the thing to remember is that if you do not want the dv dr to blow up, then you want the numerator also to become 0. So, that you obtain this kind of a 0 over 0 kind of a situation. And then you will use L Hospital's rule to evaluate dv dr. So, this is the thing.

Now, when does the numerator become 0? Well, let us look at the numerator. Suppose, I want the numerator to become 0, in other words I force 2 C s squared over r to be equal to GM over r square. And this does not happen everywhere this happens only at the r equals r sonic ok.

So, let us now ask where, is the sonic point. In other words, what is r s is that place where the numerator ok, in other words r s is equal to over 2. You can see this; you can see this directly from here. If I want the point is clearly the denominator goes to 0 when v equals C s. Now, you do not want dv dr to be blowing up.

If the denominator goes to 0, and the numerator remains non-zero right; dv dr blows up. It becomes infinite. I mean you know if that is not if you want to avoid if you want dv dr to have a well behaved nature, remember we are talking about smooth flows here we are talking about flows that do not exhibit abrupt jumps that are smooth and mathematically well behaved.

So, if you want the flow to be mathematically well behaved, you would demand that the numerator also become 0. And where the numerator becomes 0? Well, the numerator becomes 0 at r equals r s. So, if you are asking that the numerator becomes 0, 2 C s squared over r has to be equal to GM over r squared.

And that happens at an r, which we will now call the r sonic equals G M over 2 C s squared. So, this is the so this r s is the sonic point, ok. This is the answer to this. Where is the sonic point? Well, this is the sonic point, ok. Now, even there and that is where, so this is I should have I forgot to put this I should have had this slide later, but that is ok, I mean we have discussed this. So, the denominator becomes 0.

And in order to ensure that the dv dr remains well behaved, we also demanded that the numerator equal to 0 at the sonic point. And now the dv dr having a 0 over 0 kind of character is not a problem, ok. We know how to deal with it. We use L Hospital’s rule to evaluate dv dr at the sonic point, ok.

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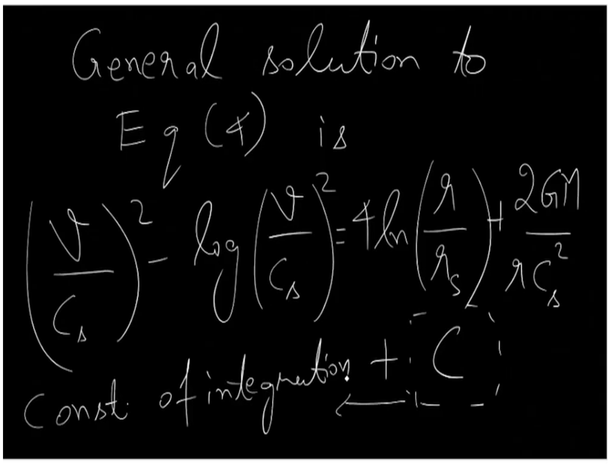

Now, so the general solution to equation 4, this is the equation 4, this is the main governing solution. The general solution to equation 4 is v over yeah minus log you can verify this r sonic plus over r ok, where this is the constant of integration. This is the, is constant of integration, ok.

So, now, depending upon what values the C takes right, depending upon what kinds of values the constant of integration takes, you can have several different kinds of solutions ok. So, essentially what I will be now showing you is just plots of this solution, you see this is essentially a solution for v ok, plots of this solution for different values of the constant of integration. And this turns out to be a very rich kind of behavior ok.

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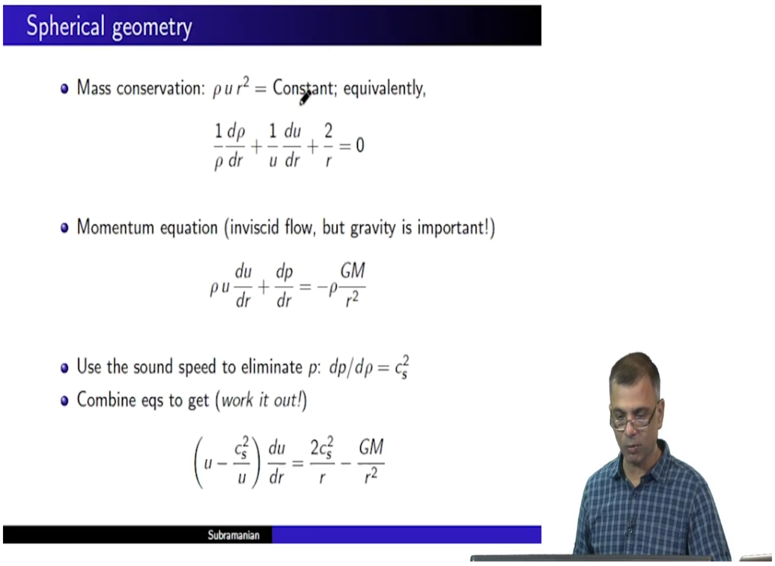

The behavior can exhibit lots of different things and. So, we discussed all these.

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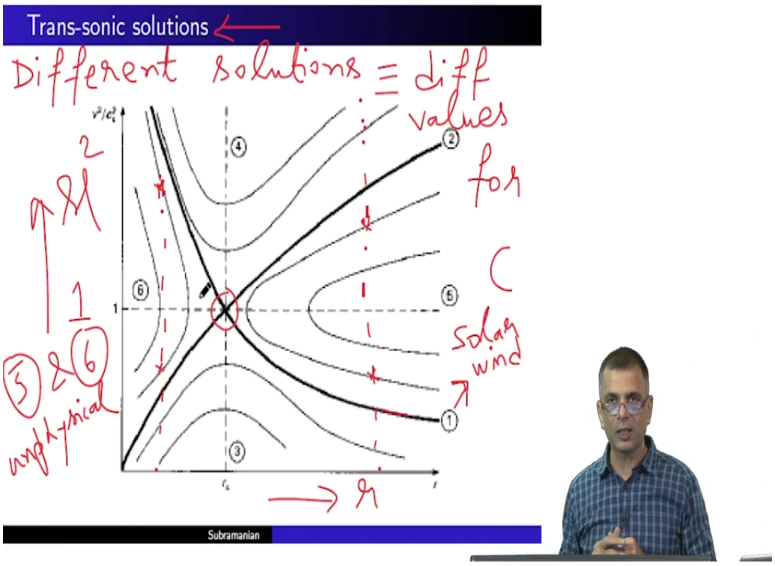

So, the different, different kinds of solutions, different values for the constant of integration C, ok. So, there are several different kinds of solutions here. And what this is? Is this is the square of the Mach number, and this is the radius that is what is plotted here. It is written here, but this is very small you cannot see it. It is v square over C s square, which is essentially the square of the Mach number and this is. And here is where the Mach number equal to 1, ok.

Now, there are different kinds of solutions, there is 6 different kinds of solutions. This is solution 1, this is solution 2, these are solutions 3, and these kinds of solutions are solutions 4, and these are solutions 5, and these are solutions 6, ok. So, let us now immediately see you see let us look at solutions 6 and 5s solutions of the kind 5 and 6 first, ok.

Now, at a given r let us now draw a little line here at a given r, ok. You see this solution 5 for instance predicts two different values for the Mach number. Suppose, I was on this curve this outer curve, solution 5 predicts two different values of the Mach number. These are double valued this says that the solution is double valued same with solution 6. I draw a little thing, ok.

And it intersect solution 6 at two, two points. In other words, both solutions 5 and solution 6, you know solutions of the kind 5 and 6 predict double valued solutions, predicts two values for the Mach number. So, 5 and that is clearly unphysical. So, solutions of the kind 5 and 6 are unphysical. They are mathematically ok, but we do not need to worry about them anymore ok.

They are unphysical. They might well be mathematically ok, but you know there is no point talking about them anymore. So, we really need to concentrate our attention only on solutions of the kind 1, these kinds of solution 1, 2, 3, and 4; only on these kinds of solutions these are the only solutions that deserve our attention from now on ok.

And so we can quickly see what are you know solution 1 for instance, you start from Mach number less than 1 ok, and you end up at Mach number greater than 1. And you pass through the sonic point, this is the sonic point. Solution 2, on the other hand, starts supersonic starts with Mach number greater than 1 and it ends up at Mach number less than 1, but it passes through the sonic point, right.

Solutions 3 and 4, however, do not pass through the sonic point, they are not transonic solutions. Only solutions 1 and 2 are transonic solutions. They are the only bona fide transonic solutions. However, this equation, this admits solutions that are not transonic. This also admits solutions of the kind 3 and 4. For that matter, it also admits solutions of the kind 5 and 6, but as we discussed solutions of the kind 5 and 6 are unphysical.

So, we do not bother about them anymore. What are solutions 3 and 4? Solutions 3 are the one corresponds to the ones which remain subsonic throughout you see they are always below

1 they remain subsonic throughout, ok. And solutions 4 remain supersonic throughout ok. So, for instance, solutions 3 would be some sort of what are generally called subsonic settling flows, and solutions 4 are solutions that remain supersonic throughout, ok.

However, having said all these, however, our attention here will be exclusively on solutions for the kind 1 and 2, ok. This is where our attention will be concentrated, because these are the only transonic solutions ok. And when we meet next, we will explore solutions of the kind 1 and 2 in some more detail. In particular, we will explore solutions of the kind 1.

So, solutions of the kind 1 would correspond to say yeah to the solar wind, solutions that start out subsonic pass through the sonic point and go on to become supersonic, except we are talking about outflows here.

So, you start from the surface of the Sun right, you have a subsonic flow right, and it passes through the sonic point smoothly, and then it becomes a supersonic flow as you go on ok. And this is a spherically symmetric solution. So, we will talk a little more about this kind of a supersonic solar wind the solutions of the kind label by 1 when we meet next.

Thank you.