Fluid Dynamics for Astrophysics
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Lecture - 27
Criterion for neglect of compressibility, method of characteristics

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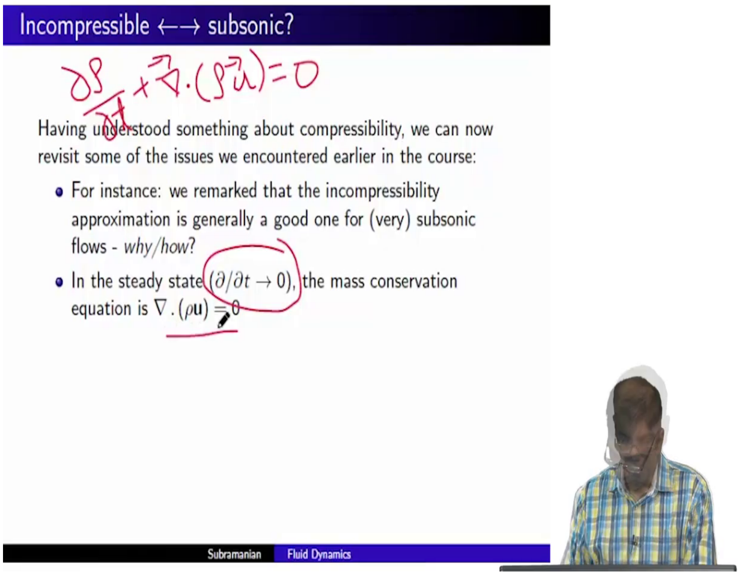

Welcome back. So, we saw the last time when we talked how we hinted and how large disturbances right in pressure and density can conceivably result in phenomena that are qualitatively different from what we encounter with small differences.

In particular we hinted at the possibility of discontinuities forming and today we will have the occasion to talk a little more about these discontinuities which are called shocks ok.

But, before all that before all that before starting to talk about large disturbances before plunging into large disturbances and shocks and things like that I thought I would touch a little bit upon one issue with regard to small disturbances with regard to sound compressibility with regard to the relation between compressibility and the Mach number.

This is something that you will hear occasionally now and then as we go along in fluid dynamics and I have also mentioned this in the passing, but I thought we would take a little bit of a detailed look at this. So, just to emphasize for the next five or ten minutes we are not yet talking about large disturbances. We are now examining the relation between compressibility and Mach number.

In particular to understand why it is that subsonic flows are generally regarded to be incompressible whereas, supersonic flows are regarded to be compressible. Actually the more the more accurate statement would be the effects of compressibility are crucial to examining super to understanding supersonic flows and conversely the converse statement is applicable for subsonic flows.

But there are some subtleties here and I thought we would examine these subtleties a little bit before we go forward right. So, what is it, right? So, we like we said we remarked that incompressibility approximation is generally a good one for in particular very subsonic flows. In other words the more subsonic the flow the better the incompressibility approximation right.

We have said this and you will you will you will also encounter this statement here and there. So, let us try to understand why and how right. For simplicity let us restrict ourselves to steady state situation.

Steady state simply means this statement right. In other words the time variations as discerned by an Eulerian observer, an observer whose, outside the flow whose, who standing in the lab frame and watching the flow go by. For that kind of an observer there are no time variations ok just for simplicity.

So, in the steady state we know that the mass conservation equation is just this. Why is that? Because in this you know in Eulerian frame the mass conservation is simply right. This is the mass conservation equation and. So, if this first part is 0, we are left just with the second part and so this is what I mean here right.

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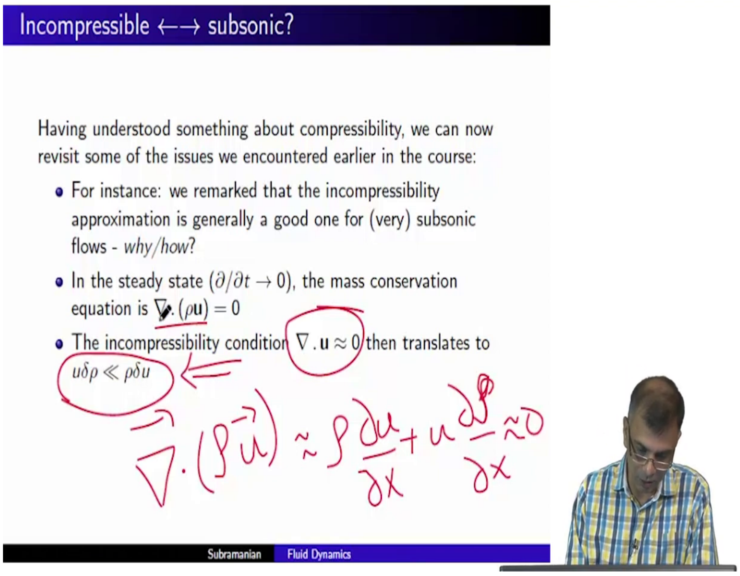

Now, we know we have repeatedly emphasized this that the incompressibility condition is this. This is the incompressibility condition. So, what does that translate to here? You just apply you know the product rule for differentiation right. So, suppose in one dimension in one dimension would be something like rho du dx plus u d rho dx. You agree with me right? So, this is just the product rule right.

So, never mind the dx is right. So, this is the kind of logic which leads to this statement. So, if this is much less than 0 or rather if this is approximately equal to 0 right and we know that this is almost equal to 0. Is not it? This is essentially this in one dimension. So, if this is 0 in which case this had better be 0, which means that u del rho is much much smaller than rho del u. In other words, this is much much smaller than this.

So, this is the main thing. This is the; this is the main statement and that we need to focus on and let us keep this in mind u del rho is much much smaller than rho del u. This follows from the steady state mass conservation equation.

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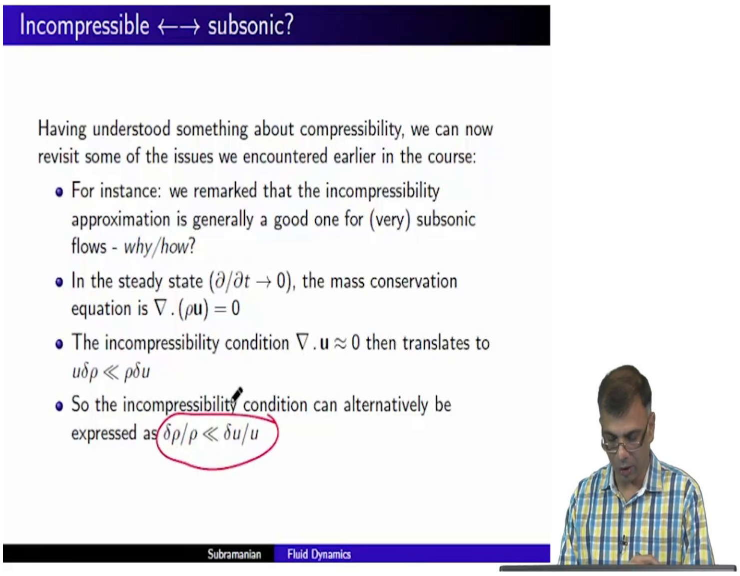

Alternatively you just you just divide by rho here. You have del rho by rho equals is much much smaller than del u by u. So, this is the other way of writing this very important condition.

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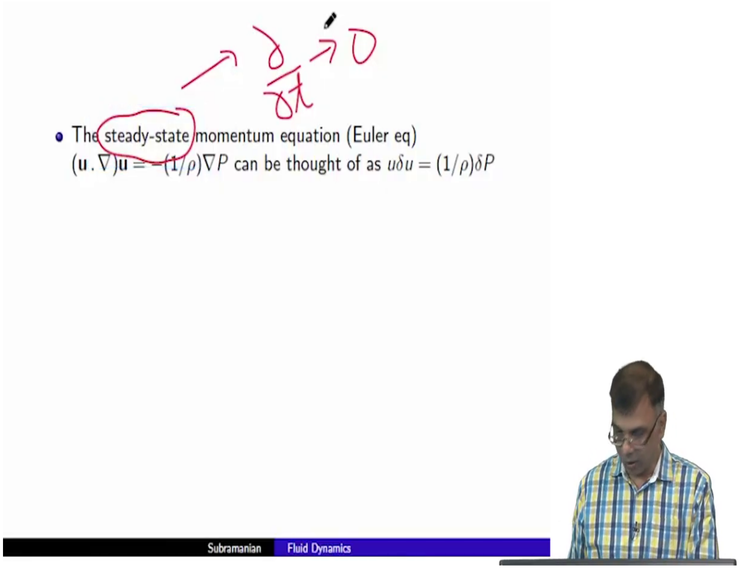

And now, let us now write the steady state momentum equation again. Just to remind you the steady state simply means this simply means right partial d partial t is equal to 0.

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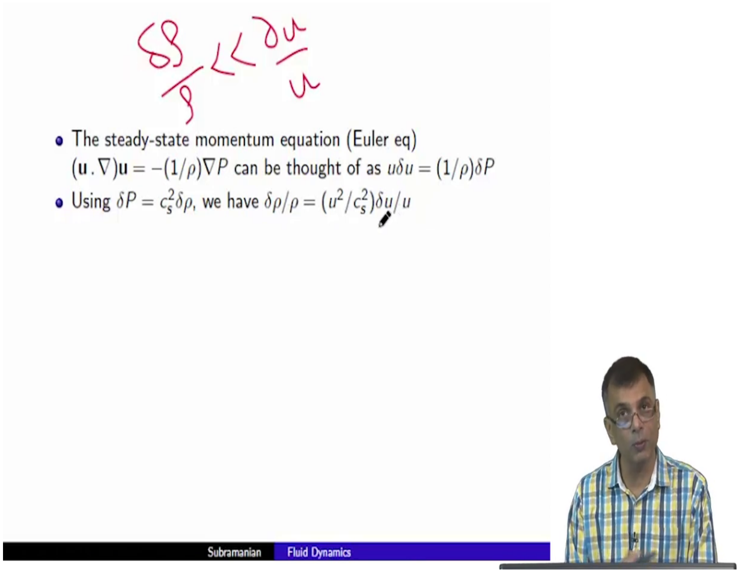

So, the steady state momentum equation now, it can be written as this the Euler equation neglect viscosity right. And using the same logic you this can essentially be written as u del u and this can be written as 1 over rho del P and we relate the del P to del rho using the sound speed right.

So, instead of del P we can write del rho here; del rho divided by c s squared and using this you can do a little bit of rearrangement to write del rho over rho is equal to u squared over c a squared del u over u. Now, if you remember what we wrote in the previous thing, we said del rho over rho is much much smaller than del u over u that is what we got from the mass conservation equation, right.

So, right that is what we got and compare it with this. If this is to be true while this is also true there is only one way it can be true right.

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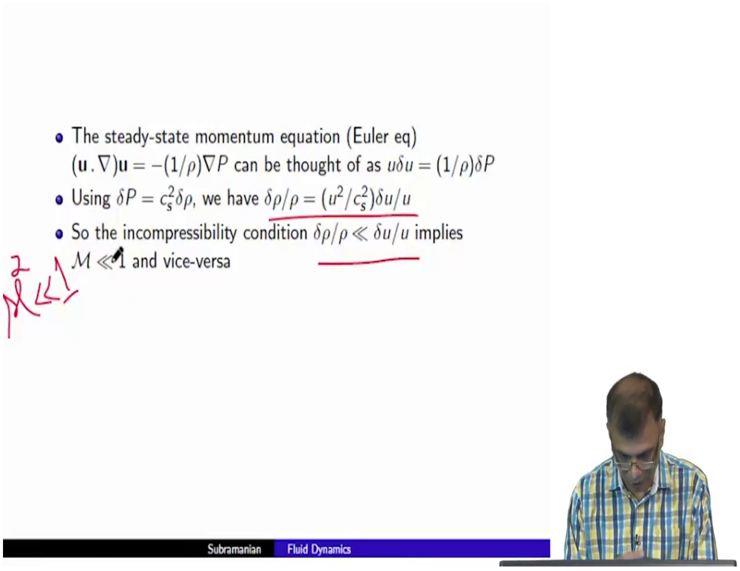

The incompressibility condition this. So, this and this together imply that strictly speaking the Mach number squared is much much less than 1 because that is what this is there is a Mach number squared right. Well, if the Mach number squared is much much less than 1, it also means that the Mach number is also much much less than 1 and vice versa.

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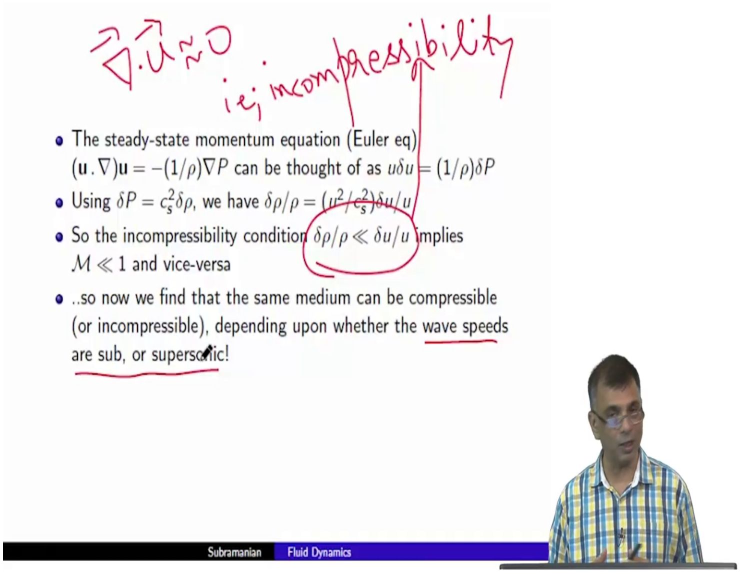

So, this is the basic thing right. So, in other words and where did where did this come from? This came from the fact that this was predicated this comes from i e incompressibility. You remember that, right. In other words incompressibility implies that the Mach number is much much less than 1 and vice versa if the Mach number is not much much less than 1. In other words, if the flows are somewhat supersonic, then the incompressibility approximation is not so good.

So, we now find that the same medium can be compressible. So, see here is the other thing. There is no such thing as incompressible or compressible. It depends is a medium largely incompressible or the is a medium largely compressible, it depends. It depends upon whether the wave speeds are sub or supersonic ok.

If the wave speeds are subsonic right then incompressibility is a good approximation, whereas, if the wave speed which you are talking about are supersonic then incompressibility is not such a good approximation. So, this is one of the main things I wanted to emphasize

However, having said this there is a conundrum right. You might wonder about this if the if the flow speed is subsonic well then you know that the medium can effectively be thought of as incompressible. But I thought you know the air in this room is essentially still, essentially still there is no velocity at all. Is not it?

So, you know the waves speeds are definitely subsonic essentially 0 and yet you are talking and sound waves are reaching me. And I thought sound waves were the were an essential I mean no compressibility, no sound waves right. Sound waves and compressibility go hand in hand. Is not it? So, they are reaching me. So, what gives, what is up?

The key here is that the wave speeds are sub or supersonic in the lab frame. As we said earlier always the perturbations the small perturbations that arise out of my speech or my clapping my hand or something these small perturbations always travel at the sound speed, exactly the sound speed and they travel at the sound speed with respect to the background flow if any.

So, to the Eulerian observer ok, what will happen? If there is a background flow then the sound speed gets added to the background flow whether its in the plus x direction or the minus x direction or anything ok and it is at added speed which should be either subsonic or supersonic. So, so this it is the speed is with respect to the Eulerian observer.

Whereas, when you are in the frame of the flow small perturbations always travel at the sound speed exactly the sound speed. There is no question of whether this these small perturbations are subsonic or supersonic they are exactly sonic. So, this is something to keep in mind.

So, this entire statement ok is true for the lab observer ok. So, when you hear this statement often that you know subsonic flows essentially incompressible supersonic flows not so, yes all that is true, but this is to do with the lab observer not within a not with a Lagrangian observer ok.

So, and for the Lagrangian observer sound waves i.e, small amplitude ah pressure and density disturbances travel at exactly the sound speed they are sonic. They are neither subsonic nor supersonic they are bound to be sonic there is no other way around it ok. So, I thought I would reemphasize this small aspect with regard to the speed at which small disturbances travel. They always travel at the sound speed with respect to the background flow ok.

So, this is something that you need to keep firmly in mind. Before we go on and start talking about large disturbances where you know this whole thing about linearization and everything breaks down. And its with regard to large disturbances that we will examine the interesting phenomenon of shocks.

So, we will do that in just a second. Thank you.

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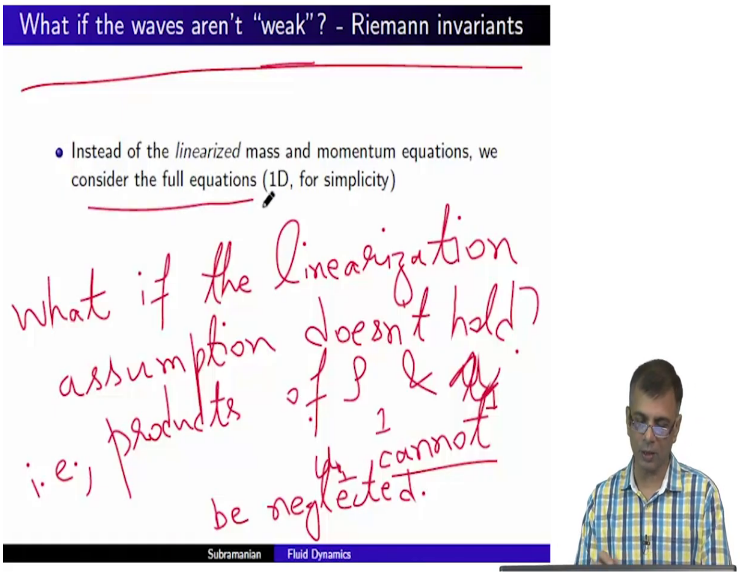

So, having talked about having clarified a little bit about the relationship between Mach number and the sonic or supersonic nature of the background flow rather the relationship between the Mach number which tells you something about the sonic or supersonic nature about the of the background flow. The relationship between that and compressibility or incompressibility having talked a little bit about that and all this is about small disturbances of course.

Let us now get back to starting to think about what the following question. What if the disturbances are not small? In other words, what if the waves are not weak ok? In other words, what if the linearization assumption does not hold? This is what I mean by this statement ok. In other words, i.e, products of quantities with subscript 1 right say rho 1 and say u 1 right. Never mind this right rho one and say this is a 1 ok. I beg your pardon for this cannot be neglected.

So, if the waves are not weak in other words if the if the disturbances u 1 and rho 1, u 1 and things like this are large then the products of quantities like rho 1 and u 1 cannot be neglected anymore ok. So, instead of the linearized mass and momentum equations we now have to consider the full equations right and for simplicity let us restrict ourselves to one dimension ok.

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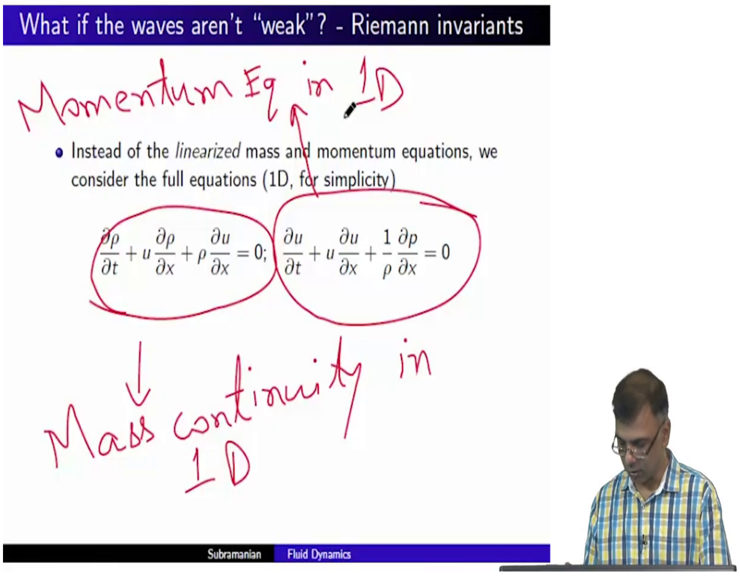

So, you have the mass continuity equation in 1D right and this is momentum ok. You can verify this.

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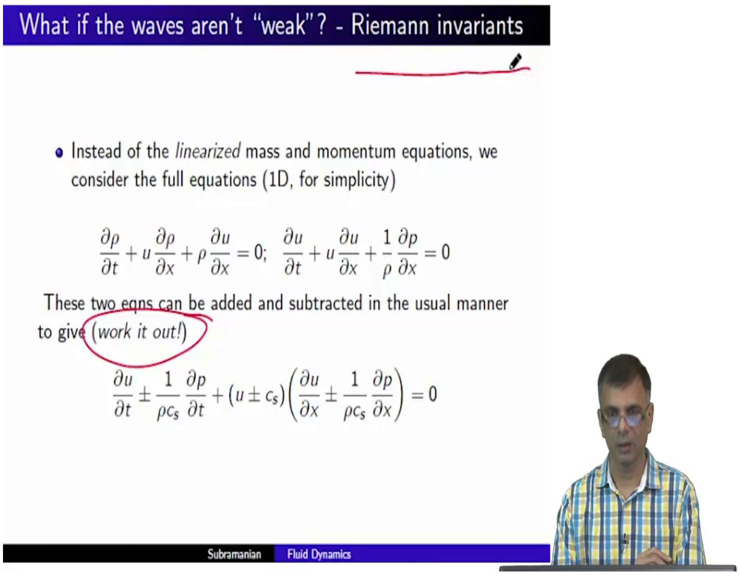

So, you have these two and what we are now going to do is this the familiar jugglery, so, these two equations you add and subtract ok. You add them first that corresponds to the plus sign here and you subtract them and that corresponds to the minus sign here ok. And so, you add the equations and you get du dt plus 1 over rho c as d dp dt plus u plus c s and here also the plus sign is to be considered equal 0. And you subtract them and this entire thing with the minus sign here, here and here that is valid that is what this means ok and I strongly urge you to work it out ok.

And you realize where we are going with this? We are trying to figure out something like a Riemann invariant. You remember the j plus and j minus that we talked about when we met last so, but those j pluses and j minuses were for the linearized mass and momentum equations also in 1D. Here there is no linearization because we are considering a condition where the perturbations are no longer weak ok.

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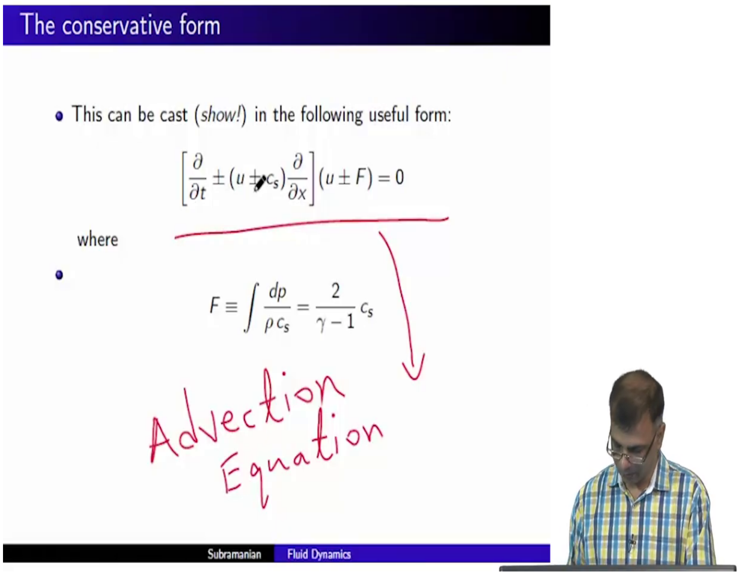

So, you have this and this can be cast again I urge you to show this in the following useful form like this equation. This entire thing can be written as this where the quantity F is nothing but dp over rho c s which is 2 over gamma minus 1. Again I urge you to carry out the algebra which you know.

So, what you can see now is its really this quantity that is at vector along. This is again an advection equation because this is a d over dt of this quantity plus u plus c s d over dx of the same quantity is equal to 0. So, if we consider just the plus signs all along this means this represents a situation where the quantity u plus F, where F is defined by this is propagated unchanged in other words its advected ok with the velocity u plus c s.

Similarly, the minus sign of the equation represents a situation where the quantity u minus F, where F is again given by this is advected without change at a velocity u minus c s that is what this means ok.

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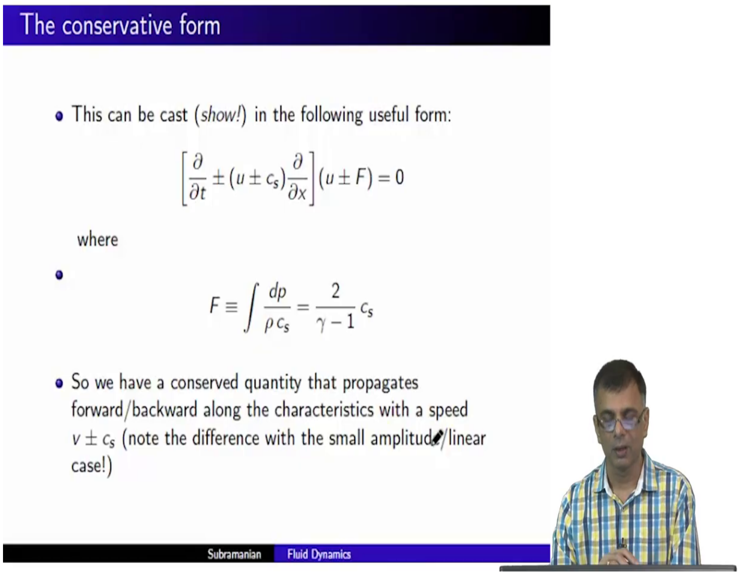

So, we in other words we have a conserved quantity that propagates forward or backward ok along the characteristics with a speed v plus or v minus c s. So, if you are talking about the forward characteristics it corresponds to v plus c s, if you are talking about the backward characteristic you are you are talking about the velocity v minus c s.

So, you can already see the wave can be either supersonic in other words it can it can be v plus c s or it can be subsonic v minus c s. And v itself can be you know anything literally. And I urge you to examine how this equation is different. This is an advection equation.

Just like the advection equation we wrote down for j plus and j minus ah the other two Riemann invariants some time ago and I urge you to compare these two equations and note the difference what are the similarities and differences. The similarities of course is that both of them are advection equations, both of them talk about conserved quantities, but the conserved quantities themselves are quite different ok. So, I urge you to think about them.

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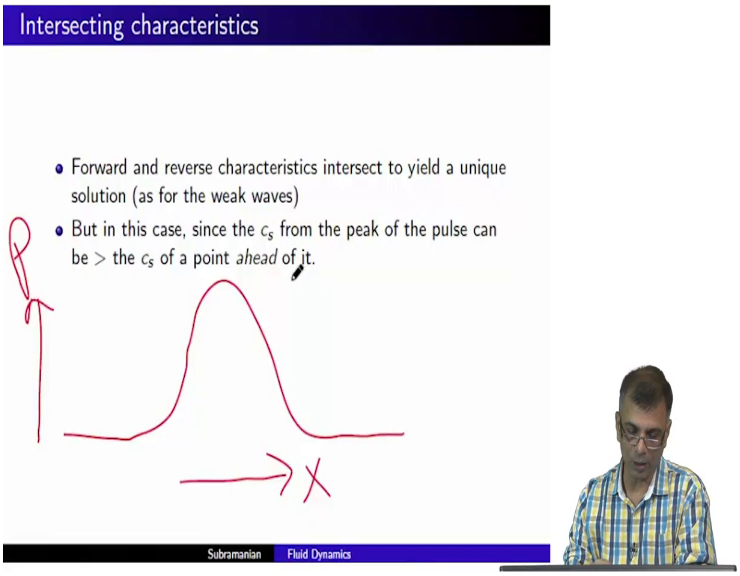

Now, what happens here? With the weak waves what you could have is the forward and the backward characteristics. The forward characteristics which is which propagates at v plus c s and the backward characteristic which propagates at v minus c s they can intersect to give you a unique solution.

So, that can happen here too as with the weak waves that that we talked about earlier that can happen now too right ok. But in this case since you know you remember that this thing that we discussed a little earlier ah. Consider a pressure pulse and the P is here and let us not talk about for simplicity.

Let us not let us not bother about you know density variation. So, let us only bother about pressure variations for the time being. And if the pressure pulse is large enough, so that the sound speed here is larger than the sound speed here. This is the leading edge and this is the peak of the pulse ok.

But, since the sound speed here can be larger than the sound speed here there is a possibility that this can overtake this. As time goes along this feature propagates faster than this feature and there is a possibility that this feature can overtake this feature and there can be a wave breaking of sorts right. So, that is what I mean by this.

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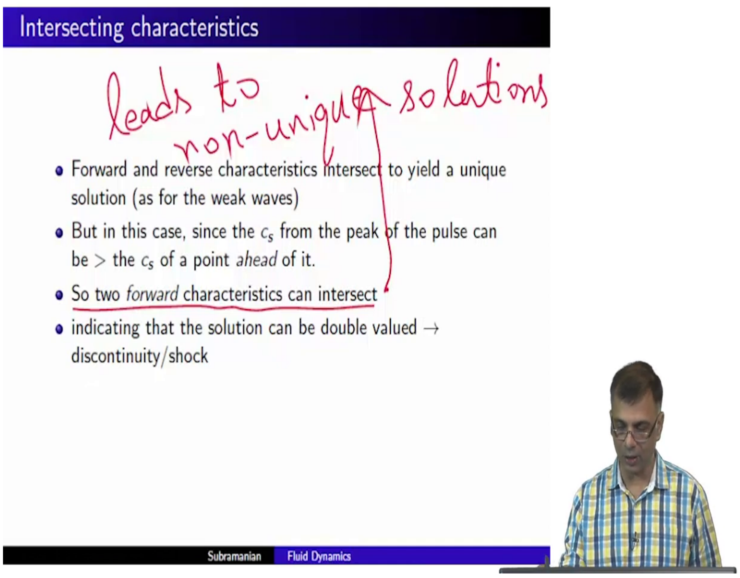

So, two forward characteristics can intersect not forward and reverse characteristics. Two forward characteristics can intersect indicating that the solution. And this is in some sense you know this is this leads to non unique solutions this thing ok.

If two forward characteristics can intersect, it leads to non unique solutions. I urge you to think about this. And what that means is that the solution can be double valued ok, which is to say that there can be it hints at the presence of a discontinuity or a shock.

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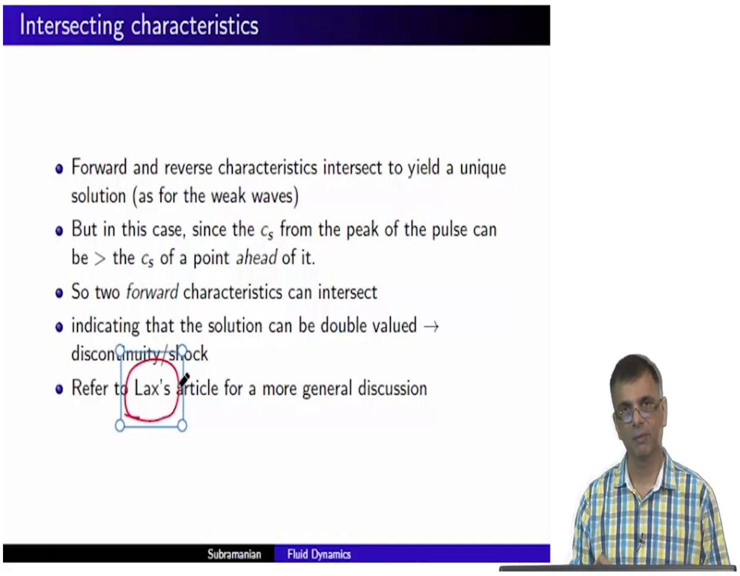

And I will put up an article by lax and this they give a little more of a general discussion. So, we will do that and I will put that up and I am only giving you sort of the essence of that discussion and there is a lot more to be said about this very interesting thing. The fact that for the situation where the disturbances are not small you can have intersecting characteristics, but not forward and backward characteristics intersecting, but two forward characteristics intersecting.

And that leads to this mathematical problem ok of non unique solutions and that essentially leads to the realization that such solutions can contain discontinuities which we will realize. And some of these discontinuities can be what are called shocks.

So, we will talk a little more about shocks in the next segment. And, but this issue about characteristics is dealt with quite extensively in this article by lax which I urge you to read. So, we will stop here for the time being.

Thank you.