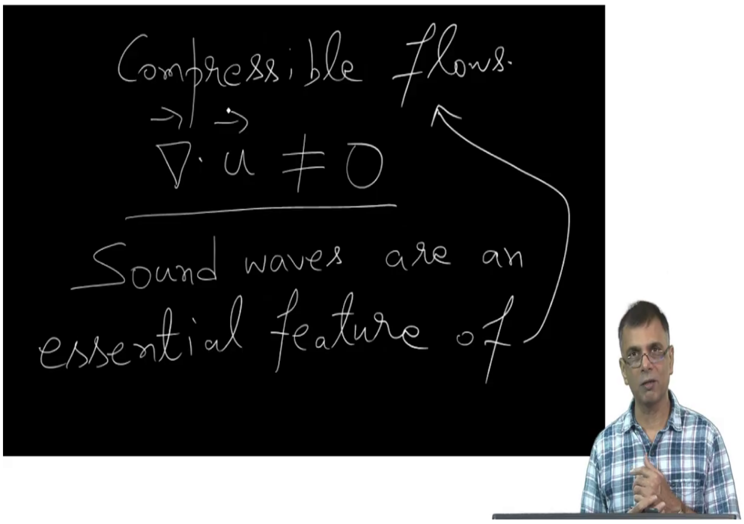
Fluid Dynamics for Astrophysics
Prof. Prasad Subramanian
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture - 24
Compressible flows: Derivation of sound speed and dispersion relation

Hi. So, we are currently talking about incompressible sorry compressible flows since we have been talking about incompressible flows for so, long I keep saying incompressible, but really we have transitioned to start talking about compressible flows right.

(Refer Slide Time: 00:34)

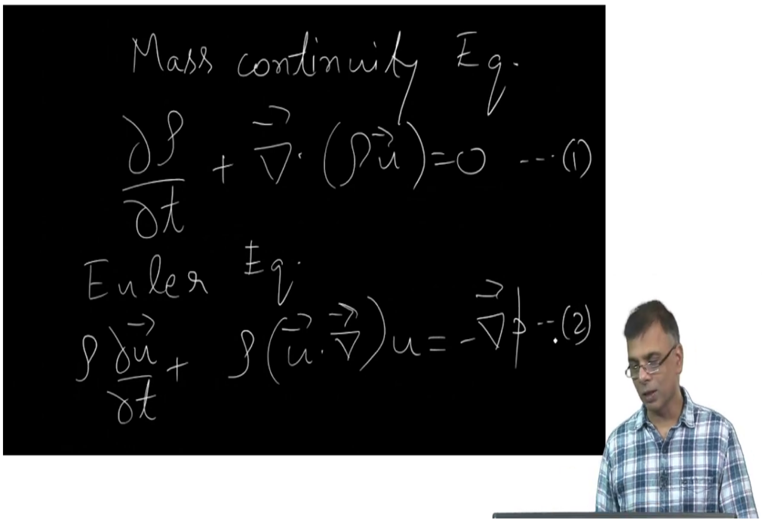


Where, the condition of incompressibility as you know we know it the generally accepted condition of incompressibility where the divergence of the velocity was taken to be equal to 0 that is no longer true ok. So, this can be taken as one thing about compressible flows.

The other thing about compressible flows is that the speed of sound waves rather sound waves are an essential feature of compressible flows right. So, these are the two important things we have sort of highlighted so, far we have done a short derivation on the speed of sound on trying to show how small disturbances in pressure and density travel at exactly one speed ok and that is the speed of sound right.

And what we did there just by way of a very short recap is you recall what we really did was we started with the mass and momentum continuity equation, we started first with the mass continuity equation written in Eulerian form for convenience and then we also wrote the Euler equation which is momentum continuity equation.

(Refer Slide Time: 02:21)

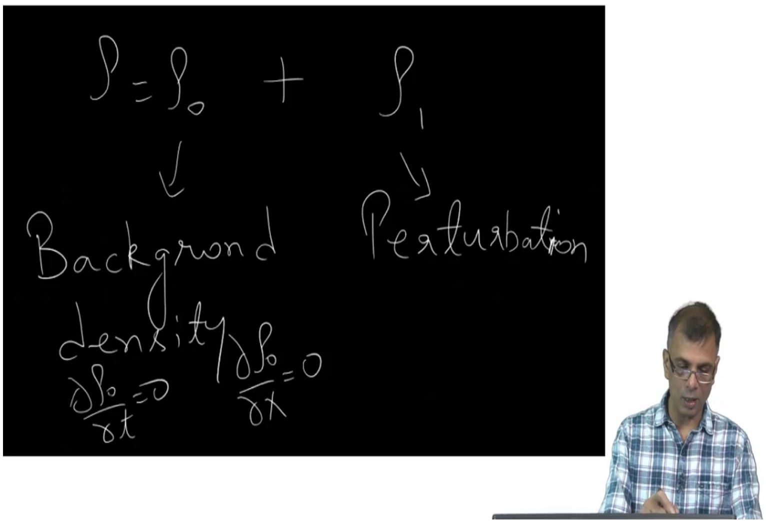


Again written in the Euler equation also written in the Eulerian form in other words in from the point of view of an observer whose standing in the lab and so, that would be this right this equal to minus gradient of pressure. So, we took these two equations, these are the two main equations we took and then we said that the basic quantities you see here you look here there are three variables right there is rho, there is u and there is p eventually there are three variables and only two equations.

So, eventually we will want to eliminate one of these variables as it happens we choose to eliminate p in favor of rho, through what is called the sound speed, but the way the sound speed comes about we will have you know let us do a very quick recap we have already done this in the when we met last, but it is useful to do a very quick recap right.

So, but you see the point is this is total density, total velocity, total pressure right, but what those total density and total velocity are something like this.

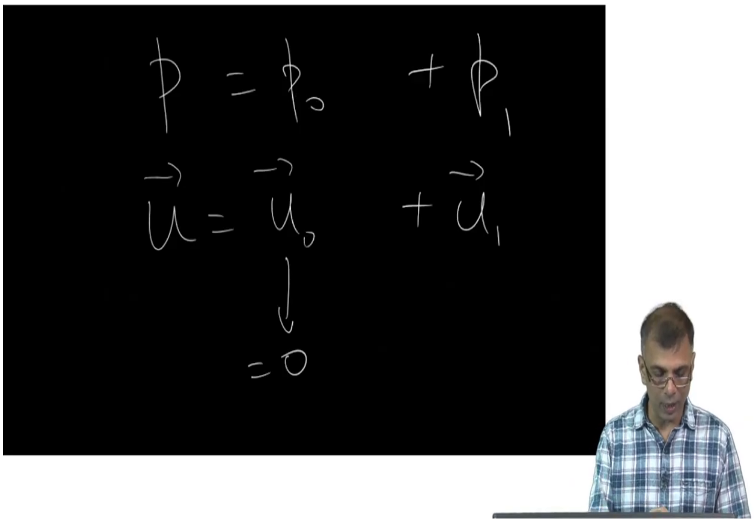
(Refer Slide Time: 04:11)



So, the total density would be the sum of a background density plus a perturbation density. This is a background density and this is a perturbation right. And what is the property of the background? The main thing about the background density is it is uniform in space and its time invariant it is not varying with time in other words d rho naught dt equals 0 and its uniform in space in other words d rho naught dx equal to 0.

And by dx I mean dy d rho naught dy d rho naught dz everything all of those are equal to 0 right not so, for the perturbation of course. So, you start out with a uniform background and the same thing we have the same philosophy for the velocity as well, we for that matter we have the same philosophy for the pressure.

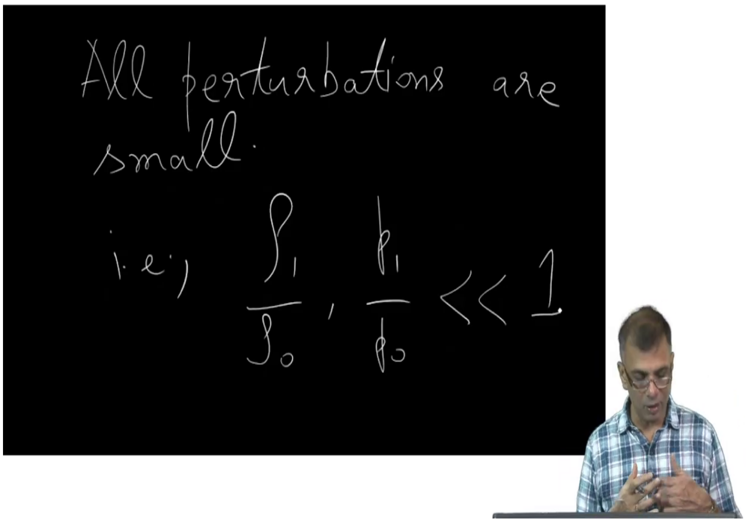
(Refer Slide Time: 05:31)



So, the total pressure is the sum of a background pressure plus a perturbed pressure and the background pressure is time invariant d over dt of p naught is 0 and it is also uniform in space so, that any spatial derivative of p naught is 0. Not so, for the perturbed pressure and the same thing for the velocity; the velocity is u naught plus u 1 except now without loss of generality we just take this to be equal to 0.

No background velocity in other words there is no breeze through the room ok. So, what we do is we substitute this u equals u naught plus u 1, p equals p naught plus p 1 and rho equals rho naught plus rho 1 into these two equations ok.

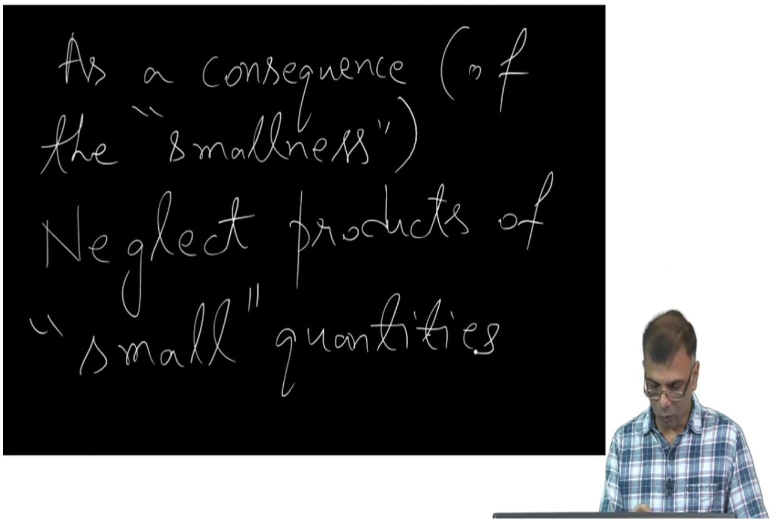
(Refer Slide Time: 06:36)



and there is one more important thing to note is that all the perturbed quantities all perturbations are small, i e, rho 1 over rho naught p 1 over p naught everything is much much less than 1 ok.

So, this is the other central assumption that we make. So any perturbation that the density perturbation that I launch or the pressure perturbation that I launch by virtue of speaking these are small perturbations. These are the density perturbations rho 1 it is small in comparison with the background, the magnitude thereof ok. So, this is another central assumption.

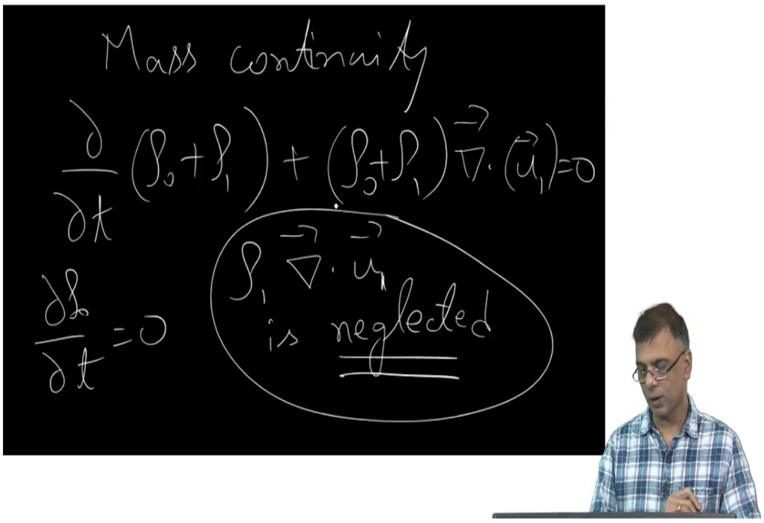
(Refer Slide Time: 07:31)



And as a consequence of this assumption as a consequence of the smallness of the perturbations and I write this in quotes ok. What happens is, you can neglect products of small quantities neglect products of small quantities.

What do I mean by this? For instance, I can I am allowed to you know neglect things like say for instance when you substitute things like rho equals rho naught plus rho 1 in the say in the mass continuity equation. So, you see you would have rho naught plus rho 1 here and u naught plus u 1 here, but u naught is of course, 0.

(Refer Slide Time: 08:43)



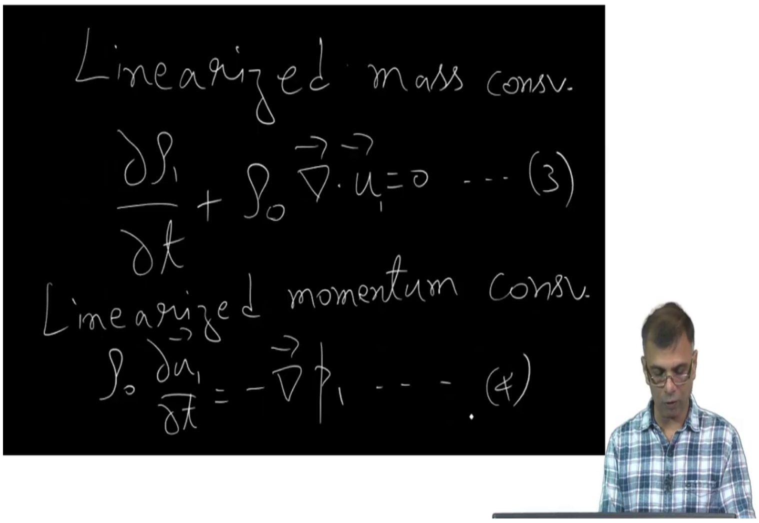
So, the mass continuity I will just give you an example becomes d over dt rho naught plus rho 1 right plus rho naught plus there used to be just a rho here and that is rho naught plus rho 1. And there used to be just a rho here and that is now you know rho naught plus rho 1 times there is to be a u here and that is u naught plus u 1, but we know that u naught is 0 therefore, it is you just have the u 1 here. Now what you do is first of all we say as a consequence of the uniformity of rho 1 d rho naught dt equals 0 of course.

Whereas, we are not saying anything about d rho 1 over dt we retain that as it is and this quantity rho 1 is neglected ok sorry here because this involves in some sense the product of two small quantities ok. So, that is what I mean by this, neglect products of small quantities. This is what I mean by that this is an example of what I mean by that ok.

And so, this process of neglecting products to small quantities is called the process of linearization. I mean you know in principle if you were to able to relate u 1 to rho 1 this would become like a non-linear combination right, but we neglect that.

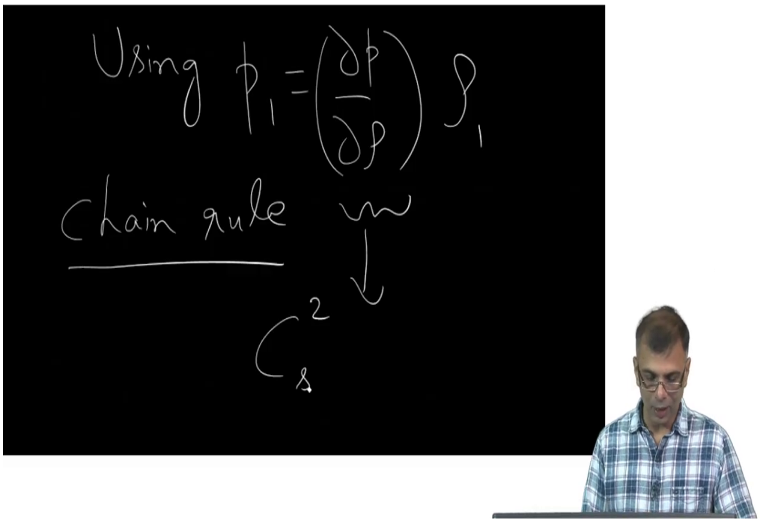
So, you only have a linear combination rho naught and u 1 rho naught is a background quantity. So, this is linear in u 1 right. So, this is essentially the process of linearization and as a consequence what we have is the linearized version of the mass and momentum conservation equations.

(Refer Slide Time: 10:57)



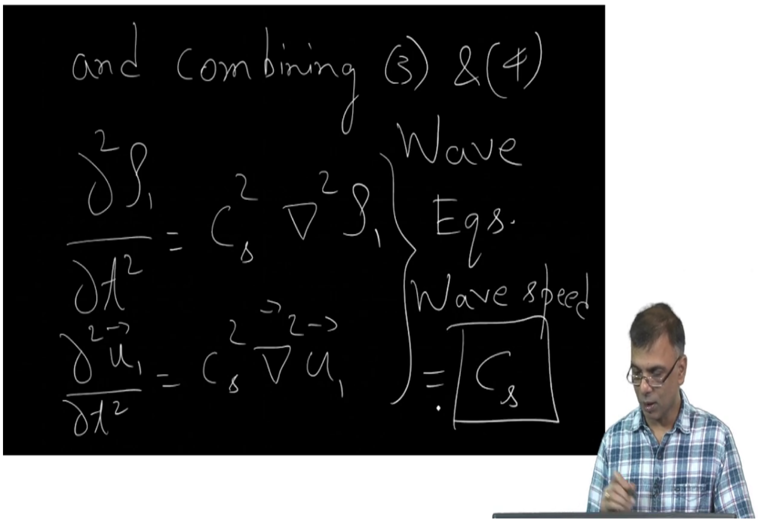
That is one thing and the linearized and linearized momentum conservation equation becomes rho naught. So, this would be equation 3 and equation 4 ok.

(Refer Slide Time: 11:49)



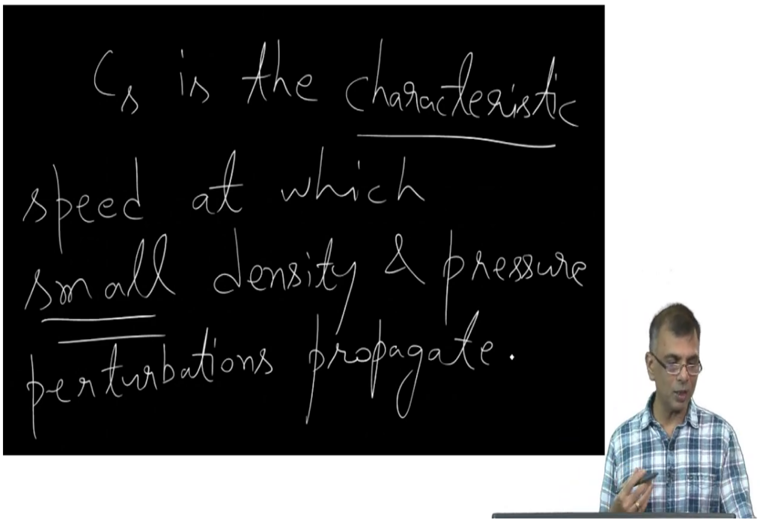
Now, using this is where we introduce the sound speed, using the chain rule this is just simply the chain rule for differentiation right that is all this is and we identify this as the we simply for the time being we just say this is C s squared. Later on we will discover that this is really the square of the speed of wave propagation using this.

(Refer Slide Time: 12:33)



And combining 3 and 4, what are 3 and 4? 3 and 4 are this and this ok. So, combining 3 and 4 we get a wave equation you are combining two first order differential equations, you are bound to get a second order equation. And same thing for u 1 like this and these are essentially wave equations and what is the speed of propagation of the wave? And the wave speed equals C sub s.

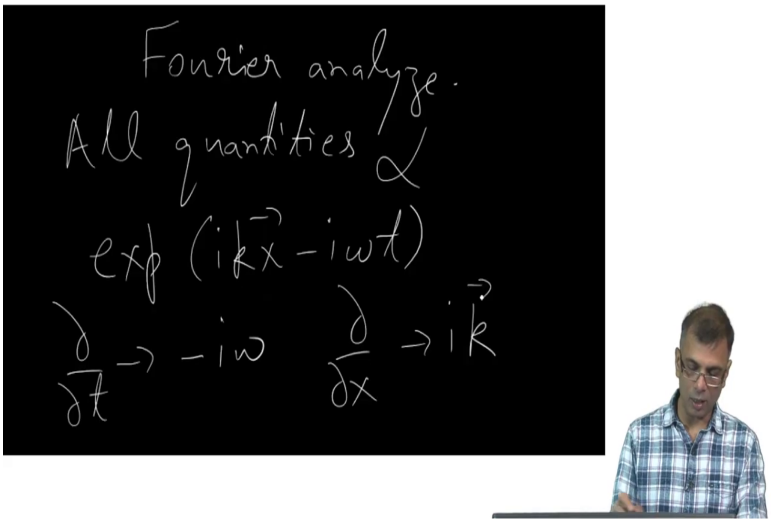
(Refer Slide Time: 13:54)



So, C sub s is the characteristic speed and I say characteristic because just to emphasize that it is the one and only speed at which small and I emphasize small, small in the sense that you know rho 1 over rho naught is much much less than 1, p 1 over p naught is much much less than 1 so on so forth.

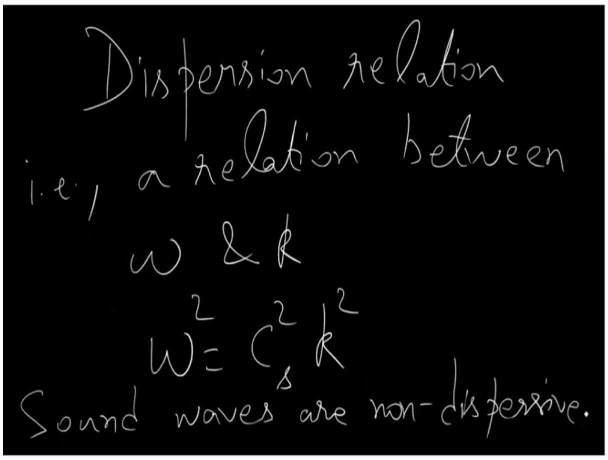
Density and pressure perturbations propagate. So, this was a very quick recap of how the speed of sound arises naturally from just the you know mass and momentum continuity equations right. So, we did this.

(Refer Slide Time: 15:08)



And then we did a little bit of Fourier analysis and where we essentially said we Fourier analyzed where we set all quantities go as exponential i k x minus i omega t. And as a consequence any kind of d over dt becomes a minus i omega right and any kind of d over dx becomes a i k right that is how it goes and using this in these two wave equations we get the dispersion relation which is always a dispersion relation is always a i e, a relation between omega and k.

(Refer Slide Time: 16:05)



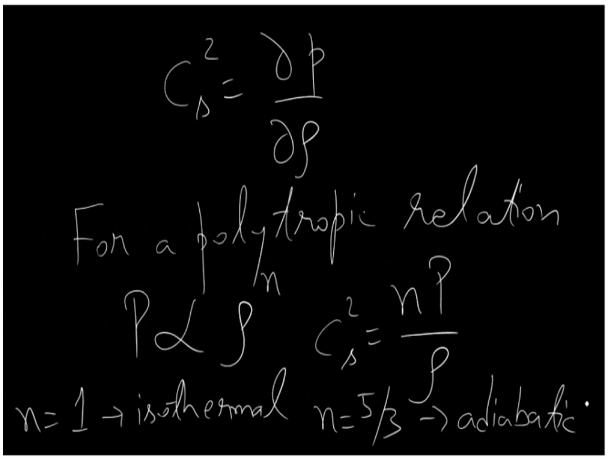
The omega here the temporal frequency and the spatial frequency that is what the dispersion relation is and for sound waves it is simply given by omega square equals C s square times k square i e, sound waves are non-dispersive. It is not like the phenomenon of light waves propagating through a prism where the phase velocity the group velocity for that matter of light of different wavelengths of light in other words the different temporal frequency of light is different inside a prism

And that is why you know monochromatic light splits up into it is components that is not the case with sound waves ok.

Different frequencies of sound waves you know propagate at the same speed whether I am speaking at a high frequency or a low frequency, the speed of propagation is always the same and that is the speed of sound. So, just to you know once again recap because this a very important thing this is a direct consequence of compressibility sound waves are a consequence of compressibility.

In a given medium characterized by a given background, density, pressure and temperature it represents the characteristic speed at which small disturbances in what? Disturbances in density and pressure as the small disturbances propagate it is a characteristic speed ok.

(Refer Slide Time: 18:13)



And we wrote down that you know we did this right we said C s squared equals this was the basic definition you remember of the speed of sound like this. So, if there is the case, then for a polytropic medium in other words for polytropic relation which goes where you have p proportional to rho raised to some n right.

If you plug this in here you get C s squared equals n P over rho right and generally and as you know n equals 1 is isothermal and n equals five third's adiabatic right. So, these are two commonly invoked limits of the polytropic relation. So, you would have two different kinds of speed of sound.

So, normally what happens is you know you are often asked ok. So, you are talking about the speed of sound you are talking about the characteristic speed at which smaller density perturbations propagate, but what is the thermodynamics of these density perturbations?

How do these small you know compressions and rare factions? How what is the thermodynamics? Is it a an isothermal compression and rarefaction or is it an adiabatic compression rarefaction?

If you can answer that question you will have the accurate definition of speed of sound. So, you can have two different kinds of speeds of sound right. Depending upon you can have an isothermal speed of sound, you can have an adiabatic speed of sound, you can have something in between as well depending upon the value of n.

As we remarked a little earlier, the value of n I mean this is really kind of a cheap energy equation n n this value of n embodies all the energy dissipation and processes that can be happening.

So, the actual speed of sound although formally is defined like this, the actual value will depend upon the thermodynamics of this of the compressions and rarefactions and the two limits are an isothermal process and an adiabatic process depending upon the value of n the speed of sound can vary ok.

And couple of other things you see the speed of sound is linked to pressure disturbances right. So, it is linked to communication I am able to communicate with you via the speed of sound right. So, communication or propagation and medium happens at one characteristic speed, the speed of sound in a philosophical way it is similar to the speed of light ok. So, the speed of light is of course, invariant whereas, the speed of sound can be quite varying as we saw.

But that apart, you know you can see that the speed of sound can be linked to the principle of causality. So, as an you cannot hear me before sound waves have a chance to propagate from me to you hence the link to the principle of causality right. And unlike you know the speed of light objects and flow speeds for instance can indeed exceed the speed of sound that is not so, with the speed of light of course.

But the dynamics in the two cases whether the object or the flow speed is traveling subsonically or supersonically you know in these two situations the dynamics of the flow will be very very different in other words subsonic flows and supersonic flows are very different in nature and when we meet next we will talk about the details of these.

Thank you.