Fluid Dynamics for Astrophysics
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Lecture – 21
Reynolds number recap, Low Re flows, and drag on a sphere (Stokes law)

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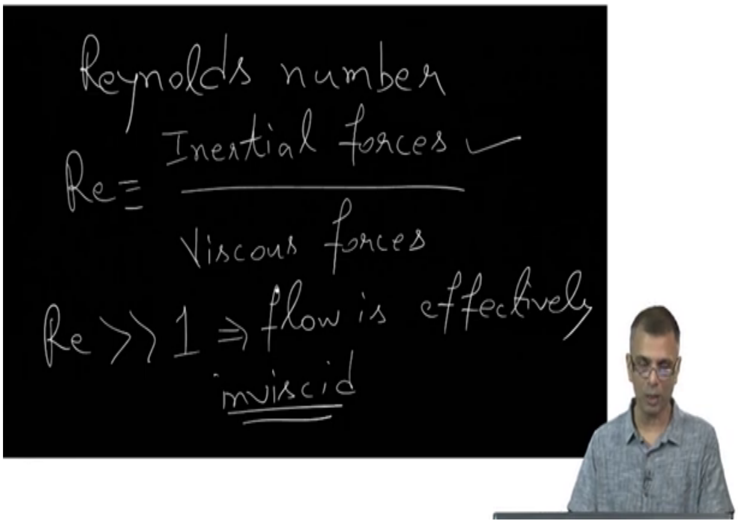

So hi, we are back and we will resume our discussion of the Reynolds number which is one of the most important dimensionless numbers in Fluid Dynamics, especially for us in the context of Astrophysics. And if just by way of a recap you remember that the Reynolds number is defined by is essentially the dimensionless ratio of inertial forces to viscous forces.

So, physically, if the Reynolds number is large, inertial forces are more important right, more important in relation to viscous forces. And therefore, the flow is relatively less, the flow is effectively inviscid right; viscous effects are not important and vice versa. If the Reynolds number is small, then viscous forces are more important. And so the problem can be simplified accordingly right. So, this is one way of looking at the Reynolds number. When we met last, we also talked a little bit about dynamic similarity.

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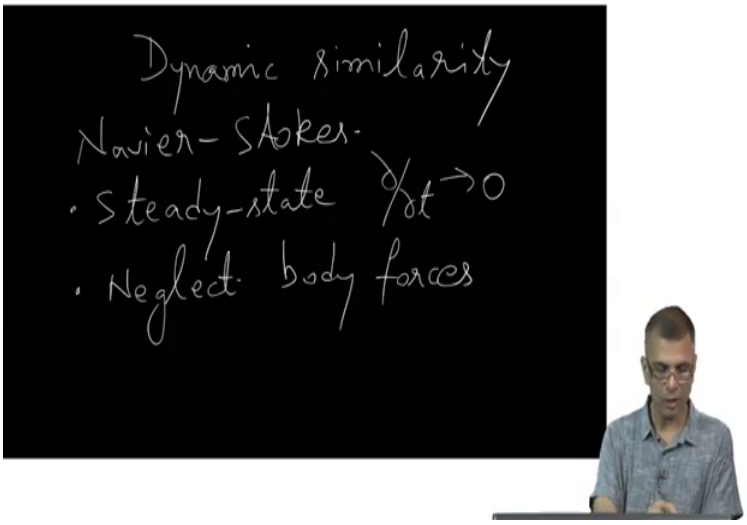

And at the heart of it, sorry, yeah, so dynamic and this is one very useful way in which Reynolds numbers that the concept of the Reynolds number is applied. And we essentially wrote down the Navier-Stokes equation. In the absence of Navier-Stokes right which is just the momentum equation, we specialize to steady state which is to say any partial derivatives. In other words time derivatives as discerned by the observer in the lab frame, it goes to 0. And we also neglected body forces such as gravity right.

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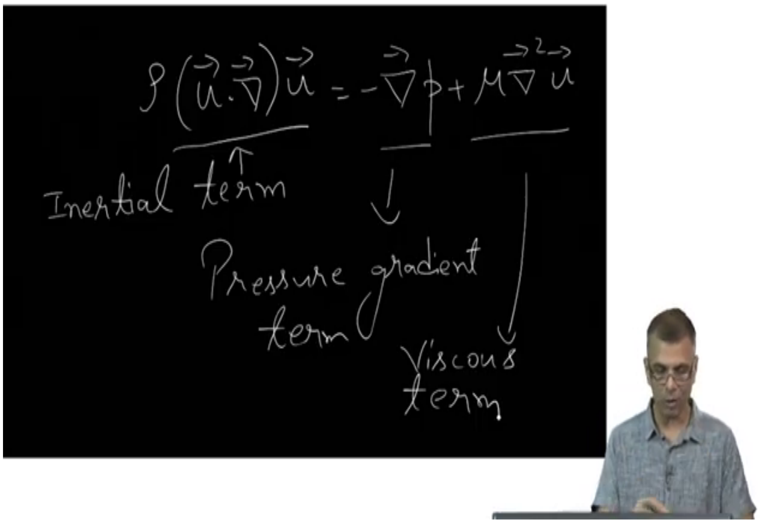

If we do that, the Navier-Stokes equation is simply rho times. So, this is m a so to speak; and m a can be due to pressure gradients, and there is also, so that is the viscous term, and this is the pressure gradient term. And this is the inertial term right, maybe it is useful to write this down inertial term pressure gradient term and the viscous term right.

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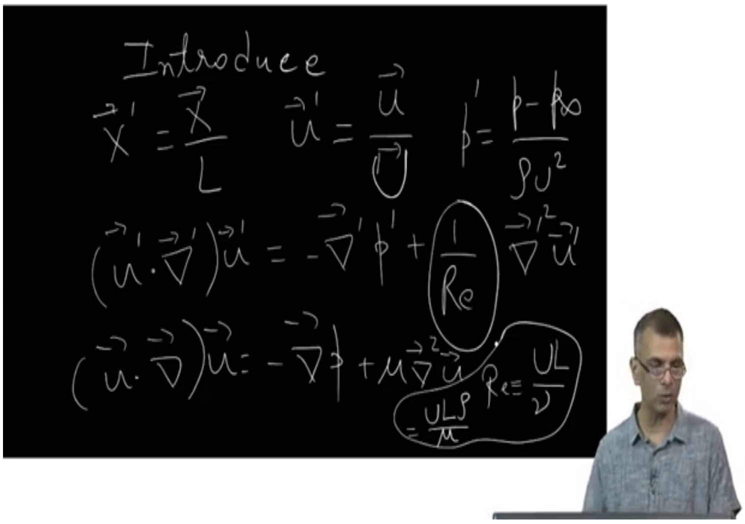

There is this and then what we did we did something very important. We said let us introduce the non-dimensionalized introduce non-dimensionalized variables X prime is X divided by L. So, you notice X prime as such is dimensionless because L has dimensions of length as does X, I messed it up. So, let me write it again, it just X right.

Similarly, the velocity is like that, and the pressure is minus sum. And what is the logic here? The quantity L is some representative microscopic length yeah, the quantity U is some representative, well, technically is undisturbed velocity ok.

The idea being that you know viscous effects can typically be considered as a perturbation. And so U would be say if you are talking about for instance flow around a sphere U would be the undisturbed velocity say far away from the sphere where the sphere has little influence on the flow, yeah.

So, you non-dimensionalize, the velocity everywhere close to the sphere as well as far away with this quantity capital U, where capital U is undisturbed velocity right, the velocity of the fluid far away from the sphere.

And you non-dimensionalize the pressure in units of rho U squared, this is the units of pressure, except you do a small tweak here. You do not quite say P over rho u squared for reason is that technically for some technical reasons which we would not go into we say P minus P infinity, where P infinity we account for the fact that even at large distances from the object there can be a finite pressure ok. There can be such situations.

If this is the case, then the steady state equation of motion you know which as we wrote down in the previous this thing right. This can now be written as in terms of these of the of these non-dimensional variables. This can now be written as u like that where notice. So, I really should be having vector signs over all of this yeah same for this yeah.

Notice even the gradient has a prime ok. We will, we talked about this when we met last and we will emphasize this once again in a minute. This is all important term like that. Now, how does this differ from this? Look at this and look at this, yeah. If you wish I can write down the other dimensional version of the equation right here. So, it will be a little more obvious grade of p plus mu like that.

So, compare this in this. It looks very very similar except for the fact that you have got primes here yeah, no primes here. These are this is the completely dimensionless equation yeah. And in term in place of the viscosity coefficient, you have this 1 over Re yeah appearing here, where Re is defined as U L over nu r U L rho over mu same thing ok. So, excuse me for the mess, but this is what it is.

So, we mentioned this fact and we also mentioned that equivalently many times we work in terms of the Navier-Stokes equation, we can also work in terms of the vorticity equation which is often you know more convenient in for certain circumstances. For the sake of completeness, I will simply write this down.

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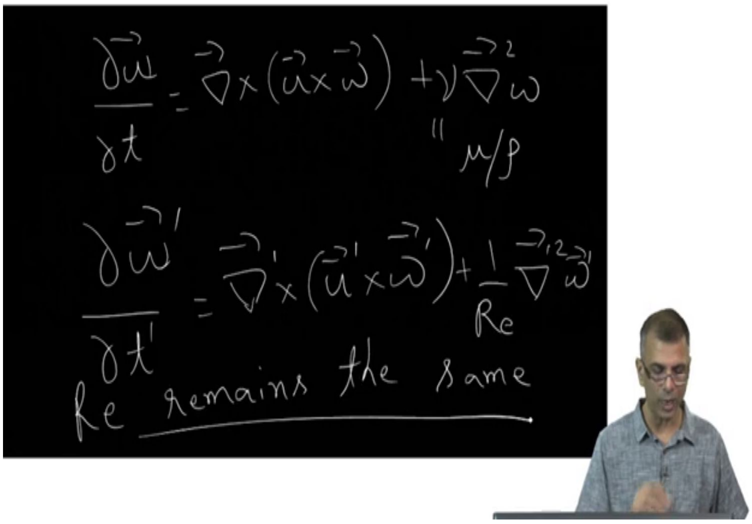

This is the vorticity equation for viscous fluids, not for inviscid fluids where the vorticity is conserved circulation is conserved more properly. It is exactly the, and, so the vorticity equation when written in a non-dimensionalized form can be written as like that. And this also is a prime, even the time variable is prime plus the familiar 1 over Re appearing again like that, yeah. So, compare this with this and this is nothing but that is what the nu is right.

So, compare this with this and so you have got the primes appearing everywhere including the time. And you have got the coefficient of viscosity being replaced by 1 over Re right. So, you have got dimension. So, so look at this and also look at this, look at this, very very similar right.

So, here you see its dimensionless right. So, you can apply it to an object that is this small or this small, this big that is the whole point. And here n is the principle you know assumption is the principle idea of this thing called dynamic similarity right.

Because this equation or this equation is written in dimensionless numbers. It does not all you need to do is change the macroscopic length, macroscopic velocity, that is all you have to change and the rest adjust themselves the vorticity and other things.

So, if your microscopic length scale was 5 centimeter for instance in one situation, in that case you non-dimensionalize the variables accordingly. Or in another situation, if you are talking about a geometrically similar object with much larger dimensions though, in other words the object is exactly similar to your miniature model, but the dimensions are much larger the dimensions are 10 meters for instance, all you got to do is plug in capital L equals 10 meters.

But this equation this dimensionless equation here this one and this one remains the same ok. And therefore, the dynamics of the flow around the miniature body will be exactly the same as the dynamics of the flow around the larger body provided of course the Reynolds number remains the same.

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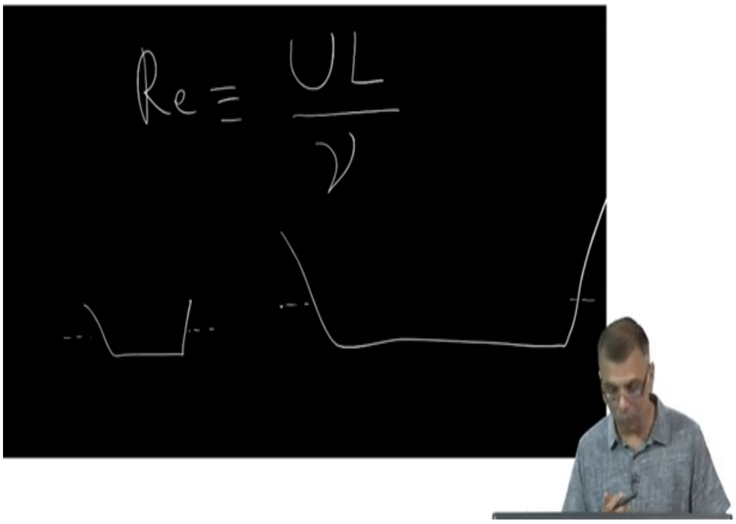

And let us go back and look at the definition of the Reynolds number. So, it is worth writing this down yet again. So, microscopic velocity times some microscopic length divided by the coefficient of viscosity right.

So, now we are talking about the situation where we insist that the dynamics of the flow will be the same in the two situations Re, one situation where you have got a miniature model of a ship, for instance a miniature model like this, and the other situation where you have got a full scale model of the ship right.

And you want to see for instance this would be water in this case; in this case it well, in the larger model the actual ship is going to be sailing in water sea water for instance. The question is what should be the medium in which this miniature ship is sailing, and is sailing and we will come to that right. So, here the L is much larger, whereas, L is much smaller here. And U the typical velocity here will be something, but here it will typically be much smaller.

But nonetheless we want to ensure that the Reynolds numbers is the same in this situation as well as in this situation only then the dynamics can be considered to be the same as per as is evident from this non-dimensionalized equation or this non-dimensionalized equation which in many situations carry the same information ok.

And so this is the principle of dynamic similarity. And this is often used I mean instead of figuring out, you design a ship on the computer for instance, but then you want to perform an experiment on it.

And so how do you do this? You do not take the full scale ship and put it in water and that will be too expensive to perform experiments on because you have not completely you know figured everything out. Instead you build a miniature model of the ship, and test it out in the lab.

But in order to do that, you need to ensure that the Reynolds numbers in the two situations are the same, and U and L are very different in these two circumstances. Therefore, nu has to be different in the two circumstances in order to ensure that the Reynolds number remains the same.

So, this is something to be kept in mind. If you want to use this clever thing about you know dynamic similarity. It is a very useful concept its used to figure out you know things like the drag on a ship, the drag on an airplane wing, the drag on a car all kinds of things ok.

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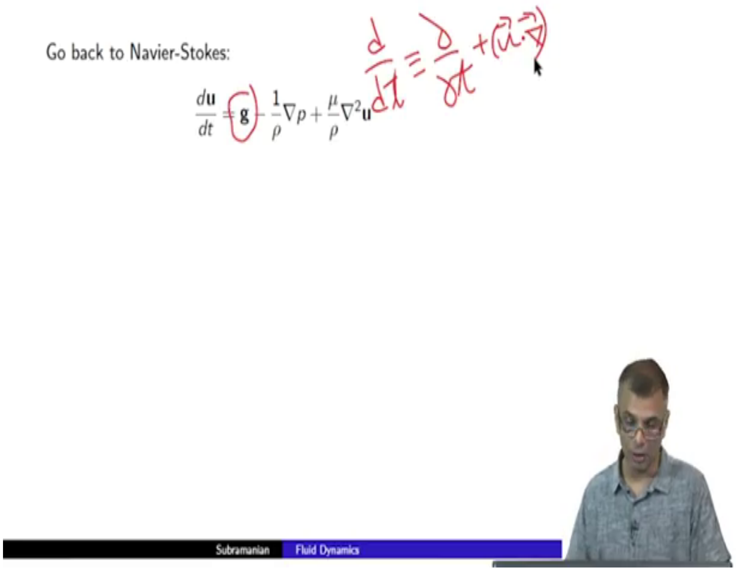

So, having reemphasized this concept of dynamic similarity, let us now go back we have already discussed this, but just to reinforce. Let us now go back to what we were talking about the to reemphasize the fact that the Reynolds number is really the ratio of the inertial term to the viscous term.

So, we go back to Navier-Stokes and so Navier-Stokes here, I have again. So, I do this sometimes, I have again reintroduced the body force term it is really not necessary might as well drop it, and without the body force term you see that it looks exactly the same as what we were talking about earlier.

Except for the fact that you know I am using the Lagrangian derivative the d over dt which is really the Eulerian derivative plus right; we are familiar with this. So, it is a little simpler if we write it I often switch between these two ways of doing things hope you can keep up with me right.

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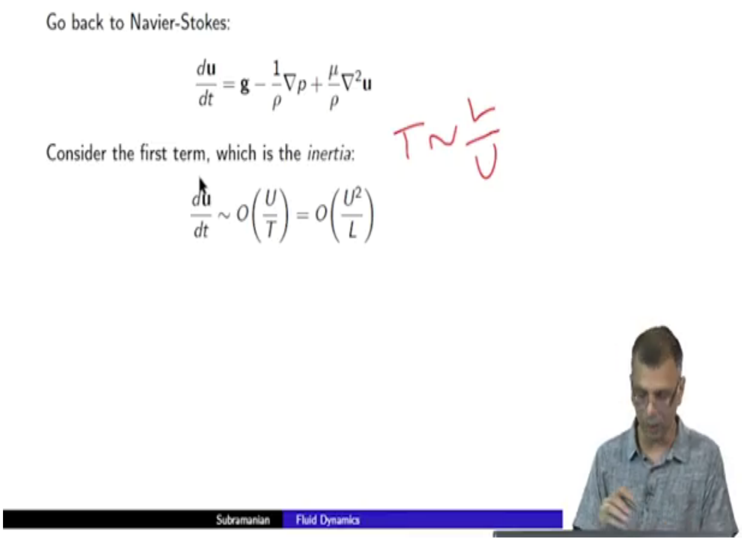

So, the very first term this one, this guy is the first term. This is simply if I multiply this row here it is you know rho a right. So, m a so to speak so that is the inertia. And what is the order of magnitude of the du dt? It is it is roughly some kind of microscopic velocity divided by some kind of microscopic time.

And the microscopic time, I write as what would time be something like L over U right centimeter over centimeter per second right. So, if I substitute for this here, I get order of U square over L, so that is the rough order of the inertial term.

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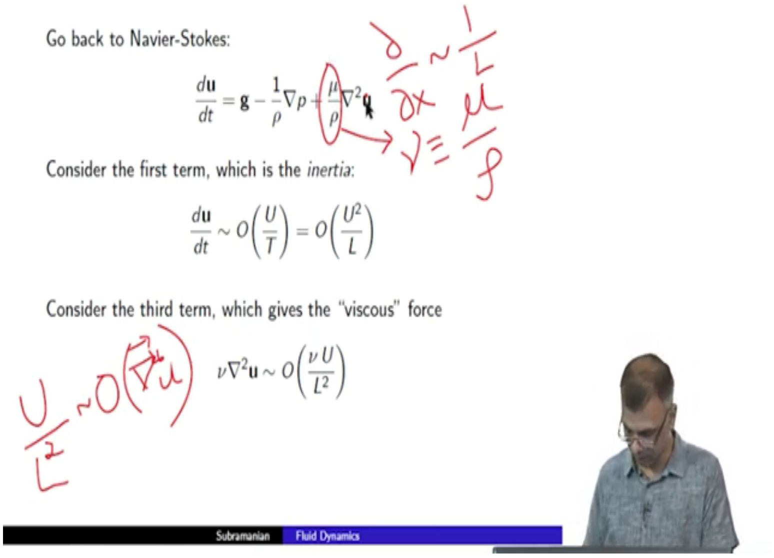

And the third term, this guy ok, this gives the viscous force. And I in order to, so here again just to reemphasize here the nu is simply ok. So this thing is nu ok, so that is how I have written things down here. And in order to get the order of magnitude of this term, this is the second derivative of U.

So, it would be something like U over L squared because the first derivative if you remember you know a d this would be something like 1 over L. So, d square over d x square would give me an L square in the denominator and so that is how I get a capital U because so the order of magnitude I replace the velocity by the undisturbed velocity.

And I replay and in order to account for the second derivative here, I have a capital L squared here. So, U over L squared essentially U over L squared this is the order magnitude of this quantity right, so that is accounts for the U over L squared. And the nu, I simply keep it as it is right. So, this is the order of magnitude of the third term.

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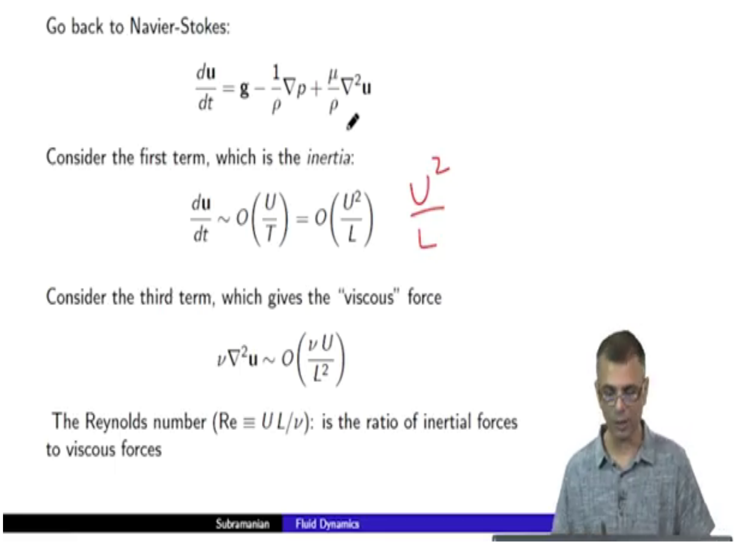

And you got that you already guessed it. The ratio of the first term to the third term is by definition the Reynolds number. And if you divide U squared over L divided by this you get the Reynolds number as UL over nu as we have been writing all along. So, this is simply to reemphasize the fact that the Reynolds number is a ratio of the inertial term to the viscous term. Broadly speaking that is one way of you know that is probably one of the most basic ways of understanding the Reynolds number ok.

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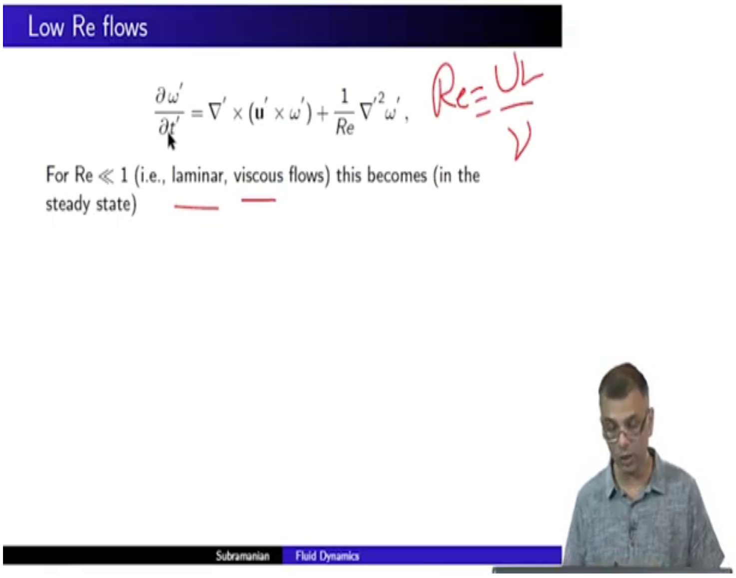

Now, let us now go on to discuss. So, having said this over and over again, let us now go what is the use right I mean you know one of the uses of the Reynolds number is of course, this principle of dynamic similarity where we can investigate flows around a small object as long as its geometrically similar to the larger object. And as long as the Reynolds number and the two situations are the same, so that is one use.

The other thing is well if for instance you know the Reynolds number is very low, yeah in which case the viscous forces predominate over the inertial forces right. So, I should derive some advantage from this from the fact that I recognize this. And so I should be able to throw some terms away and simplify the equation right.

So that is exactly what we are going to do here. And instead of considering the Navier-Stokes equation, I equivalently consider the vorticity equation which we have seen. And this is the non-dimensionalized version of the vorticity equation also something that we have seen just a few minutes ago.

And for a very low Reynolds number, this is much much less than one right Re is much much much less than 1. And this would describe generally described; describe a situation with high viscosity; why is that? Because you know the Reynolds number is U L over nu. So, if the Reynolds number is very low, that means, nu is high in comparison to what in comparison to the product of U and L right. And so this would describe relatively viscous flow say the flow of honey for instance.

And we know from everyday experience that honey the flow of honey is typically laminar right. You let honey flow and it is a little difficult to envisage a situation where the flow becomes turbulent. Whereas, it is a little easier to envisage a situation where a relatively less viscous fluid like water turns turbulent right.

So, hence these two adjectives laminar, viscous flows; viscous is obvious from here, but this laminar this adjective needs a little bit of explanation that is what we did. So, this particular equation in the situation where you know the Reynolds number is much much less than 1 only this term is important right.

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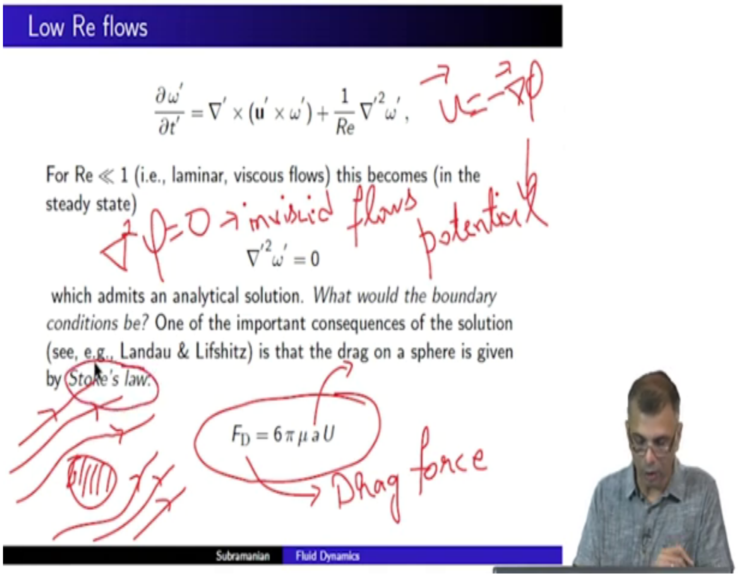

And we consider only steady state situations where you know d over d t prime goes to 0. And this becomes so this is no longer important and. So, this becomes 1 over Re nabla square nabla prime square to be precise omega prime equals 0, and because it is 0 1 minus will multiply out the Re.

And you get this very familiar looking equation right. So, this is essentially the Laplacian equation in which admits an analytical solution. So, this is one of the advantages of writing it this way. Importantly I would ask you to think about what the boundary conditions would be right. So, this has two special derivatives. So, you need two boundary conditions.

So, what would the boundary conditions be for you know highly viscous flow. I will give you the answer. You can think I will give you the answer in terms of the velocity. Both components of the velocity need to vanish on if you are considering a flow around a solid surface, both component of the velocity components of a velocity need to vanish on the solid surface. This is the, this is typically the boundary condition for viscous flow ok.

So, in terms of omega prime or omega for that matter, I will let you figure that out yeah. So, I will let you think about this question. What would the boundary conditions be? Very, very important solving the Laplacian is one thing there are general solutions right. And, but in order to pin down the constants in this in the general solution, you do need to apply the boundary condition. So, boundary conditions are all important.

One of the important consequences of this solution is that and we will not go into this. We I will simply state this fact is that the drag on a sphere is given by this ok. Now, let us let us pause for a moment and talk about what this means. If you recall; now even if you do not recall that is all right. Imagine a situation where you have got a sphere embedded in a flow.

So, the flow would look like slightly disturbed around the sphere, very far away from the sphere the flow is undisturbed with many other you know relatively less disturbed streamlines like this and so on so forth. And this is the sphere not a very good drawing, but it is a solid sphere.

Now, what is your expectation of the drag on a sphere right. You everyday experience if you are trying to either the fluid is flowing fast rate or equivalently, you are trying to move the sphere inside the fluid right. And your everyday experience tells you that there will be some drag, it is not completely frictionless, it is not a completely frictionless situation right.

Now, if you recall for a perfectly inviscid case, in other words when we a situation where you can the entire flow can be described exclusively in terms of the velocity potential which is given by this is just convention the negative, they are negative. And the velocity potential is given by this phi and this is the right.

And if you recall the solution to a perfectly inviscid situation would be 1, where would be actually the solution of the Laplacian like this, looking very much like this except its not omega there, it is actually that would be the solution in a perfectly inviscid situation right, inviscid flows.

And think of it is for an inviscid flow you allow for infinite slip at the boundary, the flow does not stick at all. The tangential velocity is allowed to slip as much as it wants to; that is the boundary condition at the surface of the sphere. You insist that the normal component of the velocity is 0.

And the normal derivative is 0, you do need two boundary conditions because you have got you know, but the tangential velocity as such is allowed to be just on it can slip infinitely right. And in this situation, we will see this when we a little later during this current discussion when we talk about you know lift on a ball on a spinning ball to be precise.

How exactly one comes about to the conclusion that I am going to state, but to recapitulate one can find one can show that in a completely inviscid situation where infinite slip is allowed between the sphere and the flow, this particular the solution to this equation with the appropriate boundary conditions. Suggests that there is no drag on the ball. This is called the D Alembert's paradox, there is no drag on the ball whatsoever ok.

In other words, you can you know the ball can slide through the fluid or the fluid can slide past the ball in a frictionless manner, no drag at all. Now, we know that this is unphysical, but this is what the solution predicts what can you do about it right. However, in the presence of a finite viscosity in particular for in our case we are talking about low Reynolds number flows and so viscous forces are important.

And when we solve this equation with appropriate boundary conditions, it can be shown that there is a finite drag as would be expected. And the drag force, this is the drag force, the drag force is proportional to the radius of the sphere, and the first power of the velocity and also to the viscosity coefficient ok.

This is calls Stokes law, very important law. This holds only for the case of a perfectly smooth sphere. And there are many you know specializations that go into this law. It is been experimentally verified to a great degree of success.

And so but this is a very important law, I am not going into the details of how exactly this drag law is derived from the solution to this ok. You can look up Landau Lifshitz or other similar text maybe Kundu the text that we you know discussed in the beginning. But so far I use to say that this is what this particular solution low Reynolds number the solution to a low Reynolds number situation predicts.

Now, when we go to the next discussion, we will also look at the other extreme in other words the Reynolds number much, much larger than one situation, and try to see what the equivalent law would be in that situation. And then we will put the two together and investigate some interesting behaviour.

Because you know in real life situations you know you often transition from a low Reynolds number situation to a high Reynolds number situation. It is not always that you can remain just in one situation for instance you know a car or an airplane for instance starts out slow and then accelerates.

So, when it is slow the Reynolds number is relatively slow, because the U is small right. But after it picks up speed, the U becomes large enough that you know the U times L becomes appreciable in comparison with the nu in the denominator. So, the Reynolds number becomes high. And so you are actually transitioning from a low Reynolds number situation to high Reynolds number situation.

And as you can imagine it is very important to figure out what the drag would be you know in one situation as opposed to the other. And there is there a reduction in drag, is there an increase in drag as you as the velocity increases, these are all very important engineering considerations, and so we will try to understand how this thing works.

So, just to summarize the result of the this or the impart of this particular discussion for low Reynolds number flows only this term is important in a steady state situation, the rest of the term by definition in a steady state situation this goes away. And since this is much more important than this the equation essentially boils down to this.

And from this ok, one can show that the drag due to a viscous fluid flowing passed a sphere, a smooth sphere, a smooth solid sphere is given by this. And the main thing about to note about this is that the drag force is linearly proportional to the velocity ok. And the viscosity coefficient also finds a place in here.

This is not the case. And when we start talking about high Reynolds number situation, we will find that this is not the case. So, we will stop here for the time being.