Fluid Dynamics for Astrophysics
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Lecture – 17
Poiseuille flow, deriving viscosity from microscopics

Hi. So, now I wanted to discuss something that we have already seen before in when we started talking about the Navier stokes equation.

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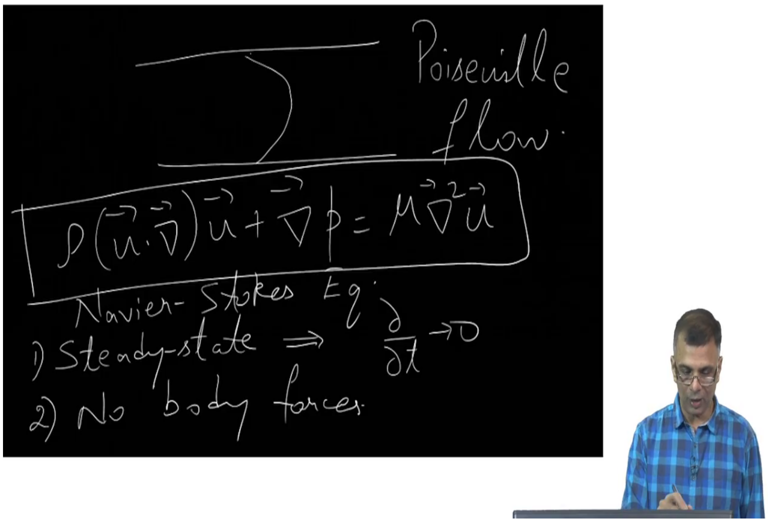

We use this very diagram to flow of a viscous fluid through a pipe and we remarked that the velocity because it is a viscous fluid the velocity would go to 0 right at the surfaces of the pipe, but the velocity would have its maximum value at the very center. And so we guessed that the, you know the profile of the velocity would look like this.

Now, turns out that there is a way of deriving this deriving the exact shape of the flow without too much trouble and in particular this problem is called that of Poiseuille flow; let me see if I can get the spelling correctly yeah ok. So, the equation here to consider is a Navier stokes equation since it is a horizontal pipe we do not have to worry about gravity as in you know and they are both at the same level and so the only terms left are plus gradient of pressure.

So, flow is so the fluid is flowing only on the under the influence of pressure and yeah so ok. So, this is the Navier stokes equation, steady state which is why you guessed it, which means the partial time derivative goes to 0 yeah the other thing is no yeah let us just say no body forces, in other words the gravity terms are not important. So, that is what this equation describes. Let us say that the velocity here is described by you know draw this again, ok.

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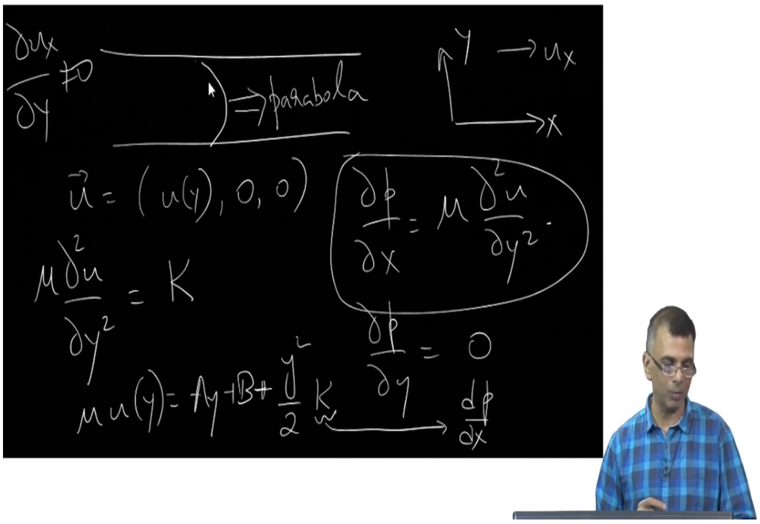

This will be the x axis and this will be the y axis. So, the u and there is a gradient of v x, there is a gradient of u x along y which means that the u is very simple it becomes u as a function of y yeah 0 0. So, there is only a u x yeah and this is the u x and that is a function of y, which means that; obviously, right; which is why viscosity is important there is a shear here right, but this is all there is which means that in this equation, only this term and that term are important the rest of the terms they all fall away ok.

So, the way to write it is it splits up into the Navier stokes equation basically splits up into dp dx is equal to right and right. This is how it splits it this equation just splits up ok. Now, you see how remarkably simple this already is beginning to look. In particular we focus only on this nothing else just on this.

Now, as far as finding out you know the y variation of u is concerned this is as good as a constant right think about this. This is oh yeah so the other thing to keep in mind is that this these plates are infinite in the direction perpendicular to the screen and also infinite in the x direction so that is another thing to keep in mind.

So, as far as this equation is concerned the d square u dy squared this is a constant this is a dp dx, as far as a y variation is concerned this one is a constant right. So, really what you are saying is that mu equals some constant K this can immediately be integrated. You take you know du dy as K times y and u would be something like K y squared in particular that the solution would be.

So, this is a very simple equation to integrate and in particular the solution would be something like equal to A y plus B plus y squared over 2 this K or where this K is really this is really a ok, it is a pressure gradient in the x direction. So, this is a plus here; I beg your pardon for my handwriting.

So, really if you look at this this is a equation of a parabola and you apply the correct boundary conditions and it looks like a parabola. The boundary conditions would be that the velocity goes to 0 here and here and so, depending upon where you take the origin if you take the origin here or if you take the origin at the center of the pipe you can even get rid of this A y term. And it looks just like you know a y squared over 2 and that is exactly the equation of a parabola.

So, this confirms our intuition you know that we started out with on this slide we said you know a viscous fluid would be something it sticks to the boundaries and you would expect the velocity to be maximum at the towards the middle and that is exactly what this looks like. So, this is the solution to what is called a Poiseuille flow this would be in cylindrical I mean in rectangular coordinates if you specialize to a cylindrical pipe right.

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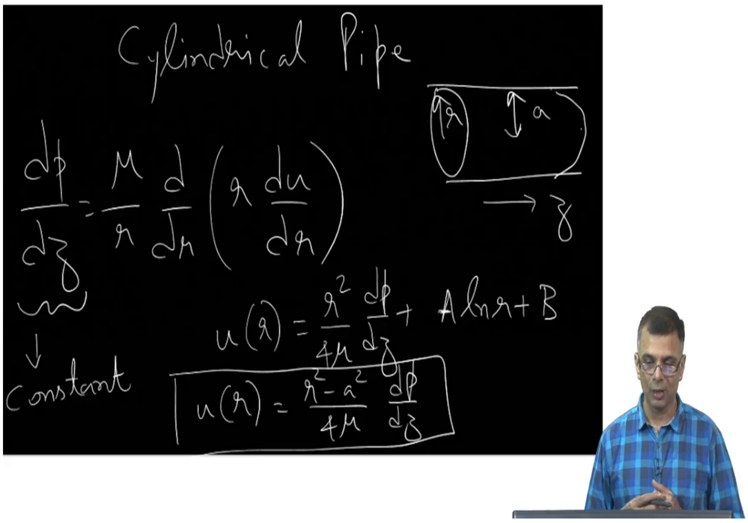

A cylindrical pipe also infinite in the x direction and so it would be something like this so, this would be the r coordinate in which case and this would be the z coordinate just making small changes in notation. The Navier stokes equation now this equation if you write this down in cylindrical coordinates it simply turns out to be mu over r.

I am doing nothing, but writing down the relevant terms of the Navier stokes equation in cylindrical coordinates ok. And again this is a constant as far as the r variations is concerned this term is a constant. So, you can see that you expand this out and you will have a first derivative in u and a second derivative in u and the, you apply the appropriate boundary conditions and turns out a radial wave.

So, again it is a parabola you will have a real the full solution actually will be something like. So, this would be a parabolic kind of thing in r as such you would have a logarithmic term, but if you apply the boundary conditions correctly you will find that a has to become 0 and the particular solution becomes this the radius of the pipe is a ok.

So, it turns out that the particular solution is squared over 4 mu d p, this is the solution we are after mind the 2 here. So, this is a solution for again and this also the profile of this also looks like this with the maximum at the center and 0 at the ends. So, this is the solution for flow of a viscous fluid inside a cylindrical pipe.

So, this is I figured out introduce these things by way of simple examples of solutions to the Navier stokes equation, ok. And I have not sketched out every single step, I urge you especially for instance you know applying the boundary conditions correctly so that you figure that A and B are both equal to 0 and those well B is not equal to 0 that is how you get the a squared right.

So, I urge you to you know do those steps. So, it is useful to demonstrate how simple solutions to the Navier stokes equation not merely talk about the equation right. So, it is in that spirit that I said this. Now, I figured I would say something a little of the mainstream of what we have been talking about.

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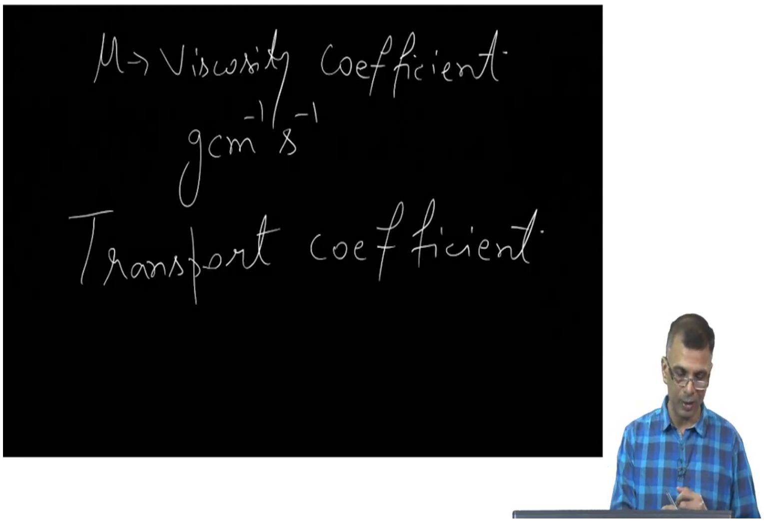

But you see we have been talking about viscosity a lot and so mu, which is the viscosity coefficient which has units of right. Is an example of what is called a transport coefficient, what does it describe it describes a transport of momentum. And this is simply taken to be a property of the fluid this is a particular number for a particular fluid just like thermal conductivity or something else right.

Now, we talked about how transport coefficients well, how bulk properties like density or velocity or things like this are derived from microscopics towards the very beginning of the course. Let us I figured we would revisit this thing about a viscosity coefficient and, in particular show how this you know this corresponds to transport of momentum in particular momentum flux between layers.

And how a viscosity coefficient is derived using the concept of a mean free path using microscopics a little bit right. It is quite simple. So, let us consider a situation where you have a two dimensional flow and same thing as before you have z like this and y like this.

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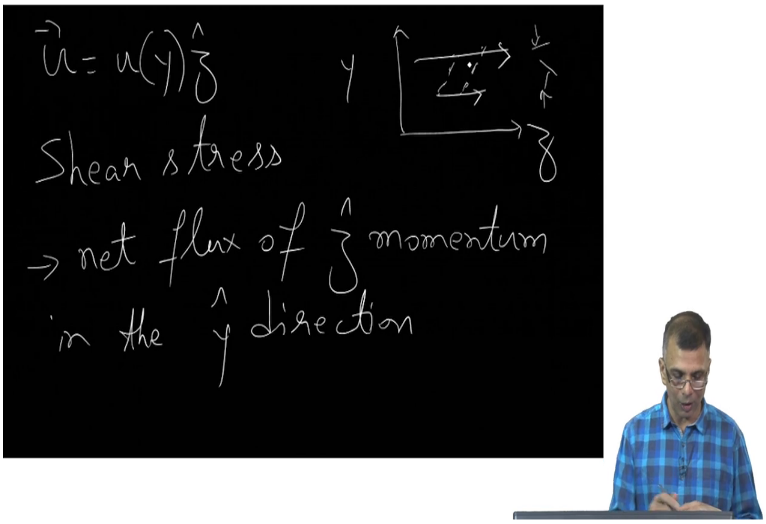

So, you have a situation where u is equal to u as a function of y in the z direction. So, the shear stress from which viscosity is derived remember is nothing, but the net flux of z directed momentum in the y direction. In other words if you have a layer here and here remember our analogy of you know suppose this was a viscous fluid and you know you had a larger velocity here and a smaller velocity here.

Viscosity remember was all about little rubber bands joining these two layers refuse that the; and what do these little rubber bands do they refuse to let you know smooth sliding happen.

Let me draw these rubber bands a little bit little better. And so what do the rubber bands really do it they carry the fact that.

So, there are particles and these would be separated by approximately one mean free path in a microscopic sense and what they would do. So, molecules or atoms as the case might be from this layer would go from here to there and carry the fact carry the information that.

Right here, there is this amount of flux of z momentum and they would and when they hit this layer, which is spaced one mean free path apart they carry that information that there is this flux of z momentum and they and they tell this layer about that information and when the collision happens this layer as a result of the collision it slows down.

And so, that is sort of where the rubber band analogy comes in ok. In particular you can write down something like a unidirectional particle say the expression for a unidirectional sorry.

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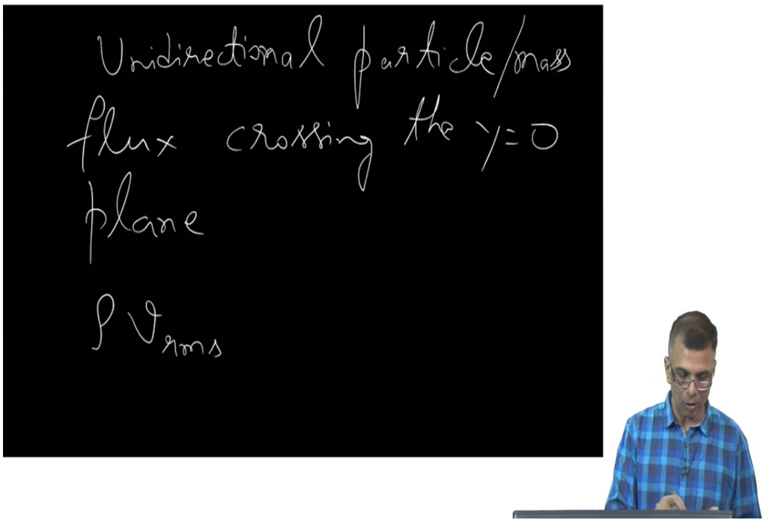

Crossing the y equals 0 plane this is something like rho this is the unidirectional particle flux well mass flux particle or I should say mass flux. If I was strictly talking about particle flux this would be N V rms, if I am talking about mass flux it is really rho V rms right and the other thing to remember here is that.

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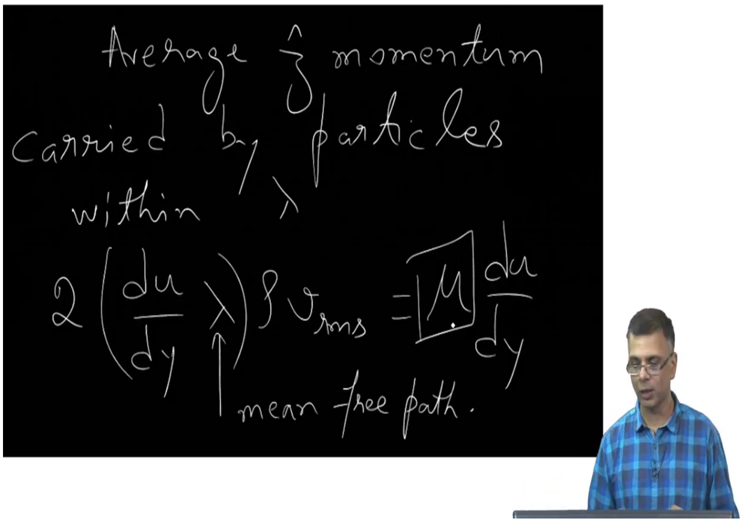

The magnitude of the average z momentum carried by particles within a mean free path lambda right and so, this is the other important thing. So, the average z momentum carried by particles within one mean free path now is something like du dy times lambda ok. So, you multiply that with this right. So, you have this times rho times V rms. And because of the fact that you have particles going on both sides of the plane you have a factor 2.

Now, remember I said the shear stress is essentially the net flux of z momentum in the y direction. So, that is this that completes this the net flux of z momentum this is the this entire thing is a net flux of z momentum in the y direction. And we arrived at this from a particle kind of picture yeah, the average z momentum carried by particles within one mean free path is this and this is the net particle flux.

And the shear stress by definition is equal to; this is how the mu is derived from the microscopics and this is equal to du dy. So, this is how you derive mu from the microscopics from things like the rms velocity of the particles that constitute the fluid. Now, remember in order to have a meaningful definition of a fluid you should have many many mean free paths in any representative macroscopic length scale.

So, all that is very much there, but I figured this is how I will give you an illustration of how a transport coefficient such as a viscosity coefficient is derived from the microscopics, right. So, in some sense yeah ok; so, this is a microscopic quantity the du dy, but the rest of it are all sort of microscopics and everything really depends upon this mean free path ok.

If the mean free path is large the viscosity would be large and vice versa; however, the mean free there are funny situation such as collision less fluids or collision less plasmas, where the mean free path is actually the regular collisional mean free path. Mean free path for regular collisions between particles is actually much larger than any meaningful microscopic length scale.

And in such situations you cannot apply this simple minded expression and you would have to find an effective mean free path that is not simply the collisional mean free path. Let me write this down since I am saying this word over and over again, I should write this mean free.

And you would have to find a more meaningful mean free path that is not simply the collisional mean free path, but that is for collisionless fluids and that is a bit of a special situation I just thought I would show how a macroscopic transport coefficient, which is regarded as a property of the fluid is derived from the microscopics.

Before going on to talk about the other aspects of the course and transitioning again to fluids and not from a microscopic point of view, but just the macroscopic fluids where we simply talk in terms of transport coefficients. So, that is it for now.

Thank you.