Fluid Dynamics for Astrophysics

Prof. Prasad Subramanian
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture – 16
Boundary conditions in Navier-Stokes equation, d'Alembert's paradox

Alright, now I figured we would having spent a fair amount of time on the Navier Stokes equation. Let us write it down again and look at a couple of important mathematical aspects just one really which has to do with boundary conditions. We have alluded to this earlier, but I figure it is useful to emphasize that a little more.

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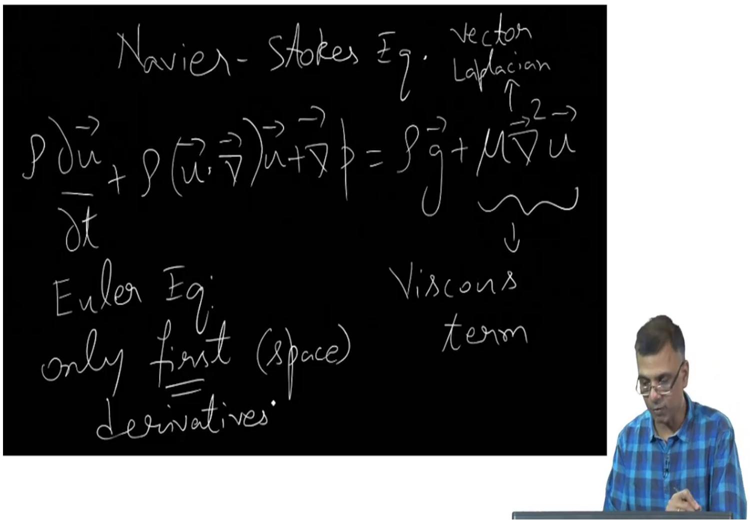

So, to begin with let me write down the Navier Stokes equation once again right. So, again in lab coordinates in the Eulerian frame plus u dot del this is the usual thing. And instead of writing the pressure term the gradient of pressure on the right hand side with a negative sign I choose to write it here yeah.

Ah This will be the body force and mu where I want to emphasize that this is a vector Laplacian right. Now, so a few thing so remember we wrote this down in a few different guises one of the guises was one where all of this this this and this were all bundled together in one term right.

Where, but this is a slightly more popular way of writing down the Navier stokes equation and this entire term as we know is the viscous term. And this is what in some sense makes it the Navier Stokes equation, if this was not there if the viscous term was not there this thing is just the Euler equation that you have seen earlier which does not take viscosity into account.

The first thing you should notice is that if the viscous term was not there this entire thing has only these are all spatial gradients right, this grad these are all spatial gradients in these would look like d over d x or d over d d y no d square x over d x square, no second derivatives only first derivatives; first derivative, first derivative, first derivative and well this these two first derivatives in space first derivative in time. So, for the you know ah. So, what I mean is that the Euler equation only first space derivatives right that is a evident.

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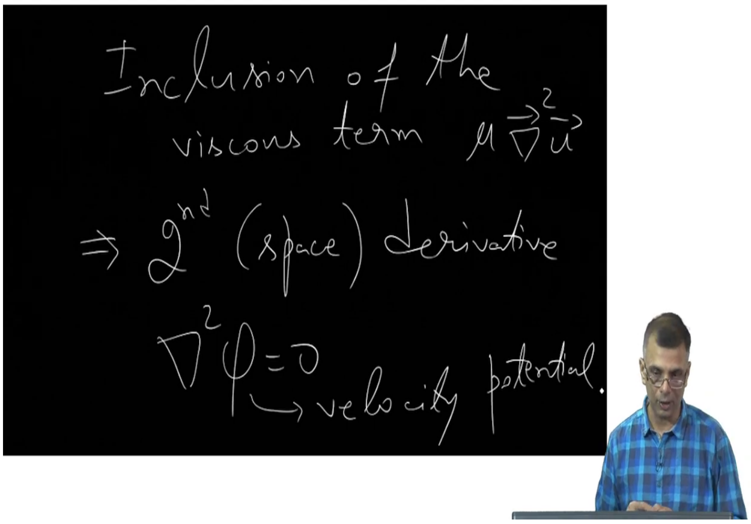

Whereas the inclusion of the viscous term mu this one this gives rise to second space derivative, if this completely changes the character of the equation. This is the main thing by way of boundary conditions you have two derivatives you need two boundary conditions right. You have one derivative you remember when we were talking about the it is a different equation, but nonetheless when we were talking about you know this equation, where this was the velocity potential that was for inviscid flows.

So, it did not have the viscous term, but nonetheless the equation was this and we had two boundary conditions you remember, we had one boundary condition on the surface of the sphere and one boundary condition at infinity. Sure enough you need two boundary conditions because you have two derivatives. Similarly, and so I will just I just say that and I will not confuse things I will just erase it here ok.

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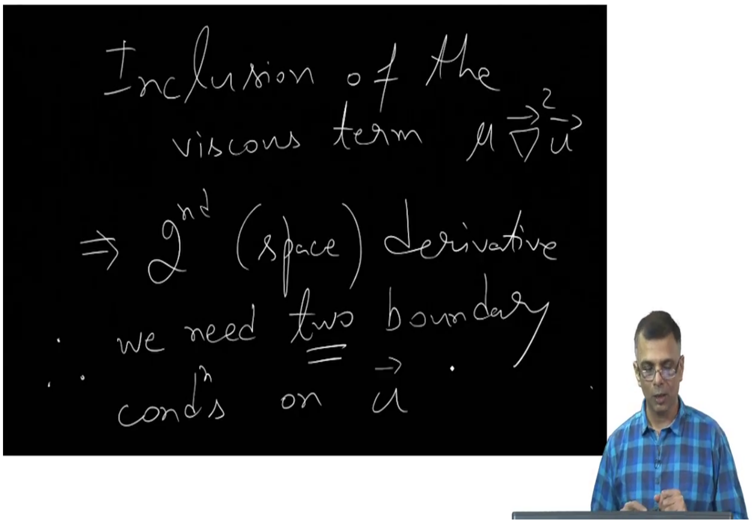

So, therefore, because you have two boundary conditions therefore, we need two boundary conditions on the velocity not on the velocity potential, but on the velocity ok. For instance you might on the surface of an object you might want to specify both the normal as well as the tangential velocity or maybe the tangential velocity and the derivative of the normal velocity or some mixture thereof, may be a dirichlet in other words a dirichlet boundary condition or a neumann boundary condition or a mix. Either way you do need two boundary conditions.

So, the inclusion of the viscous term changes the character of the momentum equation in this way right. You would suspect this in any case because you see when we were talking about purely inviscid flow you remember we were we were allowing for when we used the velocity potential and then derived a velocity from that for inviscid flow.

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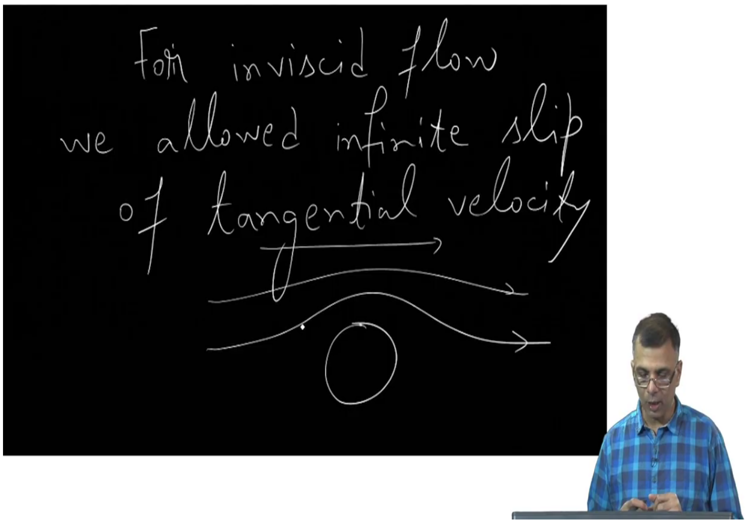

We allowed infinite slip as much infinite I will tell you what this means of tangential velocity. Like so if you have a sphere and this we have drawn many many times earlier like so like so and like so yeah. At the surface of the sphere you were allowing the fluid to slip as much as it wanted to and the, but that is allowed only for inviscid flows ok, you need to specify only one boundary condition on the velocity ok.

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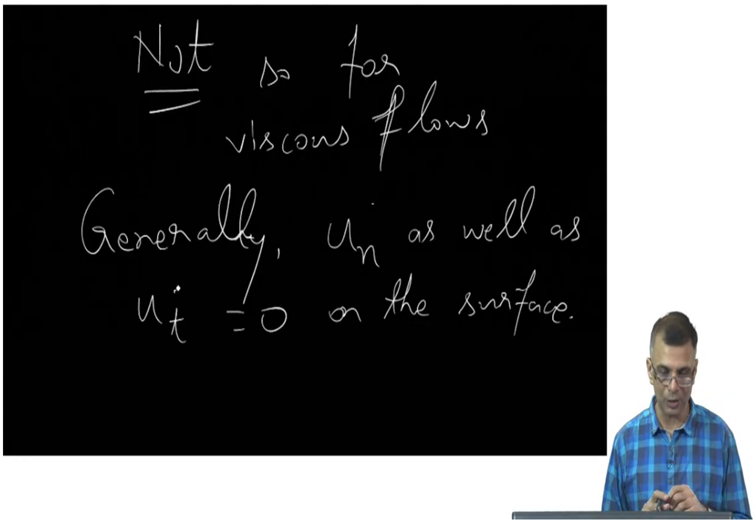

But not so for not so for viscous flows. Now this is you know intuitively obvious also right the whole point of viscosity is that the fluid sticks infinite slip of the tangential velocity this you cannot you cannot allow for a as much of a slip as you want to that the fluid sticks ah, if it was honey if knowing fastest sphere this is intuitively obvious is not it.

So, you would have to you would have to generally you insist generally, you say that ah u normal as well as u tangential equal to 0 on the surface, when viscosity is included. And you know this for instance you know on a on a fan blade right ceiling fans you know you must have seen the dust sticks on the blades of the fan right. That is because if it was a if and it all depends upon how smooth how you know the point is air does have a finite viscosity even though it is a general it is a it is a mostly inviscid in the bulk flow.

But as the blade accumulates oil or something it does I mean you know air does have a finite viscosity and when even though the blade is rotating even though the fan blade is rotating right on the surface what happens is stuff dust particles do come to a rest ok.

In other words the u normal as well as the u tangential are equal to 0 right there, and that is why you see dust sticking because of finite viscosity ok. It does not if that was not there the dust would be flying off the blades and you would not have any dust accumulation whatsoever.

Ah, but you know from everyday observations that dust does stick to the blades of a fan and that is because of finite viscosity ok. So, you do invoke I mean you know stuff does stick and so this is one aspect that I wanted to emphasize the difference between viscous flows and inviscid flows I wanted to emphasize this.

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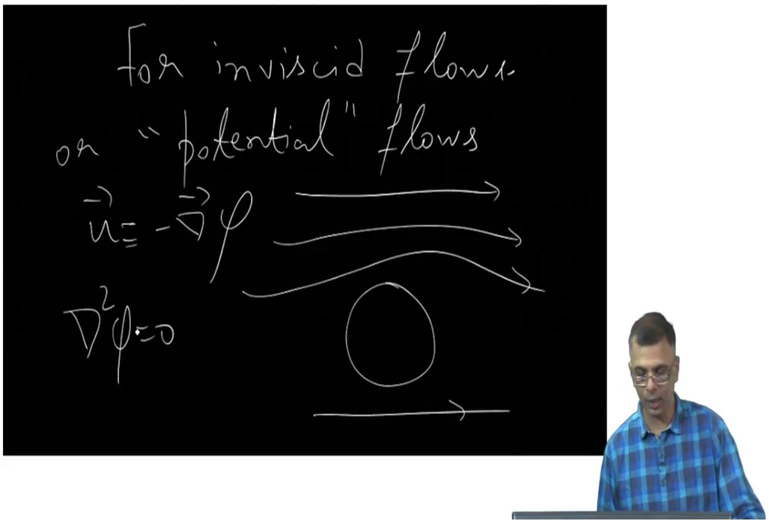

In fact, for inviscid flows for perfectly inviscid flows or perfectly inviscid flows which can be or equivalently potential flows which we have discussed. Where, u can be written as this is a velocity potential not a gravitational potential u can be written as the gradient of you know potential.

You can and this was the situation that we discussed right. Where, now we are allowing for inference slip right. So, we have we have written down if this is. So, then del square phi is equal to 0 and you recall that we wrote down the solution for phi and we applied the boundary conditions and got the particular solution for phi right.

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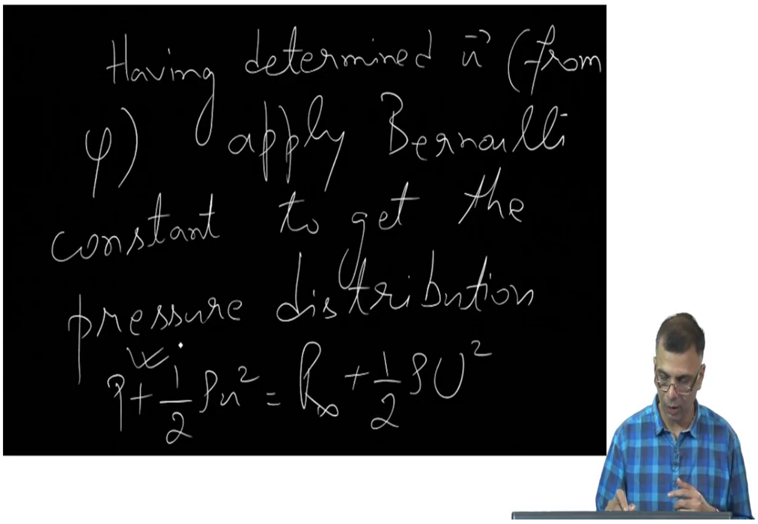

Now, if that is the case you can verify the following. You can apply I mean knowing the full solution for velocity, you can apply the having known u from phi of course, yeah. You can apply the Bernoulli constant to get the pressure distribution over the entire surface ok.

You can apply the Bernoulli constant to get the pressure distribution over the entire surface and if you do this. So, in other words what I really mean is simply say you know p, what I mean by this is p plus half rho u squared anywhere yeah is simply equal to p infinity at some very large distance plus half rho U squared where, U is this capital U is just the undisturbed velocity at infinity this one ok. If you do this you know and the p infinity p at a very large distance is also known. If you do this you can determine the pressure distribution everywhere yeah..

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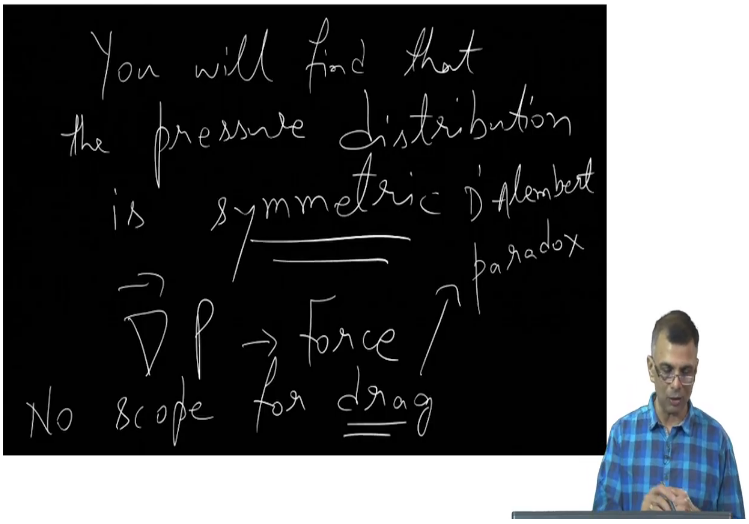

And, if you do this you will find that the pressure distribution is symmetric. In other words there is no ; you see only if there is an asymmetric pressure distribution the pressure in the front is different from the pressure at the back or vice versa there is some potential for drag for a drag force you see the gradient of pressure is what gives you know a force right.

I mean you know a gradient of pressure this is like a force, and there is no gradient of pressure if the pressure distribution is completely symmetric around a body there is no scope for force in other words there is no scope.

So, therefore, there is no scope for drag ok. In other words a sphere immersed in a perfectly inviscid fluid flowing passed it or equivalently a sphere moving through a perfectly inviscid fluid does not experience any drag at all, and this was known and this is opposite to everyday experience.

You know that it takes a certain amount of energy to it takes a certain amount of effort to move a body through a fluid no matter how inviscid it is right. So, this particular thing was called the D'Alembert's paradox the fact that because the pressure distribution is symmetric and that is simply a consequence of applying you know Bernoulli constant to the solution of Laplaces equation.

Which is applicable for perfectly inviscid flows you find that, there is no drag on the sphere and this is called the D'Alembert's paradox. And later on it was it was discovered that this is because there is no such thing as a perfectly inviscid flow there is some sticking there is always some sticking at the surface.

You cannot allow for infinite tangential slip there is always some sticking at the surface there is always some viscous effect and that is what gives rise to the drag. Now, going back it is not as if you know there is I mean it is a little misleading to say there is viscosity or there is no viscosity it is not exactly like that, it is just that you know whether this term is whether the viscous term is important or not.

Depends not so much on the well it does depend upon the magnitude of the you know viscosity constant yes it does, but it also it actually depends upon this combination ok. So, if the nabla square u, if the velocity derivatives are very large near the boundary then this term at the boundary near the boundary this term assumes importance in relation to of course, there is this this this and so on so forth.

Far away from the body of the fluid the flow might well be might well be largely inviscid or inviscid to a very good approximation. Because you know there is not much scope for change in velocity and definitely not much scope for second derivatives ok.

So, the flow might well be inviscid in the bulk, but when it comes to boundary layers when you are talking about layers that are close to the boundary that you are considering this term like for instance you know the blade of the fan right there or you know very close to this sphere that we are talking about and this term might well assume importance.

And so this gives rise to the viscous term becomes important and this was my main point when you are talking about, you know an equation with a second derivative in velocity you need to consider two boundary conditions ok. And generally the two boundary conditions are simply you know the total velocity goes to 0, both the normal velocity as well as the tangential velocity yeah both of these go to 0 right on the surface. So, this is the general sort of a formulation ok. So, we will stop here for the time being.