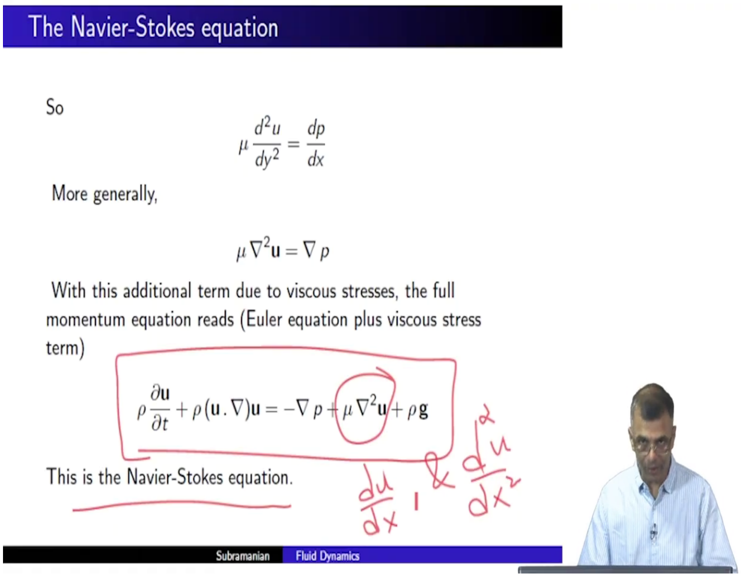
Fluid Dynamics for Astrophysics
Prof. Prasad Subramanian
Department of Physics
Indian Institute of Science Education and Research, Pune

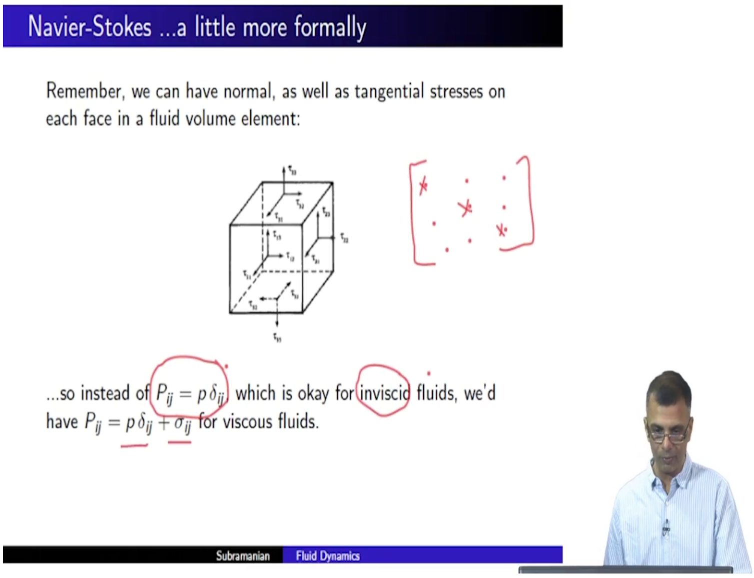
Lecture – 14
Navier-Stokes equation (contd.) and energy equation

(Refer Slide Time: 00:16)



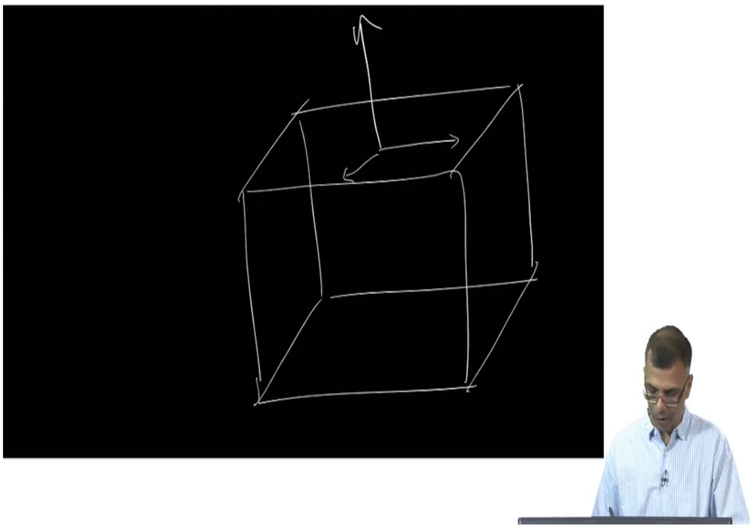
Going back to the Navier-Stokes, this is what we did when we cut off last. And so essentially we derived we physically motivated at the appearance of the viscous term in addition to the other terms of momentum equation that we were already familiar with. So, let us do it a little more formally since the Navier-Stokes equation is so important. It is a momentum equation, but since it is so central to all the fluids, it is important to look at it from a couple of different points of view.

(Refer Slide Time: 00:48)



This is one point of view that we have already come across in fact ok. Now, remember we can have normal as well as tangential stresses on each face in a fluid volume element. We have looked at this earlier as well.

(Refer Slide Time: 01:07)



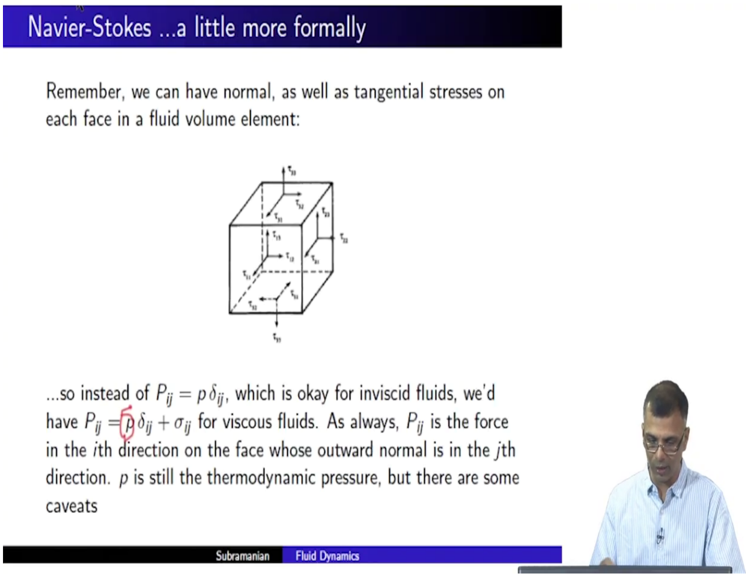
And it is just this I am going to draw this a little better like this. So, on a given face, you can have normal stresses; in other words, you can have a force per unit area in this direction as well as along the face right. And these would represent these kinds of normal forces which represent pressure.

And these kinds of tangential stresses which a normal stress would be like this and tangential stretches would be like this, or like that yeah as we have said several times before. And this volume is of course, an imaginary volume. It is not an actual physical thing.

So, instead of so we said when we were talking about in viscid fluids, remember we said you know we were specializing to up to a situation where the pressure tensor was like this. In other words, if you the pressure tensor was like essentially a matrix, you only have the diagonal elements; you only had the diagonal elements.

And the off diagonal elements were all 0 ok. So, only you only had this, this, and this, the rest of them were all 0 right, which is what this is saying. Unless i is equal to j, you have 0s; when i is equal to j, then you have p, p, p. This is for inviscid fluids, for inviscid fluids. But in general that is not always true you have a pressure tensor. So, you have the diagonal elements as they were before plus on top of that you have the shear stresses you have the sigma ijs right. So, let us now see how to go ahead.

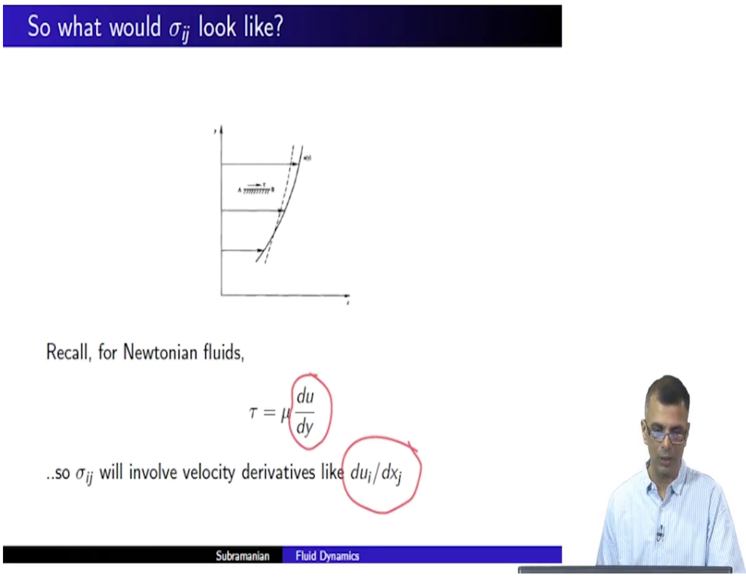
(Refer Slide Time: 03:12)



So, you right, so, yeah as always P ij is a force in the ith direction, i can be x, y, or z, on the face whose outward normal is in the jth direction. Again j can be x, y, or z. If it so happens, the j is equal to i you have p xx or p yy or p zz. If not, you have the shear stresses.

So, when i is equal to j, you get the diagonal elements of the pressure tensor; if not, you have the off diagonal elements right. So, we have said this before. And what we are now saying is no different from that. So, p is this quantity this small p and that we wrote down this one is still the thermodynamic pressure, but there are some caveats.

(Refer Slide Time: 03:59)

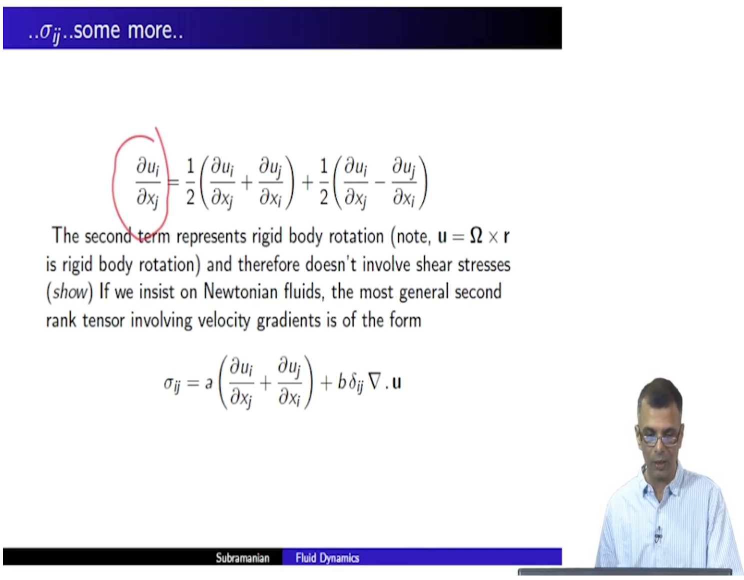


So, the question now is what would sigma ij look like ok, what would the shear stresses look like? We have written it down formally, but what would it look like? Again we go back to our definition of Newtonian fluids. Recall Newtonian fluids are one are ones where the shear stress the force per unit area which is this tau is related to the strain which is du dy via proportionality constant called mu.

Now, this mu need not be the same for x y as it is for y z it may not it need not be the same. But as such you know just to motivate the idea it is good to go back to this kind of a one-dimensional treatment right. So, therefore, clearly the sigma ijs which were these guys these off diagonal terms, they will clearly involve velocity derivatives like du i d x j where i and j can be any of x, y, and z.

So, for instance this could be du x dy right du y d z and so on so forth, i and j can take on you know any of the x, y, z, numbers right. So, but the thing is the sigma ij will involve velocity derivatives of this kind cross derivatives. If you have an x here, it would be a y here; if you have a y here, there could be an x here. There would be cross derivatives right.

(Refer Slide Time: 05:35)



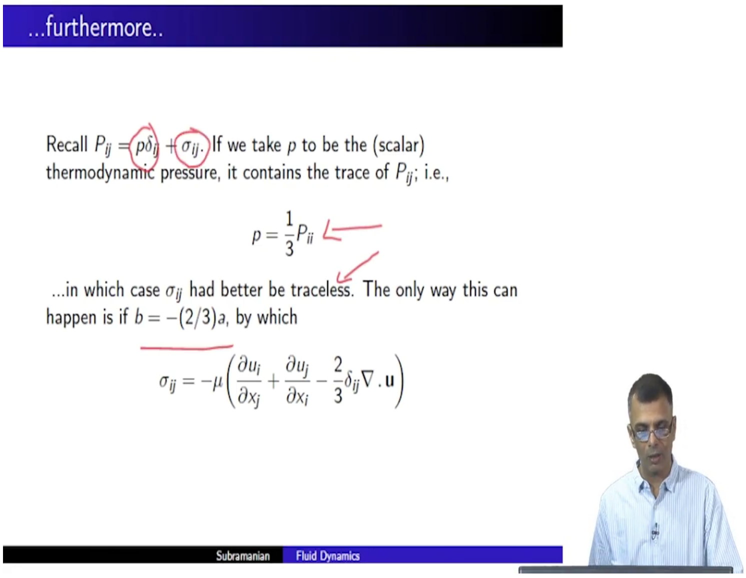
So, this is a general way of writing it. So, a du i you can immediately see that this is true just an algebra this is true. There is a reason we write it this way that is one of the most general ways of writing down du i d x j. Now, it turns out that the second term this guy represents rigid body rotation of this kind, where omega is a constant. And you can show this; it is not hard to show that this really is this kind.

So, whenever you have rigid body rotation, the rotation is rigid. There is no question of velocity stream lines looking like this. There is no question of that. These are arrows, yeah, there is no question of velocity streamlines decreasing, I mean the length of the velocity vector is decreasing. It is rigid body rotation. So, this does not involve shear stresses.

Now, if we insist on Newtonian fluids, the most general second rank tensor which is what this is a second rank. Because it has two, because it has two indices i and j involving velocity gradients is of the form is of this form. I do not want something that looks like this. I only want something that has a plus sign ok.

You can see that by way of dimensions, so the dimensions of this divergence and this are the same ok, except there is a there is a delta ij here, and there is no delta ij here that is only difference. And I stick an a here and a b, there we will see what we will do with these for the, for the time being, let us just take this and proceed.

(Refer Slide Time: 07:18)



And recall that this was my basic definition of capital P ij, I had the diagonal terms here, and the off diagonal terms here right. So, if we take p, this small p – this guy to be the scalar pressure, the scalar thermodynamic pressure. It contains a trace of P ij. In other words I add up the diagonal terms of P ij and that should add up to small p the thermodynamic pressure right.

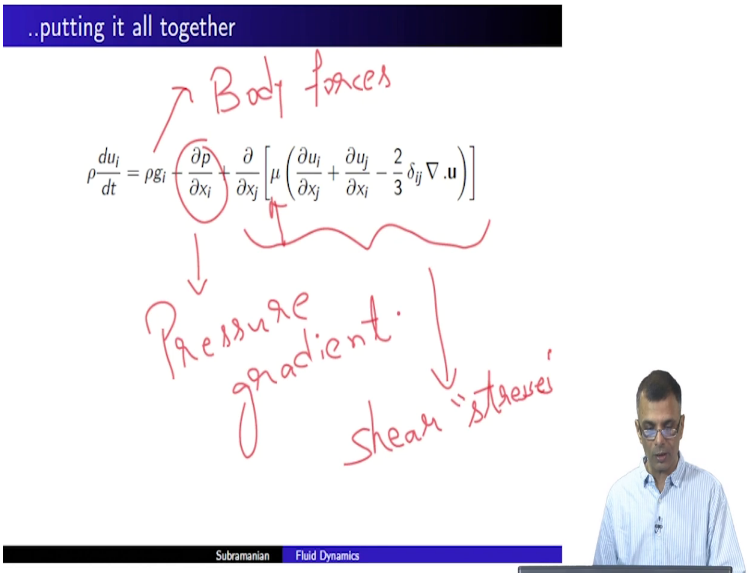
So, in other words, the small p is one-third of P i i sum of all the diagonal elements. And in which case, if this contains a trace this would better be traceless this had better, I mean in some ways this is obvious I mean this these are the off diagonal elements, and these are the diagonal elements.

So, this if you construct a matrix with only the shear terms it had better not have any diagonal elements at all. It had better be traceless that is another way of looking at it ok. And the only way this can happen is if the relation between this a and b that we wrote down here, this a and b is of the form b equals minus two-thirds a that is the only way this can happen.

That is the only way you know this and this statement can simultaneously be true; in which case the sigma ij looks like this. And here I am specializing to the slightly simpler situation where the coefficient of viscosity is the same for all combinations of i and j ok.

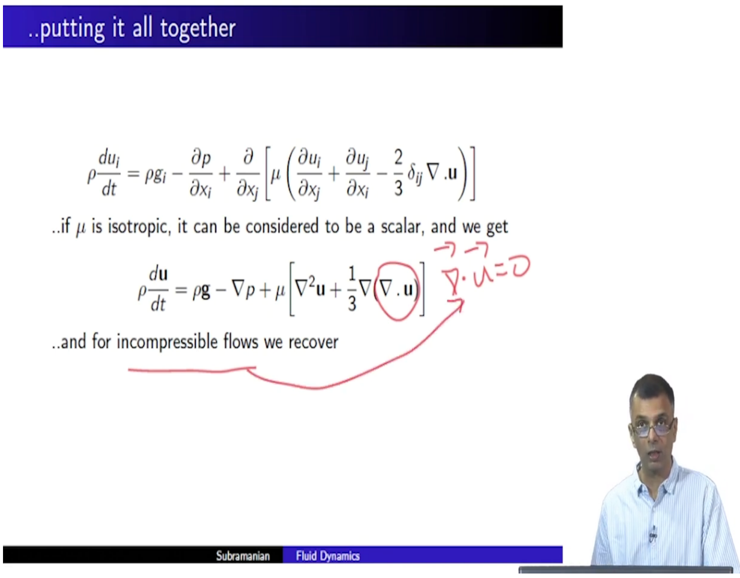
So, which is why I can write this as a scalar term I have taken it outside of the brackets. And so this is the final in some sense the most general form of the shear stress tensor ok. I, now make a distinction between the shear stresses and the normal stresses, the normal stresses are this, and the shear stresses are this ok.

(Refer Slide Time: 09:27)



So, putting it altogether, this is how it looks. So, this, these are the body forces right. This is the pressure gradient. And what are these? These are shear stresses, shear I put the stresses under quotes because this is really the force arising from shear stresses. So, it is this as such this is because of shear stresses. So, this is a force that is a force there is also a force that is arising from shear stresses ok. And this obviously has to do with viscosity, the coefficient of viscosity is this mu right.

(Refer Slide Time: 10:46)

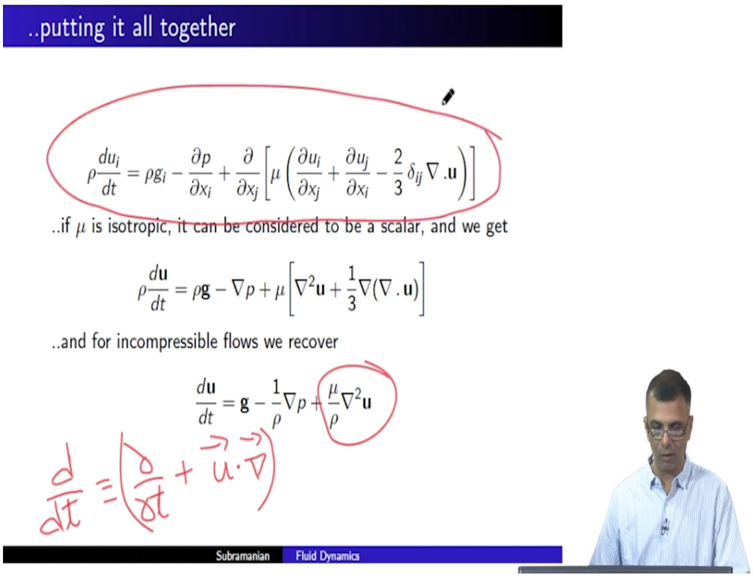


So, this is the Navier-Stokes equation and it is yeah. So, if mu is isotropic, it can be considered to be a scalar. And we get if that is the case instead of this pesky notation if you wanted to write it down in regular you know in terms of gradients or divergences, this is how it would look yeah.

And recall so, this looks very similar to what we had written down a few slides earlier, the shear term looked or the viscosity term looked just like this. And this was missing ok. This is what is called bulk viscosity. This is arising due to bulk viscosity.

And this term well you know you can write this down if only if you consider this way of doing things slightly more formal way. And for incompressible flows for which the divergence of u vanishes ok, incompressible flows divergence remember that is what an incompressible flow looks like.

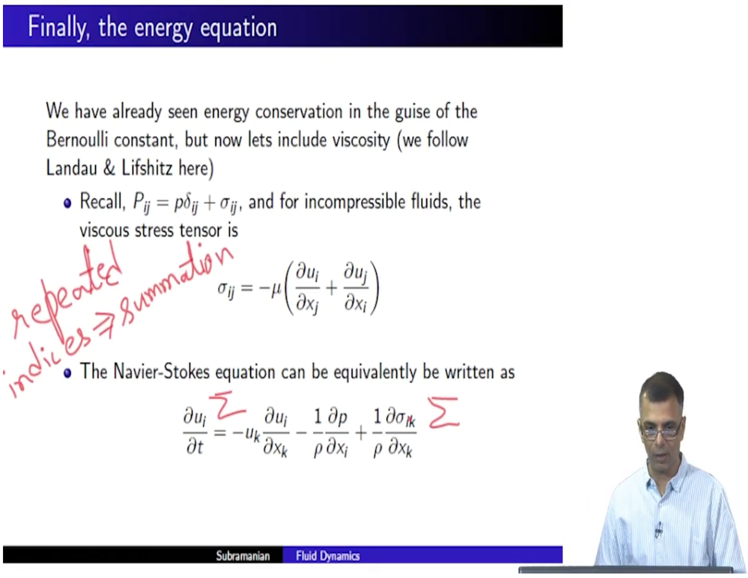
(Refer Slide Time: 11:51)



So, you recover this which is what we have written down from somewhat more intuitive considerations. And you recall that without this, this looks just like that this is essentially the Euler equation and the straight du dt this is of course, a Lagrangian way of writing it.

And if you wanted to write it down in (Refer Time: 12:15) frame you would have the partial derivatives which we know how to do. You remember it is the same as right. So, if you do not like this way of writing things, you can always write in this way same thing right.

(Refer Slide Time: 12:38)



So, and finally, let us talk a little bit about the energy conservation. We have already seen energy conservation in the guys of the Bernoulli constant, but now let us include viscosity ok. So, recall this we have already seen P ij, this we have already seen. And the viscous stress tensor is this. We just wrote this down. And the Navier-Stokes equation can equivalently be written as this.

It is really the same as what we wrote down here. It is really the same as this ok, having talked about the mass conservation equation and the momentum conservation equation. Mass conservation was simple enough we did not have to worry about viscosity at all. In momentum conservation, we have to pay you know a fair amount of attention to viscosity.

For an inviscid fluid, we talked about you know the momentum conservation in the absence of viscosity which was called the Euler equation. And for a viscous fluid we have to include you know viscous stresses. And the momentum conservation in equation in that case is generally called the Navier-Stokes equation.

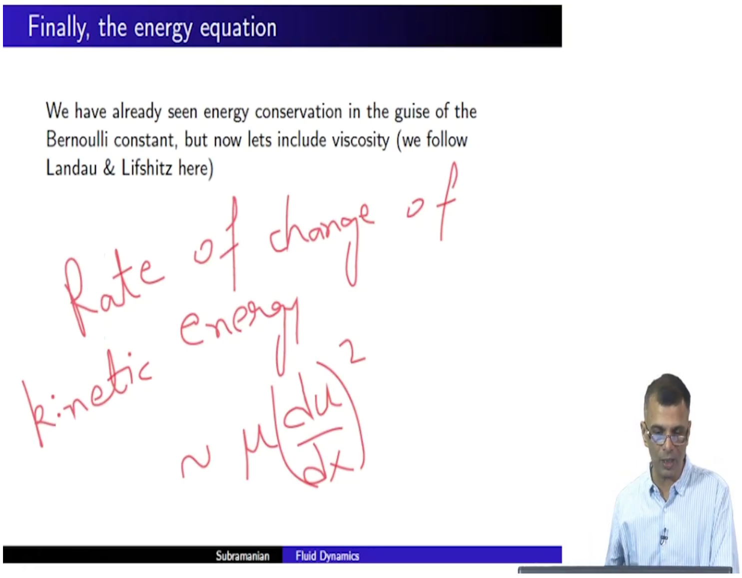
So, we discussed that in some detail in a couple of different guises. And, now we will do a little bit of talking about the energy equation. Just I want to mention that we have really already seen energy conservation in the guys of you know the Bernoulli constant. But if you recall the Bernoulli constant when we talked about the Bernoulli constant, we were talking about inviscid liquids where viscosity was not included.

And we were also talking about incompressible flows which we will in what we are going to be discussing right now. We will still be talking about incompressible flows viscosity I mean compressibility is not included that introduces additional complications. And we would not bother about that for the time being ok.

But let us see what viscosity does ok. Let us see what viscosity does to energy conservation. And we will concentrate only on that part. Since we have just gotten off talking about viscosity in some detail in the context of the momentum conservation equation, let us see what viscosity does by way of energy conservation.

And just to jump to the bottom line, what viscosity will do? Viscosity you know you as we have already said earlier viscosity is kind of a frictional kind of force, yeah, it dissipates energy. So, we will anticipate that result.

(Refer Slide Time: 15:35)



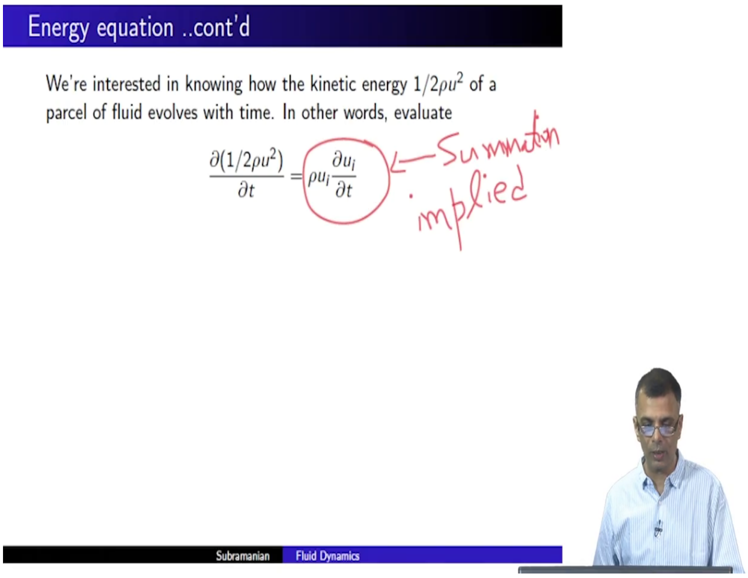
And we will see that the kinetic energy rate of change of kinetic energy, rate of change of kinetic energy of a fluid parcel, it will turn out to be some kind of it will turn out to be an expression like mu. This is what viscosity does for you as you know mu is the coefficient of dynamic viscosity. So, this will be our main result after this discussion ok.

So, let us see how it comes about. We will follow Landau-Lifshitz here, yeah. So, now, recall the pressure tensor we split it up into the diagonal part and the off diagonal parts right. And, this includes the off diagonal parts and this includes the regular thermodynamic pressure. And for incompressible fluids, the viscous this the off diagonal part was just this, yeah, this is what we did in the previous slides. And the Navier-Stokes equation which we wrote down can equivalently be written as this yeah.

So, what we have done is we have divided by you know the density throughout. And it is really the same thing wherever you see a repeated index, it implies a summation. In other words, this would be say u 1, and i is constant. So, du 1 dx 1 plus u 2 du 1 dx 2 plus so on so forth, where x 1 could be x, x 2 could be y and so on so forth. So, where wherever you see an index which is repeated, here you see the k is repeated. It implies a summation ok.

So, repeated indices summation is implied ok, so that is the only here there is really a sigma here, and here too there is really a sigma. Because you see you have repeated in this indices here the k is repeated. It is just not written for the sake of brevity right. It is just the same Navier-Stokes equation that you have seen earlier, no different. And then yeah, so just keep the repeated summation in mind, and let us proceed.

(Refer Slide Time: 18:27)

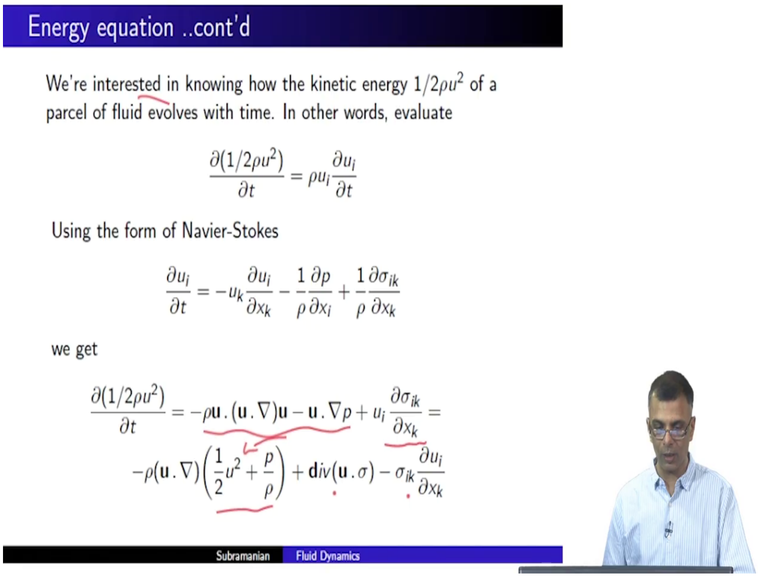


Now, we are interested in knowing how the kinetic energy half rho u squared energy density to be precise. So, half mv squared will be the kinetic energy, and half rho u squared is a energy per unit volume ok. How the kinetic energy of a parcel of fluid evolves over time? What viscosity does to it in particular ok? Because we have already seen the energy equation for you know an inviscid partial fluid.

That is essentially the Bernoulli constant right; although, it is strictly speaking valid only over a streamline ok. So, in other words, what we wanted what we want to evaluate is this quantity right, d over dt d over dt in the lab frame as seen by an Eulerian observer of this quantity half rho u squared. Another way of writing this is this ok.

This is easy enough to see you see you know d over dt of u squared is 2 u du dt, and that is essentially what this is ok. And the 2 and the half cancel and that is all this is ok. And since, again since there is a repeated index, it implies a summation. So, let me write this down summation implied ok, the summation is implied.

(Refer Slide Time: 19:54)



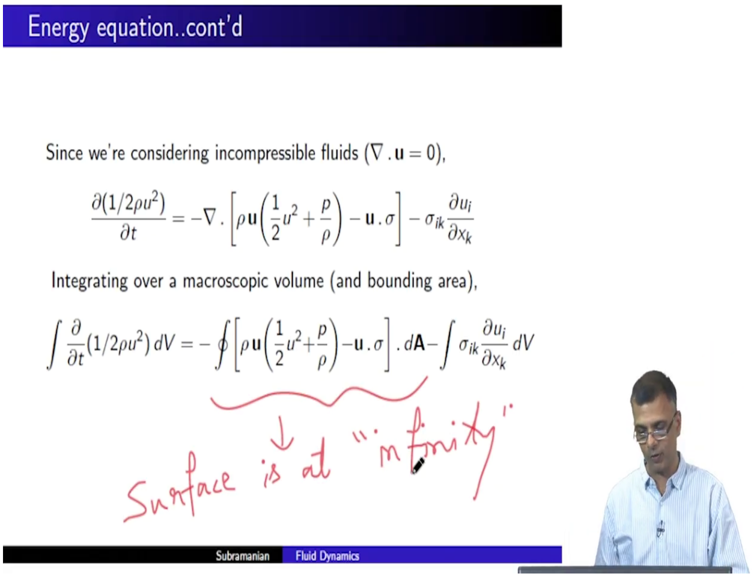
Now, using the form of Navier-Stokes which is this which we wrote down in the previous slide, so I am going to be obviously substituting for this right, this is just this whole thing right. So, it gets a little messy. So, please bear with me. So, you get. And, now I switch back to well a mixture of vector and tensor notations, I keep this as it is ok.

And the rest of them these guys I, write them down in usual vector notation. There is a reason for doing it this way. It is about convenience. And I will it will become evident as we go along right. It is a bit of a mixed notation, yeah. And this again can be simplified as this. So, it turns out that right.

So, this can be split up into two terms like so yeah. And these guys can be written as this ok. You recognize this immediately. It looks very much like the Bernoulli constant right. It is and then there is a divergence of u dot sigma where sigma is you know the viscous stress tensor.

And again you know since this is sigma i k du i dk, there is a repetition of k, so there is a summation implied ok. Just like there was a summation implied here, this is summation implied here as well ok. So, again all of these is simply this the time derivative of the evolution, well the time derivative of the kinetic energy density of a fluid parcel. In other words, how the kinetic energy density of a fluid parcel evolves, and how it is related specifically how it is related to viscosity right.

(Refer Slide Time: 21:57)

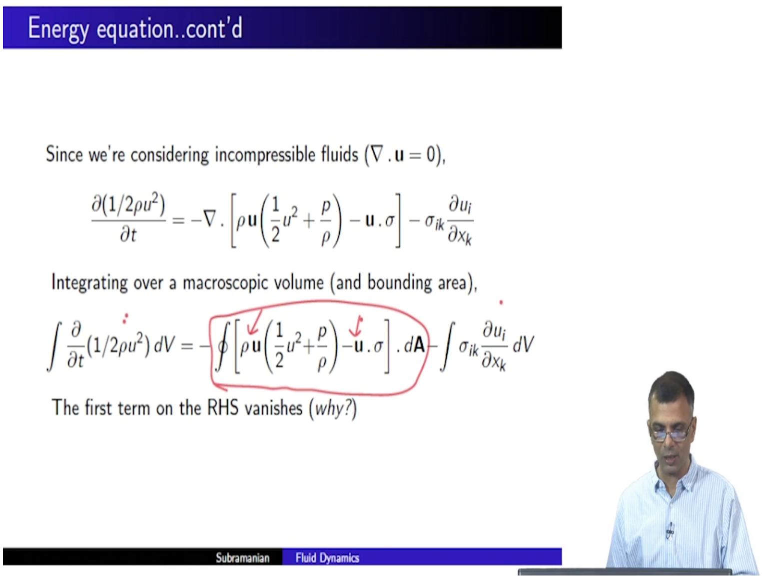


So, now we specialize to incompressible fluids, divergence of u is 0, and that makes that simplifies this term a little bit. And we get the same thing it is a little simplified, and it becomes like this ok. Now we are going to do something interesting very important, and we integrate over a macroscopic volume and bounding area.

What I mean is that we integrate over the entire body of the fluid the fluid is essentially unbounded ok. And so the bounding area is essentially taken all the way up to infinity. And I am going to be using Gauss divergence theorem here. We will see, yeah. So, the integration is over the volume yeah. And here since this is the divergence, it becomes an integration over the surface area, except now the surface area the surface area is at infinity, and this is important ok.

So, in going from in going from here to here, this is also you take a volume integral of this term. And you use Gauss divergence theorem to turn it into a surface integral here, this we keep as a volume integral. And we are going to do something clever with this and that is why and that is also why I emphasize the fact that in this surface integral I am at the surface, I am integrating over the entire fluid and the bounding surface I take it to be at infinity.

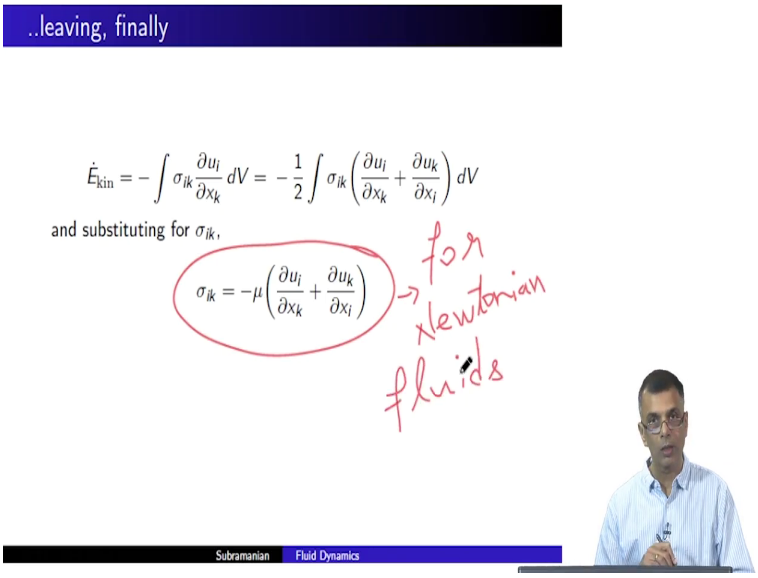
(Refer Slide Time: 23:45)



And as a consequence of that, I am allowed to throw this entire thing away, this entire thing vanishes. Why is that? Notice that we have a u here. And for any well behaved flow, you assume that the velocity of the flow vanishes at infinity. So, you are integrating only over the surface which is at infinity and the u essentially vanishes.

So that allows me to throw this entire thing away, so that I am only left with this term and this term that makes that is pretty nifty and that makes that simplifies my life greatly. Therefore, what I finally, have is the E dot kinetic, yeah this is kinetic energy this is kinetic energy density, is it not?

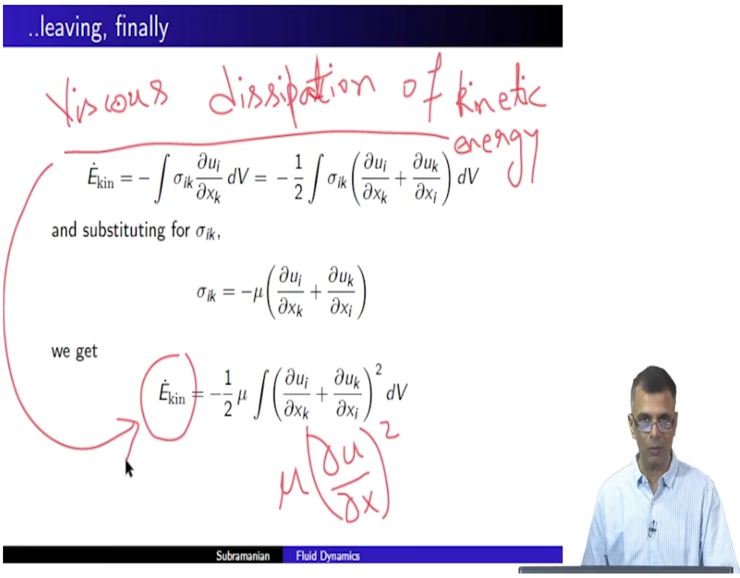
(Refer Slide Time: 24:36)



And the E dot kinetic in other words the time derivative of the kinetic energy density is equal to this quantity you know integrated over volume which is this simply you know substituting for this right. And substituting for sigma ik, well here I have not yet substituted, I am simply writing this as in the same spirit that we did earlier.

And substituting for sigma ik we know that this is the definition in other words this is for only for Newtonian fluids mind you. This is the definition so to speak of a Newtonian fluid I get.

(Refer Slide Time: 25:31)



So, this quantity gets squared because the in inside the sigma ik, I have another thing another you know inside the sigma ik, I have another thing that this bracket thing that appears again here in sigma ik. So, it gets squared right. And this is my final result. So, this looks very much like if the flow were just one dimension, this looks very much like mu times du dx quantity squared ok.

So, therefore, the time rate of change, this is the final result this is in some sense how the kinetic energy density of a fluid parcel is dissipated due to viscosity. And we understand this, we disagrees with our intuitive understanding of viscosity as a frictional kind of force. And so, you know the if you are considering only inviscid fluids, then there is no dissipation at all the introduction of viscosity results and dissipation.

So, the bottom line of all of these is essentially this expresses viscous dissipation sorry of kinetic energy – this whole thing ok. So, there you have it this is how the energy equation is modified in the presence of viscosity. And we are done with this section of the course. And we will proceed to other topics from now on.

Thank you.