Fluid Dynamics for Astrophysics
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Lecture – 13
Navier-Stokes equation

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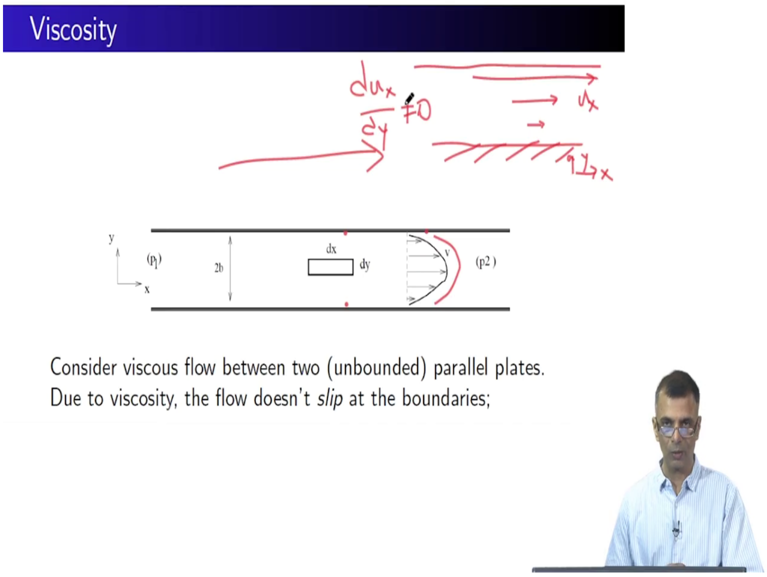

Hi. So, as promised we will, we will begin our discussion of viscosity and the final point of this discussion will be the Navier Stokes equation which is essentially the momentum equation with viscosity terms included. So, we will start with a bit of a intuitive you know derivation of the Navier Stokes equation especially with the viscosity terms included.

So, consider viscous flow between two unbounded parallel plates like so, and unbounded in the following sense the this dimension the x dimension is infinite and the z dimension which is into the plane of the screen; this is also infinite ok. So, consider a viscous fluid like say honey or something honey or maybe motor oil flowing through this flowing in this direction yeah. And intuitively you can appreciate the fact that the velocity profile of such a fluid would be something like this; like this yeah.

So, the velocity is largest at the center of the pipe. It progressively decreases, it decreases and the fluid sticks to the boundaries. This is the whole point of viscosity right. It sticks; the flow does not slip at the boundaries. There is no slippage right; there is a whole point motor oil you know you as the oil heats up the people do all kinds of engineering to ensure that the viscosity stays because you want you want the motor oil to be lubricating the piston right.

And so, even when the oil heats up as you drive the vehicle, you want the lubrication to remain intact right. So, you want the oil to form a protective layer here and here right. So, that the piston which is moving through is properly lubricated and the protective layer is all about sticking no slipping right. So, because there is sticking and no slipping and the velocity vectors at the edges are much smaller than what they are in the middle.

In other words, the velocity profile looks like this and it is not flat ok. What does this immediately remind you of you? You recall this diagram that we used to draw all the time, here is an unbounded surface and here is a; here is a free surface of the fluid for some reason. The fluid is flowing this way and the velocity the and this would be x and that would be y and the velocity stream lines would look like this.

So, there would be a gradient. We have drawn this so many different times right. We have said this so many times. So, it is essentially the same thing that we are talking about.

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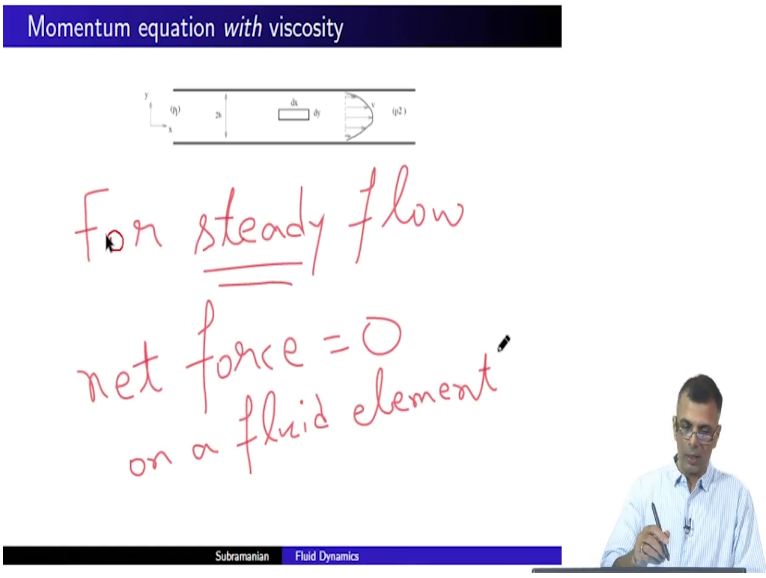

And so, now let us write down the momentum equation with viscosity for the same situation which is the Navier stokes equation right. So, instead of now writing down f equals m a, let me consider simply the two component whatever components of f there are ok

And let me consider a steady state flow in which there is no acceleration; in other words, the velocity is constant. So, a is 0. So, therefore, the forces have to balance right. The net f if that is the case for steady flow, the net force equals 0 by definition.

And by steady I do not mean non turbulent or anything I mean a flow which is not accelerating; which is just flowing at a constant velocity. The net force on a fluid element of course, when I say net force I mean net force on a fluid element right, this is equal to 0; this is you know intuitively obvious ok.

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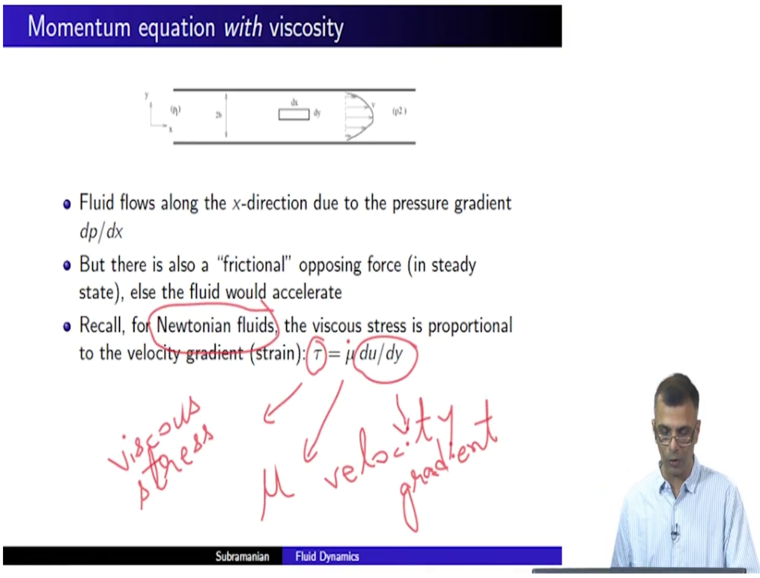

So, what about it? What are the different kinds of forces that could be acting on this kind of on let me blow this up a little bit on this kind of on this kind of a fluid element right? So, fluid would flow along the x direction due to the pressure gradient; pressure is a dp dx right. So, if for instance the fluid is flowing in this direction, it would be because the pressure here is larger than the pressure

Here in other words there is a non zero dp dx. So, this would be one kind of force and this would have to be balanced by another kind of force if the net force on this fluid element on this fluid element here is equal to 0 right if as we said that is the definition of a steady flow right. So, one kind of force is dp dx and what is it balanced by? It has to be balanced by something having to do with viscosity you get the hint

We are talking about viscosity and so, there is also and when we talk of this the other thing intuitively viscosity has to do with friction has to do with heating ok; viscous effects contribute to heating and heating in our mind is generally associated with friction ok. So, it is in it is in that sense that I use the word frictional force ok; this word. There is also a frictional opposing force

So, the first force we talked about was simply due to the pressure gradient that is making the flow; that is that is making the flow go from the left to right. And, but there is also a frictional opposing force in steady state because that is that is the situation that we are considering one where the fluid element is not accelerating and these two forces have to balance else the fluid would accelerate right.

So, now what is the frictional opposing force? This is the other big question right. Recall that all our discussions are confined to what are called Newtonian fluids. Once and for Newton the definition of a Newtonian fluid is one where the viscous stress is proportional to the velocity gradient; in other words the strain. So, this would be the velocity gradient, this quantity is the velocity gradient right.

And this quantity is the viscous stress and this is very similar to pressure. It is also the force per unit area except it is not the normal force per unit area; it is the shear force per unit area. Hence it is a shear stress and this is the velocity gradient and this is essentially the strain. It is a stress versus strain relationship just like what you are familiar with from elasticity and the constant of constant of proportionality is this quantity. This is the constant of proportionality this mu.

This is the constant of proportionality and this has to do with viscosity ok. So, this we have already discussed I just wanted to remind you of this and. So, this is the kind of thing that we are talking about.

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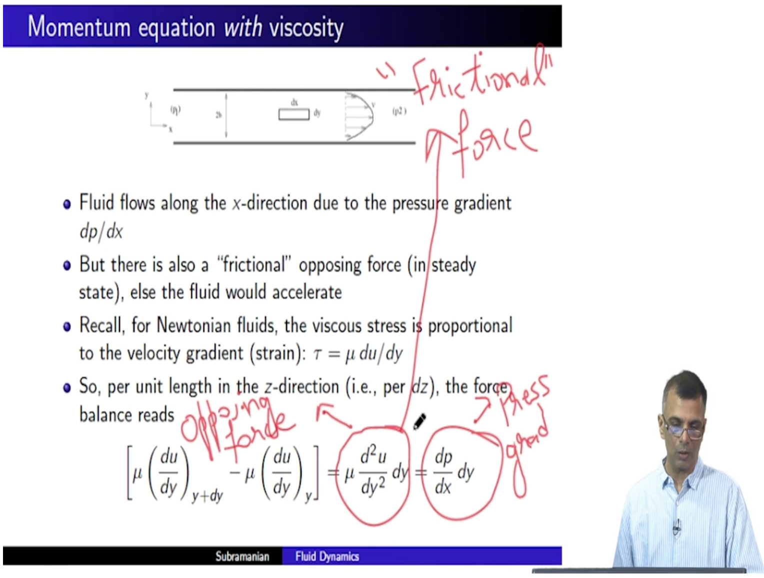

Therefore per unit length in the z direction in the z direction is into the plane of the screen per dz the force balance reads this you see mu times d u d y, this is a viscous stress this is force per unit area yeah and we multiply a dz right

So, we have essentially what I am trying to do is I am trying to get rid of the area dz is one length element and then, I will be eventually multiplying with d y as well ok. So, bear with me for a second, but what we are saying is mu times d u d y at some y plus d y minus mu times d u d y at y this is equal to this. This quantity is equal to d square u d d u squared times d y by the definition of a derivative ok.

This is this you already have a first derivative here and you divide this by a d y and you have d squared u d y squared and I multiply that because that it was not here. So, this is essentially mu times d square u d y squared times d y and this I am writing down.

So, this is the viscous force; this is the opposing. It is the opposing sort of a another way of writing it is the frictional force and this is a pressure gradient force right. And the two are balancing each other because the fluid element is not accelerating right there is no ma.

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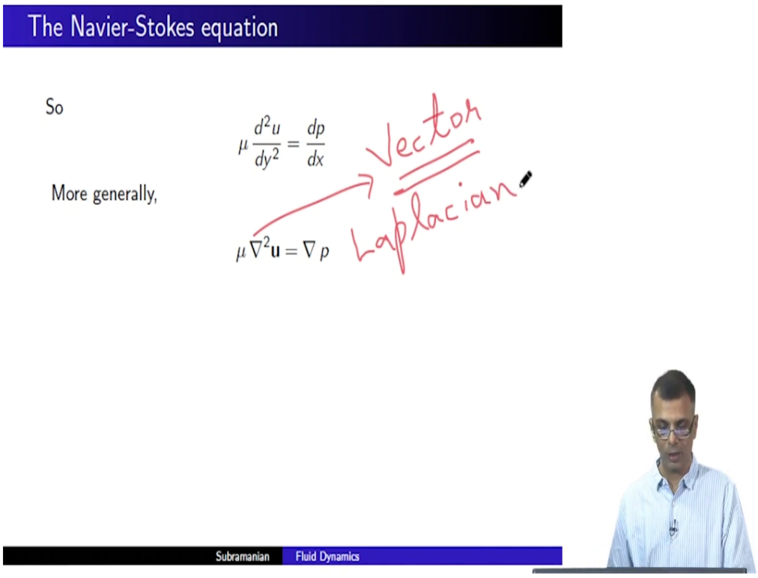

So, you can write it like this. There you go simpler way, the d ys cancel out the dz was already implicit and therefore, mu times d square u dv dy squared is equal to dp dx. In a way in some sense this is it I mean you know we have already written down the Navier Stokes equation except this is only the force part yeah.

And you really should be putting in you should be allowing for the fact that the fluid element can be accelerating ok. We wrote this down just to motivate the Navier Stokes equation yeah

So, so, more generally we just consider one dimension more, generally it is not a d square u d v square d y squared. It is a Laplacian it is a grad squad except please remember that this is actually a vector Laplacian. This is not the scalar Laplacian that your generally familiar with. This is a vector Laplacian.

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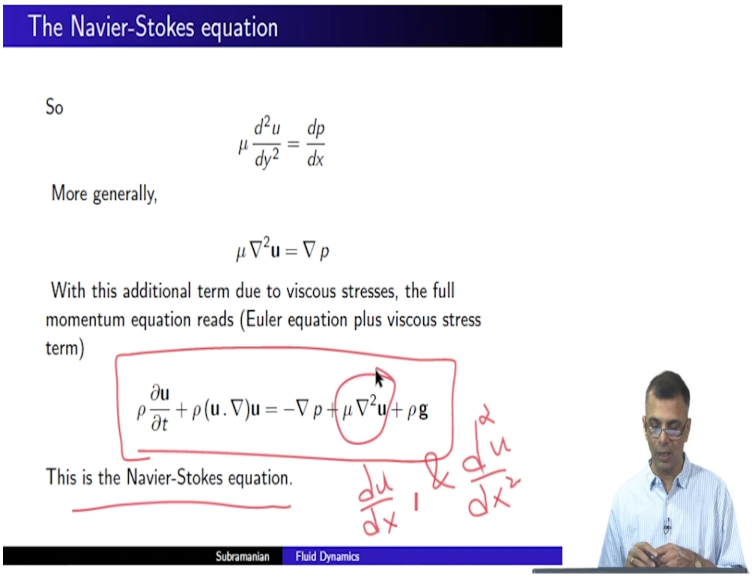

So, with this additional term due to viscous stresses, this is what was absent in the Euler equation that we have been discussing so far right. And you see this term has a opposite sign to the to the because the two are appearing on opposite sides of the equation. The sign is different and so, the full momentum equation in other words the Euler equation plus the viscous stress term which we have now introduced here the Euler equation plus the viscous stress equation term which is nothing, but the Navier Stokes equation.

This now reads we were already familiar with this; this is the m a term and these are the forces. And notice the sign of this term is opposite to the sign of this term. This is due to pressure gradient, this is due to body forces; these two we already knew from our discussion of the Euler equation, the additional term is this yeah right. So, so there you have it and just by way of an aside so, this essentially is the Navier Stokes equation; this thing ok.

Now, remember I when we were talking about the Kelvin's vorticity theorem, I alluded to this and I pointed out I alluded to specifically this term ok. Remember what we were talking about there? We were talking about water from a hose right which is being pointed at a wall and generally you think of water as an inviscid fluid or at least a very low viscosity fluid and the water is flowing out in a laminar fashion.

But we made the observation that when this laminar flow hits a wall, vortices are generated right whereas, there is no evidence for vertices in the bulk flow somehow vertices are generated. And since water is you know more or less an inviscid fluid, this seems to defy Kelvin's vorticity theorem which says that for an inviscid fluid incompressible, water also is incompressible to a very large to a considerable extent.

And so, all the assumptions that were inherent in the derivation of a Kelvin's vorticity theorem hold pretty well in this situation. But clearly vorticity seems to be generated right at the wall yeah where there are lots of swirling motions that are observed. So, what gives? The answer lies in this term. Although in general in the bulk flow up until it hits the wall, this term would not be important because mu itself is very small. Right at the boundary, yeah the water is coming to a crashing halt.

In other words, the d u d x and to a larger extent d square u dx square which is what this is the these kinds of terms are like. These become very appreciable at the boundary ok. So, even though mu might be small the combination of these terms, the multiplication of these two terms become important. And so, this term assumes importance in comparison to what in physics we always have to ask; this term assumes importance. Yes, but in comparison to what in comparison to the pressure gradient of the body forces.

In this case for a horizontal flow of water, the body forces are not important. I mean as in as in it is the same, but it becomes important in comparison to the pressure gradient forces. So, one of the basic assumptions underlying Kelvin’s vorticity theorem which is that viscosity is unimportant; it breaks down just near the boundary because of the way in which the viscous terms manifest themselves in the equation ok.

So, therefore, it is not true that vorticity cannot be generated, vorticity can be generated and that is true only at that boundary and that is why the vertices start forming. So, we will wrap up this discussion here and we will continue with our discussion of Navier Stokes equation.