Fluid Dynamics for Astrophysics
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Lecture – 12
Vorticity dynamics: Kelvin's voracity theorem and Magnus effect

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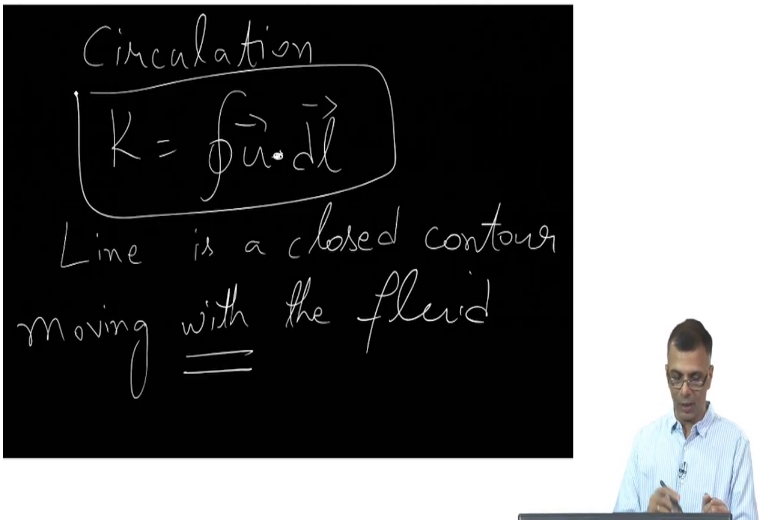

Hi, so as promised we will talk about this quantity called the circulation. Which is defined as u is a velocity field as before and there is a dot product in between. It is important to; so this seems like a line integral a closed line integral since we have written it like this yeah.

Now the line l, the line is a closed contour moving with the fluid this is important to keep in mind. As if the contour is frozen into the fluid ok. And so it is in this particular situation that this definition of circulation is valid.

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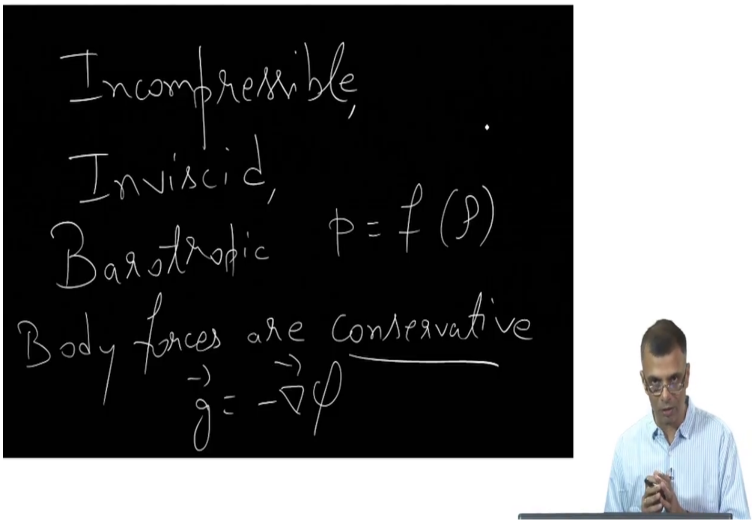

And the next thing I want to say is that this entire discussion is valid for incompressible meaning, the divergence of u is 0, right. Inviscid meaning the viscosity is negligible, barotropic we will maybe I have said this earlier, but essentially saying that the pressure p is a function of rho fluids, under the influence of conservative body forces ok; body forces are conservative.

In other words, if there is a body force g; it can be expressed as the gradient of a scalar potential phi with this, just want to emphasize once again here with this potential phi refers to the potential which describes the body force is a gravitational potential it is not the velocity potential. We use the same symbol for both.

So, I figured I would emphasize this. So, at any rate all of I; all of our you know discussion which we are going to have right now takes all these restrictions into account. And the definition of circulation that we will be talking about is this where this d l is a closed contour that is moving with the fluid, in some sense frozen into the fluid right.

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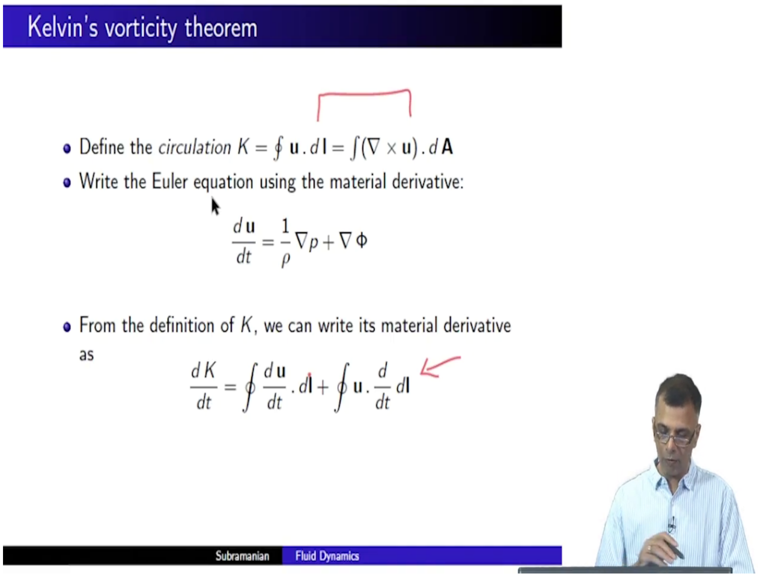

So, with those caveats let us move ahead and so this is the definition of the circulation that we just said circulation K is this. And so, what we do now is the Euler equation that we seen earlier. Instead of writing it in the lab frame we write it in the Lagrangian frame, it is a little more convenient. Because you see this contour that we are talking about is moving with the fluid.

So, naturally it makes sense to be sitting on top of a fluid parcel. So, that this contour is not changing for you. You are sitting on the top of top of the fluid parcel and you are watching stuff go by. So, you write down f equals ma in that frame. So, that you know defining the contour is also easier. You are sitting on top of the fluid parcel and the contours not changing as far as your concern still a closed contour.

So, the Euler equation using the material derivative is this and from the definition of K this definition. You do this operation u dot d l all through the equation and this connection as you might already have guessed, this connection is made using Stokes theorem.

You relate the line integral to the surface integral using the curl right. I sometimes write the area element as d s, sometimes I write it as d a; I am hoping that the context will make it clear, what I am talking about. In this case anyway we are not yet talking about the surface integral.

We just do a you know a line integral of all terms here right. Before that the d u d t I write down as you know the line integral of d u d t is you know I dot it like. So, there is this component and then there is this component. In other words, the circulation can change because the velocity itself is changing or the line element is being stretched or compressed; is moving with the fluid, but you are allowing for the fact that the shape might change and that is this term here ok.

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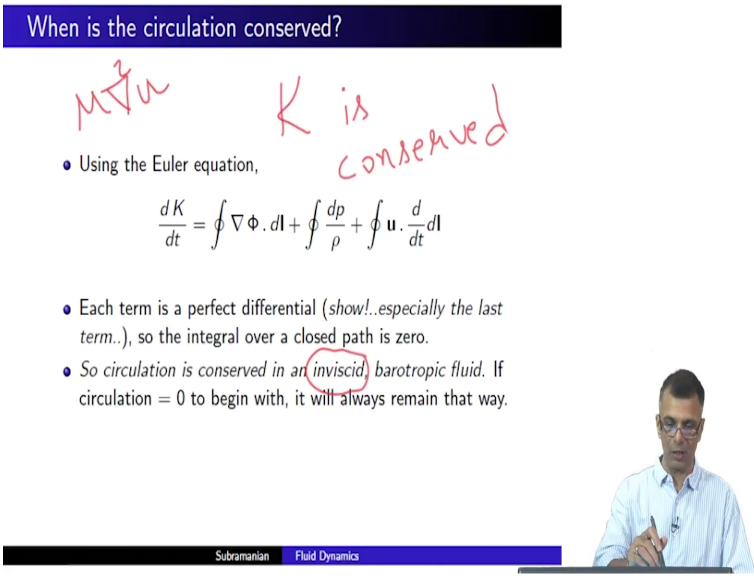

So, that is d K d t on the left hand side. And on the right hand side you have remember you have the gravitational potential and you have the gradient of p. And so, you do the line integral again because that is what we were doing to all the terms of the equation and you have d phi gradient of phi dot d l.

And this is just a perfect differential and we got this is a perfect differential because you see there is already a gradient of p yeah. And because we want a retain just this term on the left hand side this one this fellow it goes over to the right hand side right. So, that is where you get this yeah.

So, as such there is nothing there is no mystery. It is just a simple rearrangement of terms and the important thing here is that each term is a perfect differential ok. You see there is a gradient of phi and it is dotted with d l. So, that becomes a perfect differential, this is already a perfect differential by definition. And I claim that this is also a perfect differential this term, why is that? d over d t of d l is a u right.

So, it is a half u squared. So, therefore, the [vocalized-noise] integral of any perfect differential along a closed path is 0 ok. That is the whole point of a perfect differential it is a you integrate it along around a closed path and the integration is 0 and which means that d K d t is equal to 0.

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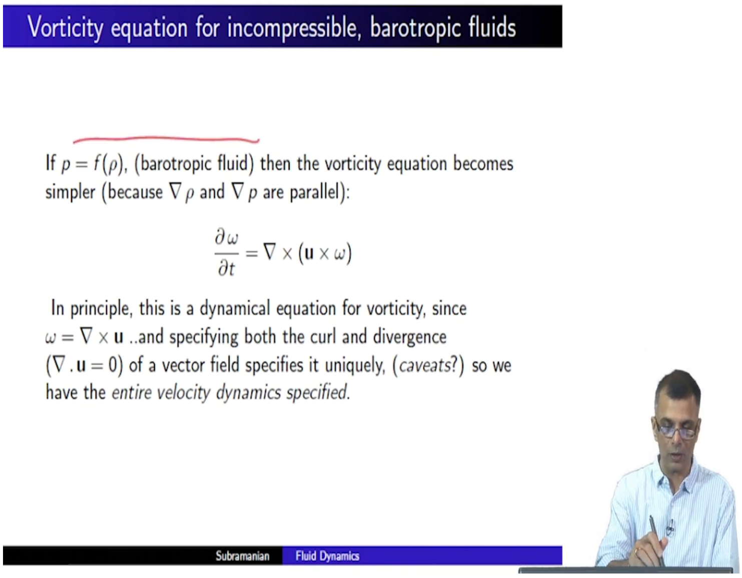

In other words the by the way before saying that you remember just wanted to emphasize that this whole thing was you know you could write. It started from the vorticity equation and you could write this down you could write down the vorticity equation in this form, only because the gradient of rho and gradient of p are parallel; which is true only for barotropic fluids, I want to emphasize this once again ok.

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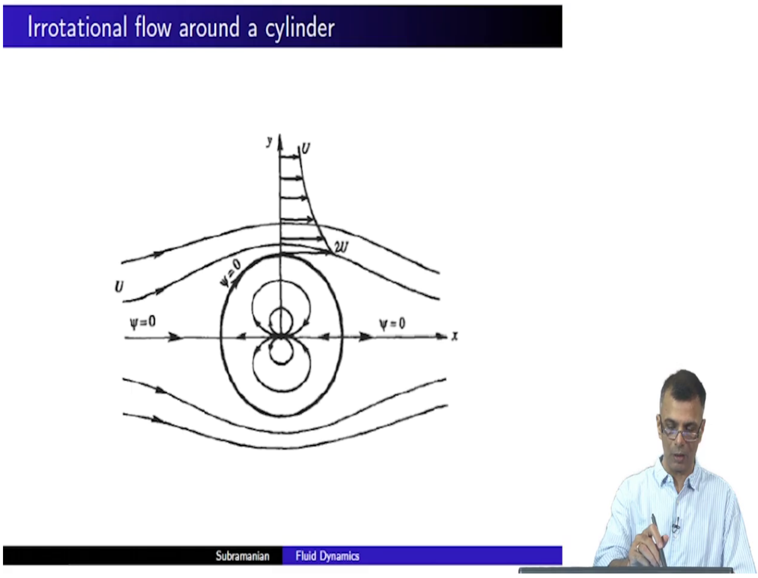

So, what we now saying is that the time rate of change of vorticity is 0. In other words the circulation this quantity K is conserved right. So, circulation is conserved in an inviscid barotropic fluid also importantly it is true only for situations where either the body force is absent or the body force is can be described using a conservative potential ok. That is not written here, but I want to keep I want you to keep that in mind firmly.

So, if the circulation was 0 to begin with it will always remain that way right. So, now, here is a here is a bit of a conundrum. You must have I mean water right I mean there is no. Before I launch on this description on this example I just want to say that, there is no such thing as a perfectly inviscid fluid these are all approximations. But nonetheless water you think of water as a pretty much inviscid fluid the viscosity is not very high.

Especially in comparison to a fluid like say honey right. Now, we are saying that the circulation is conserved, vorticity is conserved right. In other words, if there are no vortices to begin with you know vorticity cannot be generated. But you must have noticed you take a hose and of water and the flow coming out of that hose is fairly laminar, it is not a very turbulent flow.

So, that the streamlines are nice and well behaved there is no evidence of vortices right. Now you take a hose of water; so that the flow out that the water flowing out of the pipe is fairly laminar and you aim at a wall right.

And what happens at the wall? All hell breaks loose right, I mean you know there is a lot of vorticity and lots of these vortices are generated. So, what gives? Right, how come vorticity is being generated you know at the wall? And so, that seems to violate this right, it seems to be generating vorticity from nothing. The answer to that lies in the fact that there are very large velocity gradients that are being generated right at the wall ok.

In other words, the fluid is coming to an abrupt halt at the wall and so you know the diverge the gradients of u and so on so forth are being, there are very very large gradients of velocity that are being generated right at the wall. And as we will see once we start talking about the Navier Stokes equation we will see that, there is no such thing as and well I said that in the beginning there is no such thing as an inviscid fluid while viscosity might well not be important in the bulk flow.

It becomes important in places where the gradients of the velocity are large. This is because the kinds of terms that we will see when we discuss a Navier Stokes equation in a little bit. The viscosity appears in this combination like that ok. So, the del square is a second derivative ok.

So, mu might well be very small, the coefficient of viscosity might well be very small ok. But when the gradient of velocity is large as it would be when the water is coming to an abrupt halt. To say nothing of the second gradient or the second derivative this one, the second derivative is even larger.

So, as a result of the fact that the second derivative is large, the multiplication of these two this assumes importance. So, this term which is essentially the only combination in which the viscosity appears in the momentum equation. This term assumes importance even though mu is small in the bulk of the fluid. This term assumes importance at boundary layers such as the one where you know the water jet is meeting the wall ok.

In that thin boundary, where velocity is going to 0 over a very small thickness. So, d u d x, so to speak is large and d square u d x square which is what this is becomes even larger.

So, the multiplication of these two terms assumes importance and so, viscosity the one of the basic things that is implicit in Kelvin's vorticity theorem which is that of an inviscid fluid, this gets broken down even for water, that is why vertices can be generated ok. So, this is again a very simple explanation to an apparent conundrum and, but nonetheless this is the general you know statement.

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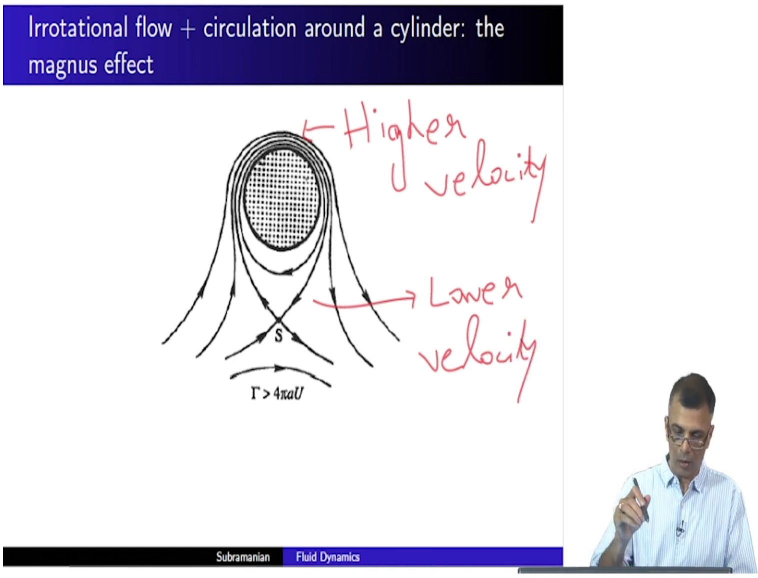

If vorticity is a 0, it will always remain that way. If on the other hand there is some vorticity, it remains that way also ok. Now, we come to this as was the case with the Bernoulli constant we used it to demonstrate a couple of interesting real life examples. Here is a real life example that I would like to illustrate, that uses the concept of circulation if not the concept of Kelvin's vorticity theorem it at least.

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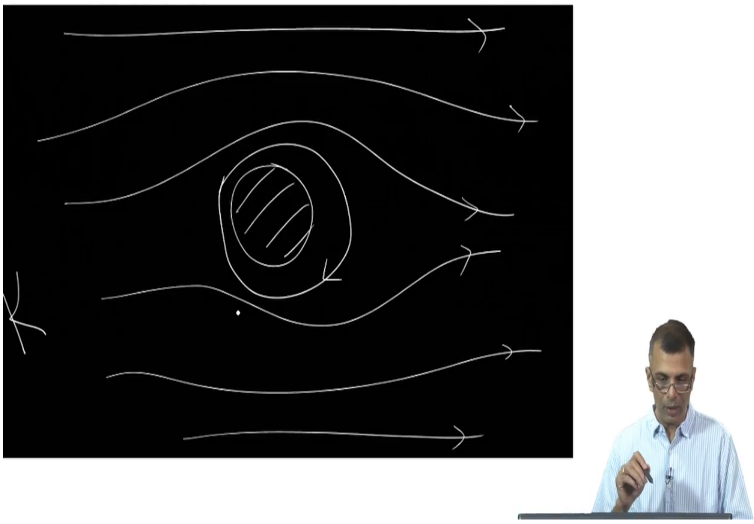

So, you recall the potential flow around a cylinder, you recall this. This is what the flow look like right yeah and around a balls yeah and so like so with the and by the time you reach you are very far away it is as good as straight. Now there is no circulation here; if for some if on top of this I just because I say so let me impose a circulation, in other words let me impose a streamline that looks like this.

So, bear with me I will tell you why ok, let me impose a circulation on top of this. I add this kind of a circulation to the flow by hand ok. In other words I add a K; this is a nonzero K, the fact that I have added this kind of circulation to the flow yeah. In this case the streamlines now start looking like this in other words the stream lines because this circulation adds here and subtracts here.

You see here down here it is you know this streamline is opposite to the direction of this streamline. Whereas, here up here the streamline is in the same direction of this streamline right. So, it adds up here and subtracts down here.

So, as a consequence the streamlines now start looking like this where they are more densely packed at the top and sparsely at the bottom ok. Now what is this saying? What is the density of streamlines? What does it say? In other words the velocity is higher here right. So, it is and right.

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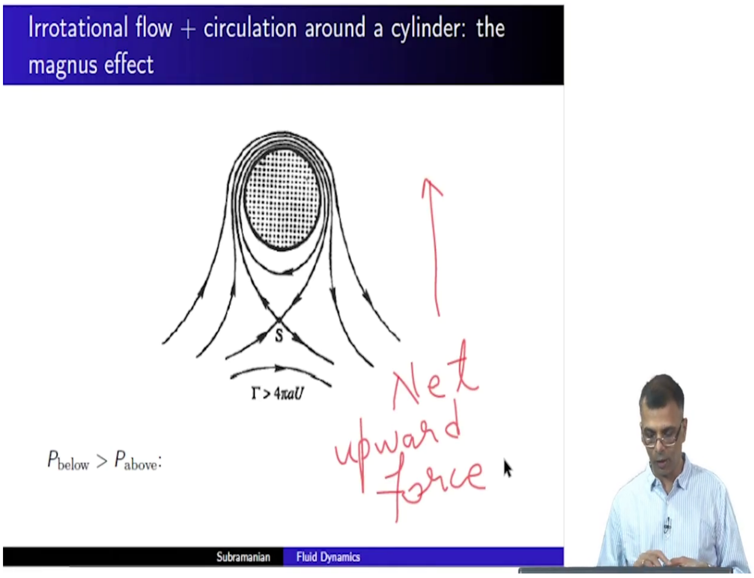

So, as a result of the fact that this lower velocity here as opposed to here, the pressure below will be larger than the pressure above yeah. In other words there is a net upward force in this kind of a situation right. Where, what kind of a situation are we talking about?

One where I have arbitrarily added a circulation of this kind to a potential flow and I will tell you why this is useful in a minute, but the bottom line is that. In this kind of a situation this body will experience a net upward force.

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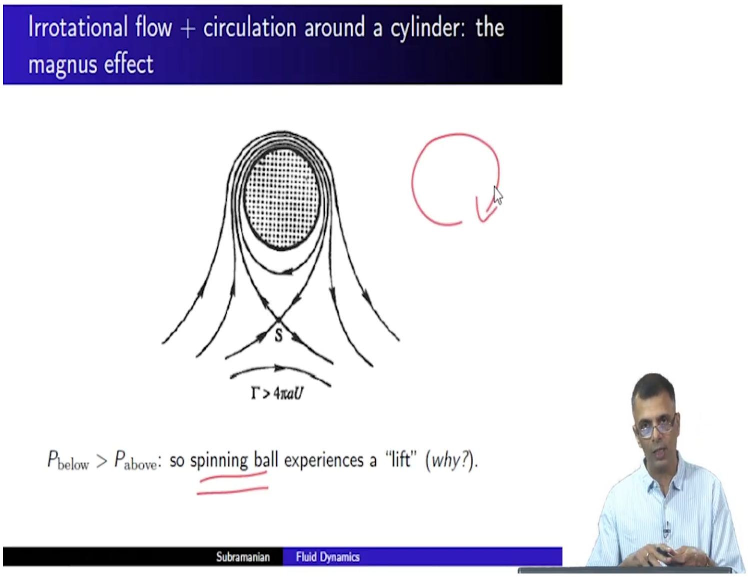

So, now I have mentioned the word spinning ball. This obviously, has to do with the fact that I have included a circulation here right. Now, although we are talking about an inviscid flow right, lets for a moment think about a real life situation where the viscosity is not negligible.

So, let us think about a situation where the ball is spinning in this direction right. I impart a spin to the ball, this ball is spinning in this direction. Now viscosity is all about sticking, sticking a fluid right. So, the fluid sticks here and because the ball is spinning it tends to drag the fluid along with it yeah. So, the fact that this viscosity generate circulation.

So, it is the same thing in effect it is the same thing as adding circulation by hand. So, if the ball is spinning it generates this kind of circulation. If you generate that kind of circulation well then you have this kind of situation where the velocity up the velocity of air above is smaller than the velocity sorry; the velocity of air above is larger than the velocity below and therefore, the pressure below is larger than the pressure above and therefore, the ball experiences a lift.

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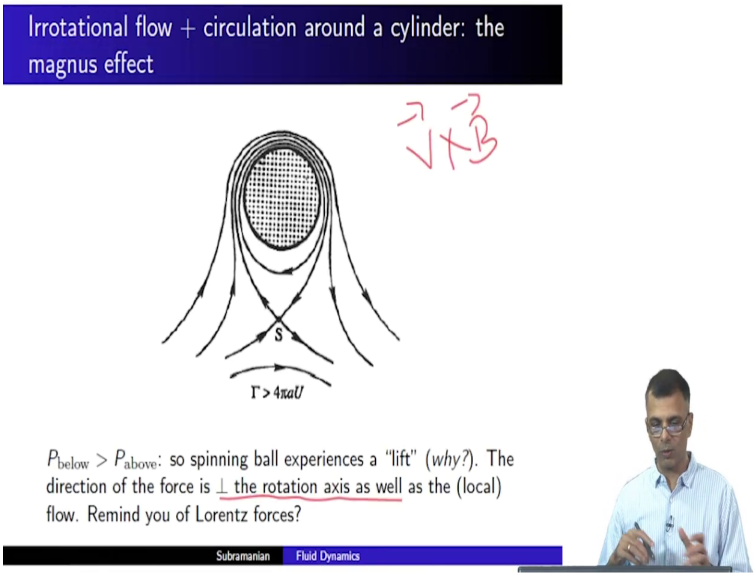

The direction of force is perpendicular the direction of the lift is perpendicular to the rotation axis. The rotation axis as you know is inward like this into the plane and as well as a local flow ok. This is very similar to Lorentz forces which are you know this kinds of very similar to that in electrodynamics.

So, this kind of thing is used by tennis players when they impart a top spin in the ball. You know I mean when a tennis player wants to make sure the that the ball does not go outside of the court right, he or she imparts a topspin ok. So, when they impart a topspin like so what happens is because of this Magnus effect there is a downward force on the ball.

So, the ball kind of curves and it falls within the base line and it does not go out and that is because it is a downward force and then there is a curve. And this it is the same thing with a cricket spinner and this is not as common mostly it is used by leg spinners, wrist spinners.

And what happens is when the wrist spinner gives a vicious spin to the ball there is often a dip and that deceives the batsman. And this is what the Magnus force is all about this is not swing ok, this is not the swing effect that fast bowlers use and that is much more complicated ok.

Swing is when there is a sideways movement here this is a case where there is a downward or upward. In this case the direction of the force is upward, but you know and in tennis you know you either give a topspin or you give a slice ok. When you give a slice you are doing this and then there is an upward force and when you give a top spin there is a downward force ok.

You can work out the direction of the forces and it is essentially this, the Magnus effect. So, before we go on I figured so we will be going on to discuss viscosity from now on. And, so all this stuff that we talked about right now had to do with inviscid fluids, inviscid incompressible fluids also subject to body forces that could be described by a conservative potential.

Also when we when we are talking about you know circulation and conservation of circulation we were also talking about barotropic fluids. So, this kind of wraps up you know this section of the course. And from now on we will start talking about viscosity. We will pay some attention to viscosity and we will introduce the Navier Stokes equation.

And as you will see we have already seen the Navier Stokes equation earlier. We specialized from there on to the Euler equation where viscosity was 0, but we had actually seen the Navier Stokes equation in a slightly different guys. Because we were already talking about shear forces a shear yeah. So, shear stresses and shear stresses intrinsically have to do with viscosity. So, we will just introduce the Navier Stokes equation in a minute and so, for the time being, we will close this discussion here.

Thank you.