Fluid Dynamics for Astrophysics
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Lecture – 11
Recap - Potential flows, Bernoulli constant and its applications

Hi so we are back, so as a prelude to discussing the Navier-Stokes equation which we have really seen in an overall guise. What we did was we did a few things with regard to streamlines and Bernoulli constant and vorticity and things like this. So, I figured out recap things a little bit before moving on we kind of stopped at vorticity and what we are going to do next is talk about circulation and how this is a quantity that is conserved.

And then we will talk about an a very interesting effect called the Magnus effect so, but before that I figured we would do a very quick recap of what we have done so far specially during the last lecture over 2.

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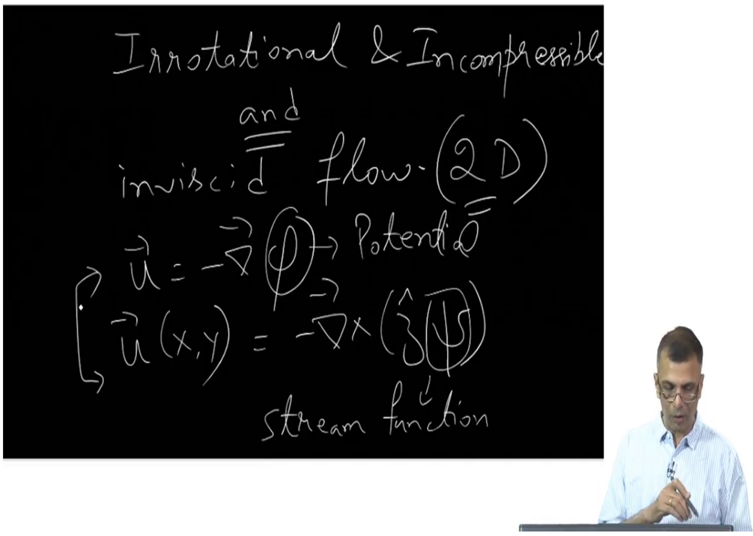

So, what we first said was that if we are talking about an irrotational and incompressible and of course inviscid flow. In this case the velocity and for now let us specify let us specialize to 2 D flows the velocity can be expressed as a the gradient of a scalar velocity potential and we can also express especially for 2 D flows we can also express the velocity.

Since its 2 D I will explicitly write this as x y as the curl of something called a stream function. So, this is the potential this thing and this thing is called the stream function. So, if all these conditions irrotationality, incompressibility, inviscid and 2 D you know all these conditions are satisfied and they are very frequently are both these hold ok.

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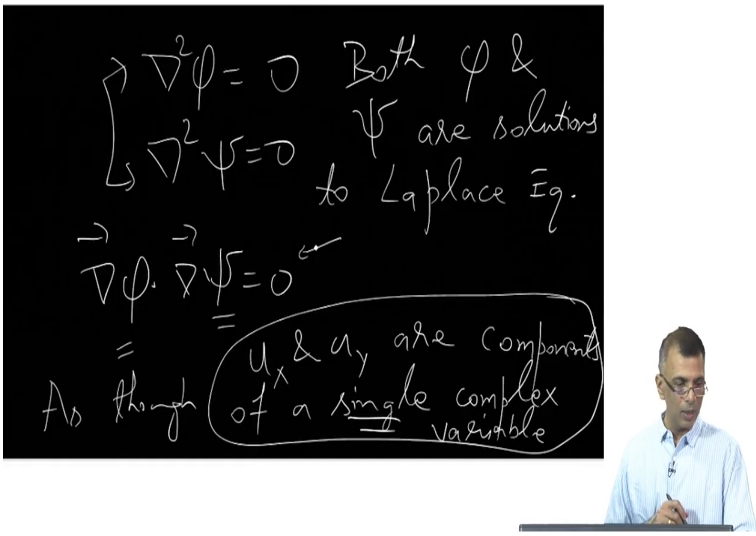

And in which case it turns out very interestingly that also ok. In other words both phi and psi are solutions to the Laplace equation both of these. What is more these are orthogonal, in other words you know the potential lines and the stream lines are orthogonal very much like electrostatics.

So, this is one thing we said and what is more as a consequence of this it is as though u x and u y are components of a single complex variable and this property enables us to make to apply the theory of complex variables you do not have to solve for u x and u y separately and this simplifies life a lot and this comes from the Cauchy Riemann conditions which is essentially this ok.

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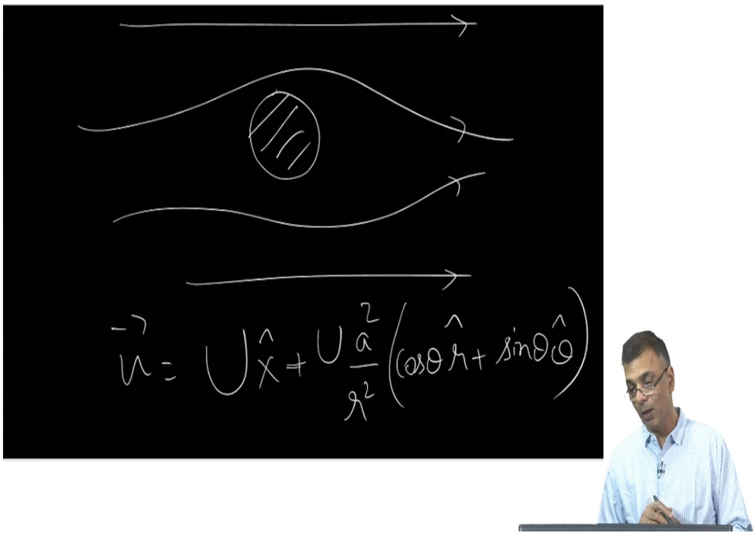

So, this is something that we said and we showed that as a result of this as an example we looked at the problem of you know a smooth sphere and velocity streamlines around the smooth sphere, which would look something like this. We say where we have drawn these and the velocity stream lines very far away would simply be undisturbed as though the sphere was not there and ok.

So, this is fine in viscid fluids and so applying the appropriate boundary conditions which is that the normal component of velocity here on the surface of the sphere is equal to 0. And the tangential component of velocity very far away from the sphere is undisturbed it is what it would have been in the absence you know of the sphere.

And so we need 2 2 boundary conditions in order to solve for either this or this right. So, there is a second order differential equations you need 2 boundary conditions and these are the 2 boundary conditions and turns out that you know the solution is something like this.

Some like so plus U a square over r square, where a is the radius of the sphere plus right. So, this is written in polar coordinates and there is a little bit of mixing here between coordinates, but you understand what we talking about this is really x is really r cosine theta.

And so this solution which is essentially which can simply write down knowing the solution to this in polar coordinates which is well known and applying the 2 boundary conditions that we talked about you can immediately write down the solution one emphasize. So, this is an example of how you know how these are very useful, you can immediately write down the solutions to these from very well known solutions.

And one emphasize that this is really only an exterior solution not the interior one, the interior solution interior to the sphere in any case we are not interested in so we are only interested in the exterior solution, but just want to emphasize that right. So, the other important the very other very important point we discussed was the Bernoulli constant right. So, the Bernoulli constant is essentially how should I say this is essentially an energy equation ok.

So, we start with the Euler equation which if you remember it is momentum equation in other words its f equals m a in the absence of viscosity ok. So, you write down an m a on the left hand side and the f on the right hand side has to be due to pressure gradients, in other words a gradient of p or due to body forces these are only 2 things possible.

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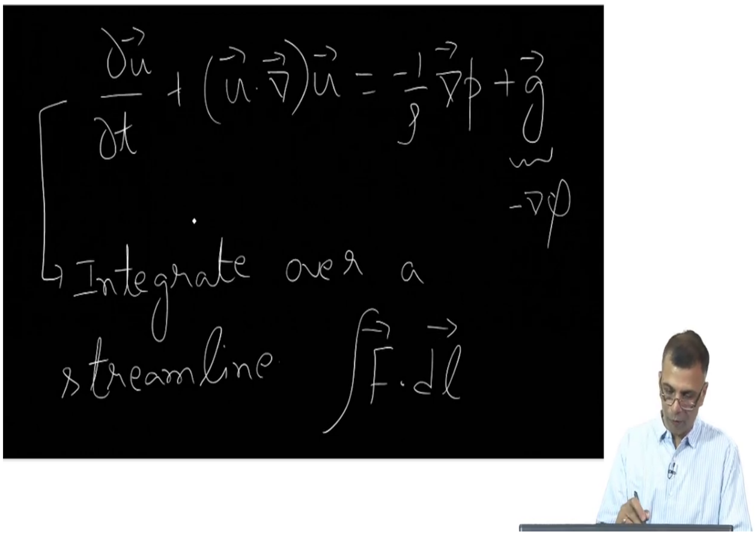

So writing it down exactly in that spirit the Euler equation is which is the momentum equation for inviscid fluids and you should ask yourself the moment I you see a partial u partial t you should know that we are writing this down in the lab frame in the Euler in frame. So, this would be m a and the f would be gradient of a scalar pressure plus any possible body forces.

Which we will assume that this can be written as minus phi or mind you this phi is different from the phi we talked about earlier this is the gravitational potential not the velocity potential ok. So, if we can write this down and do a little bit of trickery with this and you integrate over a streamline. So, this is what we do.

So, mind you this is I well this is f or that is f either way the way it is written here this is m a and that is f. But what you are really doing is right this is what you are doing and what do you get when you integrate force over distance you get work energy all of these things right. So, that is exactly what you get you get a quantity that is energy per gram ok.

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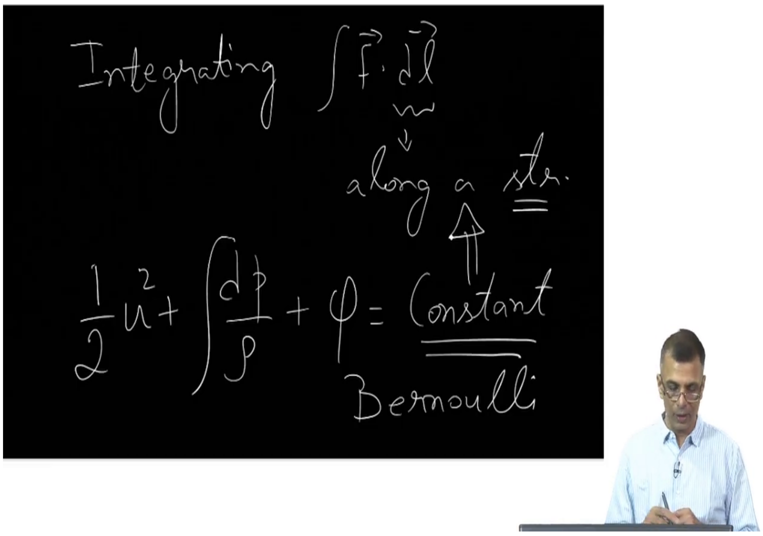

And it turns out that the result of this integration is this quantity, I write this down again just to emphasize how important it is and I want to write this along a streamline ok. We get this quantity we get this particular property which is half u squared plus let me write this down technically correctly, but you know I just want to emphasize that many times people just write this as p over rho.

We get this very important result which is the Bernoulli constant. The Bernoulli constant and so this is essentially the Bernoulli constant right here just to remind you the phi is not the velocity potential its the gravitational potential. So, let us go back to this slide here right here see it is this ok.

So, this you get this by integrating you arrive at this result by integrating the f along a streamline and you get the Bernoulli constant and there are many interesting. So, the other thing is this is constant along a stream line it is one constant along one stream line it is another constant along another stream line.

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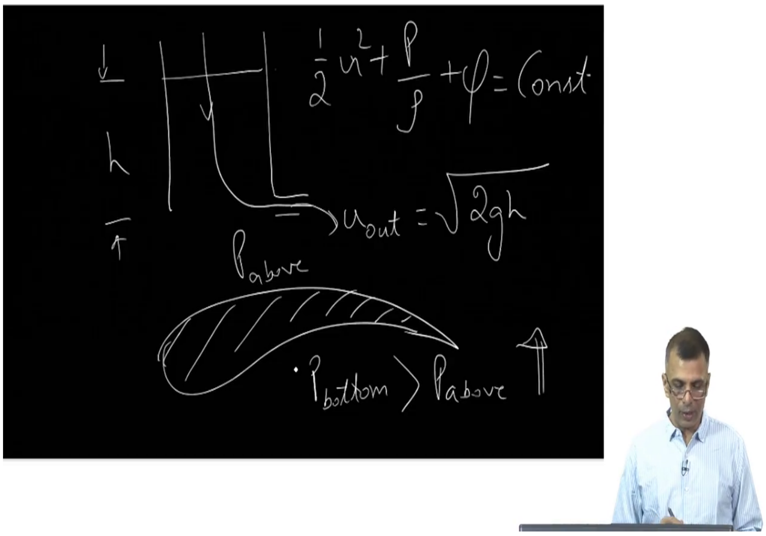

So, I want to emphasize that and this enabled us to look at some nifty things such as the following the water out of an orifice like. So, right you followed a stream line you followed one stream line all the way like this right and made use of the fact that one half u squared plus p over rho plus phi equals constant right. You made use of the fact of this fact and the fact that p does not vary much between here and here and said that you know phi is essentially g times h where h is the is the height.

And this enabled us to figure out that figure out the well known result, that the velocity of the fluid flowing out is square root of 2 g h this is one thing. The other thing we did was and I will say this very briefly, we consider an airfoil that looks somewhat like this.

It is essentially a section of an airplane wing and by considering one stream line that goes all the way; all the way across the airfoil. And knowing that the Bernoulli constant is constant all along the stream line and the fact that the airfoil is designed such again, since the airline the airfoil is very very thin phi is pretty much the same above and below.

But unlike this situation the airfoil is designed in such a way in other words the airfoil is cupped like. So, such that the velocity here is lower than the velocity up here. As a result in order for this quantity we to be conserved the pressure here, the pressure at the bottom p bottom and p above the p bottom is greater than the p above.

As a result of which you get a net upward force, this is a very simplistic very elementary you know explanation for why there would be a lift on an air wing on an aero plane wing. A properly designed aero plane wing one which ensures that the velocity at the bottom of the wing, the velocity of air at the bottom of the wing is lower than that on the top of the wing if that is the case.

Then the you know going by the Bernoulli constant the pressure the bottom will be larger than the pressure above and therefore the wing experiences an upward lift right. So, this is the other thing we discussed and finally we came to the concept of vorticity ok.

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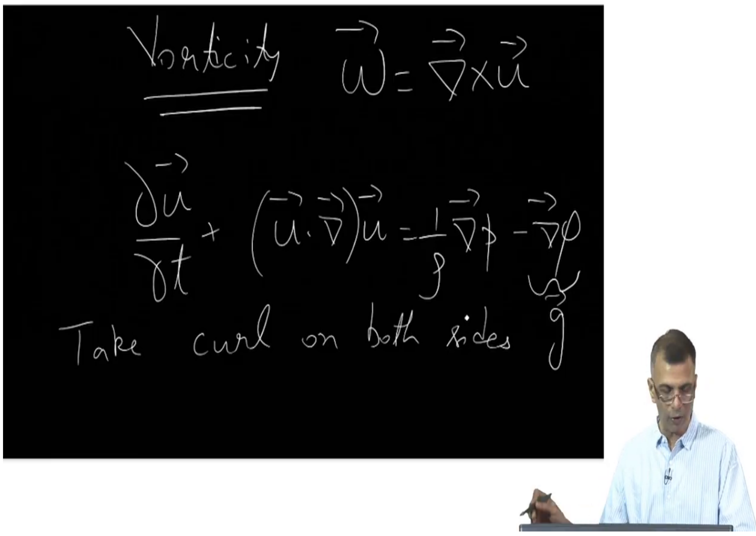

Which we know is a curl of u it measures a rotationality and as we have discussed several times before. Really it is only in a viscous flow that you expect the flow to be rotational and this brings us to so I mean just this is enough to sort of anticipate one of the results that is coming up very soon. Which is that this there is an there is a closely related quantity called circulation and this quantity is conserved in an inviscid flow.

If there is no viscosity in the flow if for whatever reason there is a; there is a rotation if for whatever reason there is a circulation in the flow, its conserved it does not change there is no scope for generation or suppression of vorticity in an inviscid flow ok. So, what we sort of did was again we started with the Euler equation, again written down in the lab frame.

And now we are simply writing this down this is essentially your g as the previous example and we took a curl of the entire equation ok. And then you take curl on both sides that is what we do ok.

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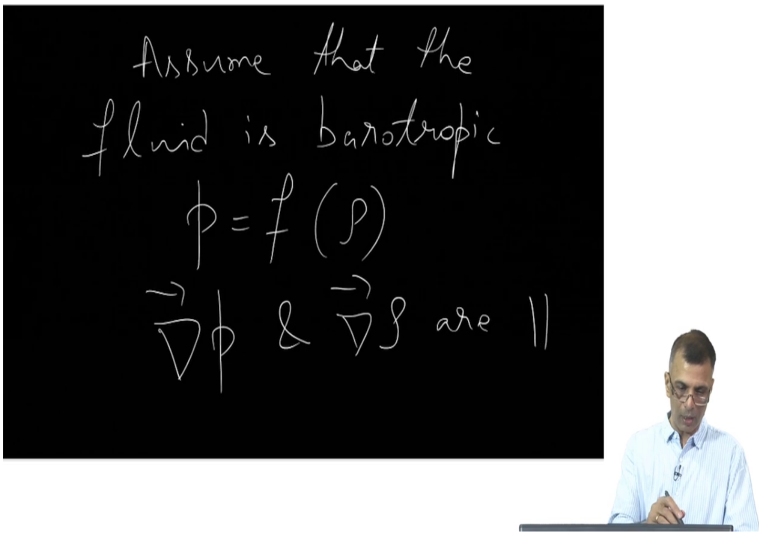

So, take curl of everything and assume that the fluid is barotropic. In other words the pressure straightforward function of you know the density. In this case it turns out that this one and this one are parallel and that leads to a very important simplification and that will lead us to equation to a dynamical equation for the vorticity.

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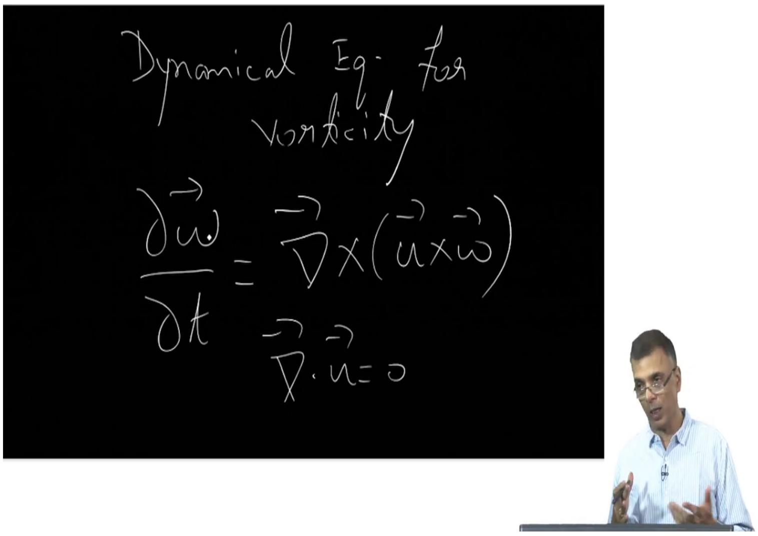

Which is I really should be putting a vector sign on top of this is equal to. So, this is an equation although there is a u here which is sort of the on curl of omega, but still this is really a dynamical equation for vorticity in that it relates the time derivative of the vorticity with space derivatives of the vorticity.

So, you start out with a certain distribution of vorticity in the flow for whatever reason and this equation gives you with proper boundary conditions. Of course, this equation completely tells you how the vorticity is going to behave as time progresses right.

And what is more you specify the and since this is all of this you know is for incompressible fluids in other words ok. So, that already specifies the divergence of the flow and this specifies the curl of the flow, because omega is the curl of u.

Specifying both the divergence and the curl of a vector field completely specifies it for certain you know for well behaved fields which die off nicely at infinity and or if you are not willing to consider boundary conditions at infinity you have to specify both these equations on the boundary, subject to those caveats and.

It is a well known fact that specifying the curl and divergence of a well behaved vector field specifies it completely. So, it is in that sense that we say that the entire velocity dynamics is completely specified by this equation by the dynamical equation for vorticity.

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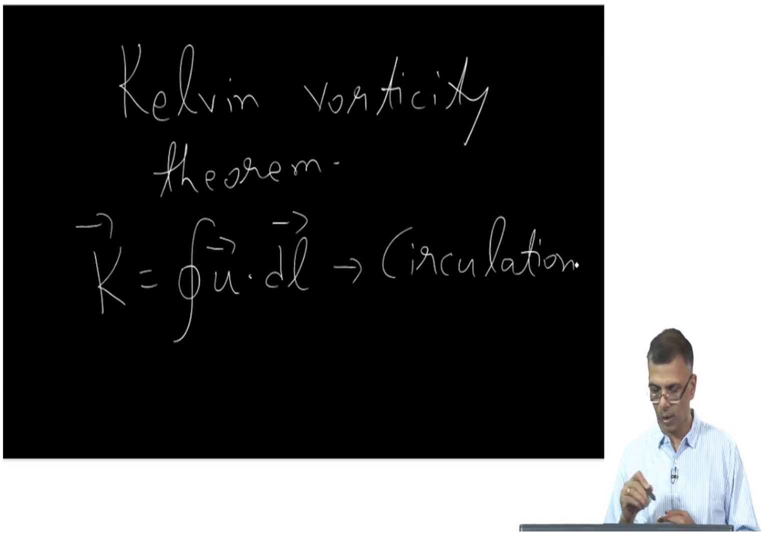

And from here on we will discuss what is called the Kelvin's Kelvin vorticity theorem, which essentially says that there is this quantity that will specify which is called K which is the circulation.

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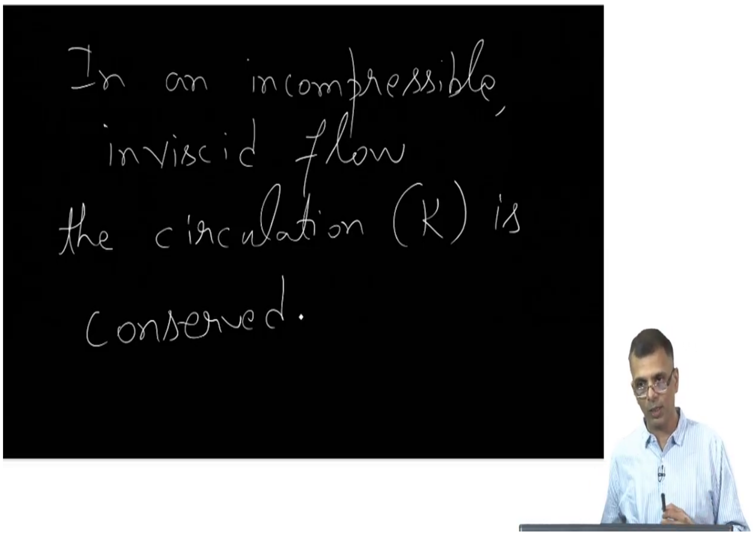

And this theorem essentially says that an incompressible, inviscid flow the circulation K is conserved. If for whatever reason there is a circulation in the flow it remains as it is if and if there is no circulation in the flow you cannot generate circulation ok. So, this is essentially what Kelvin's vorticity's theorem tells you and we will consider that when we meet next ok. So, that is it for now and.

Thank you.