Fluid Dynamics for Astrophysics
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Lecture - 10
Bernoulli constant, its applications and vorticity equation

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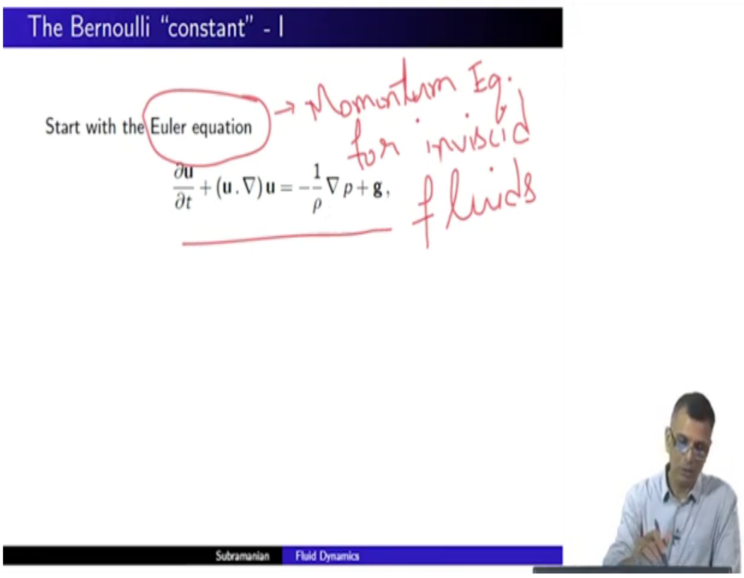

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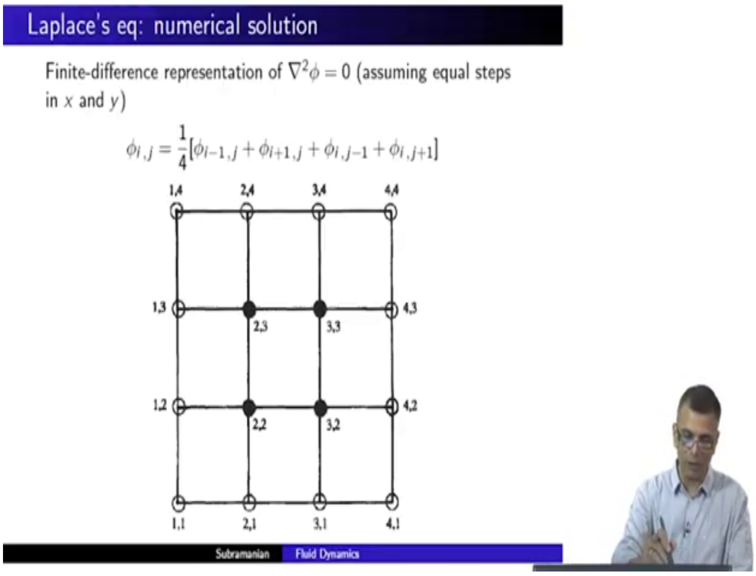

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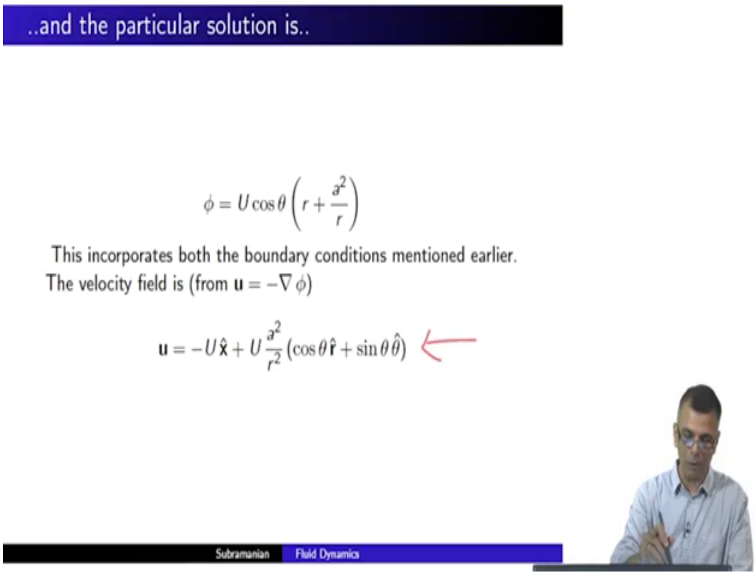

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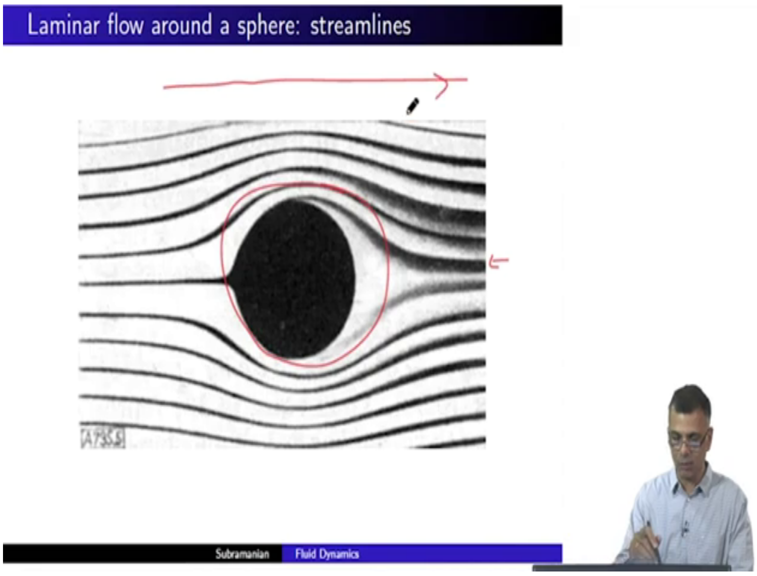

Hi, so I am having investigated maybe before we talk about the Bernoulli constant I would point out a couple of features of this solution. You see the point is the sphere disturbs the flow in its vicinity around here around here say, that is because. So, this is evident from the fact that you know the streamlines are slightly distorted due to this presence of a sphere far away it is undisturbed.

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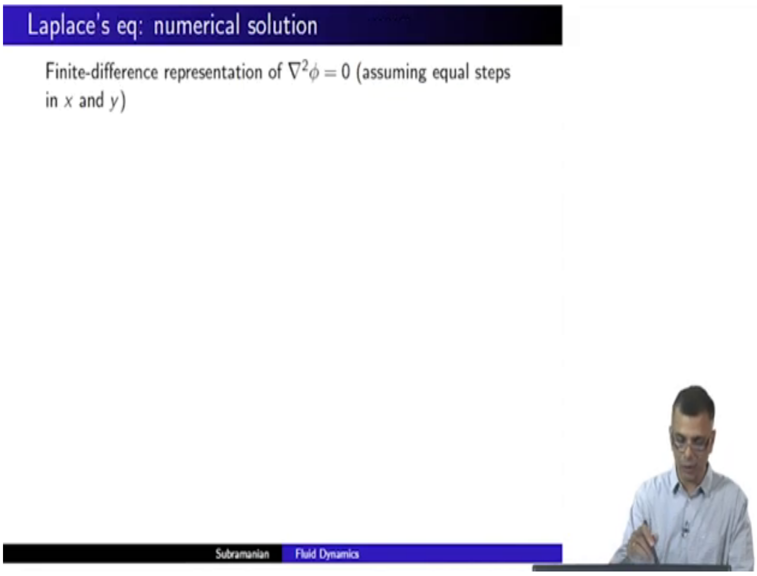

If you look at the solution it looks just like that, if you look at the solution this is the solution far away, this is what dominates far away and this is what dominates closing. And there are some funny features to this solution right at r equals 0, there are some funny features that. Essentially what I am trying to say here is that this is an exterior solution, this is a solution that is valid for the exterior of the sphere we never.

So, the boundary conditions we applied are valid on the surface of the sphere and far away from the surfaces of the sphere. We are not saying anything about the interior of the sphere, if you want to consider that too there are some other things called doublets which we do not have to you know bother with at the moment in the interest of moving on right. So, moving on I would not bother too much about this.

Let us talk about something that is called the Bernoulli constant you might have heard about this and the reason I am bringing this up is because in some ways this is our first encounter with the energy equation ok. In some ways this is an energy equation so to speak and it also utilizes the concept of a stream line or something that we have been talking about quite a bit.

And you can also derive some fairly nifty results that have to do with physical applications using the concept of a Bernoulli constant right. So, let us plunge right ahead let us start with the Euler equation, which if you remember. The Euler equation is the momentum equation for fluids that are inviscid right. Momentum equation for inviscid fluids; so, we are sticking to situations where the fluid is; where viscosity is neglected right. So, we and the Euler equation is something that we spend some time on.

So, we are familiar with and just to emphasize this is the Euler equation written down in the Eulerian frame; in other words the frame of the lab observer right. So, that is the Euler equation written down in the frame of the lab observer the left is m a and the right is f; two different kinds of fs, these are body forces, these are pressure forces right.

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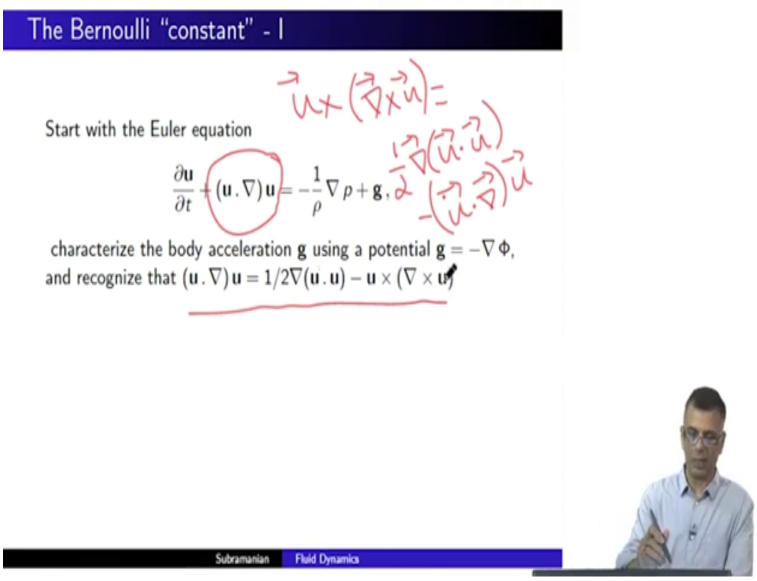

So, let us move ahead let us now characterize the body force using a potential. If it is you know a gravitational potential, then it is easy enough to write there is a reason right. So, and you can already see from mathematically if you write this down as a gradient of a potential, this is a gradient of a scalar pressure and this is the gradient of another scalar field.

So, you can absorb both the gradients. So, there is a reason also there is a physical reason you know acceleration due to gravity it can indeed be written down in terms of a gravitational potential. So, there is the utility and you recognize that this term from vector algebra can be written down like so. This really follows from the definition of this.

You would have seen this; this kind of a thing as an expansion of this quantity. This is equal to you take this to the left this is equal to one half minus. This is how this is the form of the identity that you might be you know a little more familiar with and mind you this u dot del is not the same as del dot u; u dot del it is the directional derivative ux d over dx plus u y d over dy and so on so forth ok.

So, you just rearrange this and its simply a way of reformulating this one. And there is a reason; one of the reasons I can already point out is you see you see this u dot u its like u squared yeah. So, half u squared is like the kinetic energy per unit mass right and this we can get rid of. So, bear with me for a minute and let us go ahead right.

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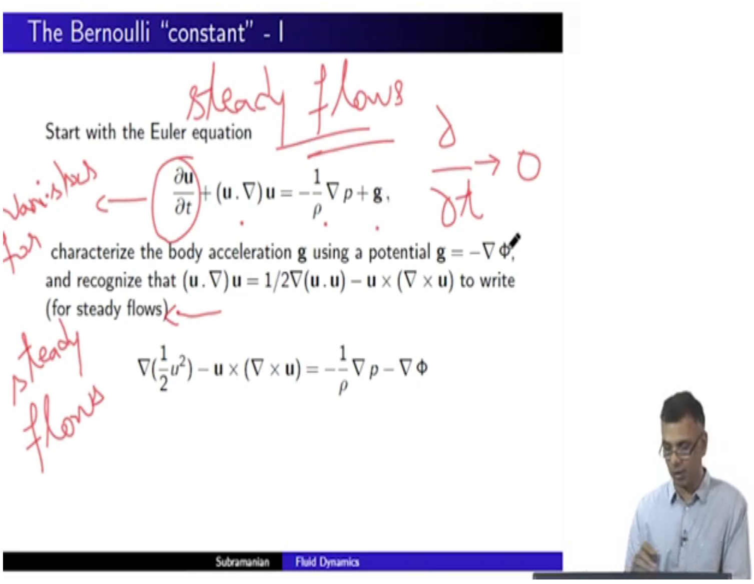

So, and now we say for steady flows we emphasize the word steady flows. So, steady flows are ones where; are ones where the time derivative is 0. The time derivative as discerned by the lab observer as discerned by the Eulerian observers there are.

Therefore, this entire term vanishes we do not have to worry about this; vanishes for steady flows. So, we do not have to worry about this term at all we only worry about this and this and in addition we take this to be the gradient of a scalar potential right.

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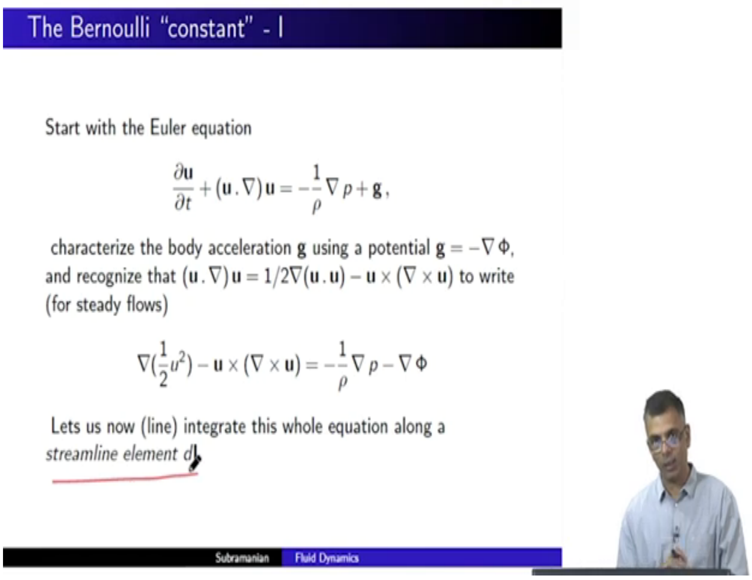

Now, so, throw that away and all we are left with is this where we have made use of this result right. Now, let us integrate this entire equation along a streamline element dl. So, you are not following you are not going along x or along y you are going along a stream line ok, you are integrating along a stream line very important to keep in mind ok.

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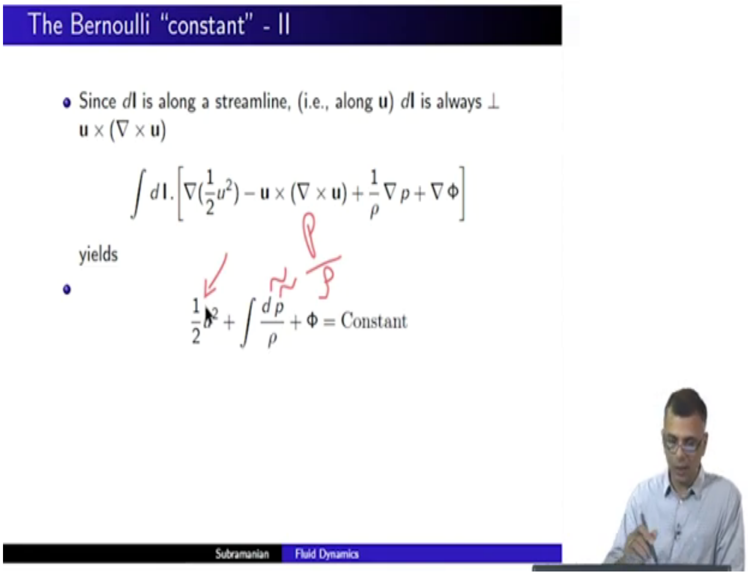

Now, dl by definition dl the you know length coordinate is along a stream line in other words is along u. That is a definition of a stream line right; the stream line is a line that points along u right. Therefore dl is always perpendicular to u cross curl of u, why is that? Well whatever this is curl of u whatever direction it points in you are crossing it with u right.

So, this entire thing; this entire thing is always perpendicular to u in other words its always perpendicular to dl, yeah first of all I mean apart from this well I mean this thing is simply; this is simply integrating; this is simply this. So, this line is simply expressed here and then we make use of the of this one to.

So, dl is always perpendicular to this entire thing yeah. So, the dot product vanishes right. So, we can throw in once you integrate you can throw this the integration of this entire middle term away gone because of this property and that yields, you see dl dot you know gradient of half u squared.

So, this is a perfect differential right. So, you simply get half u squared very simple and this is also kind of a perfect differential. So, it is really dp over rho and you are integrating over dp, sometimes many people get a little careless and just write this as p over rho ok. This again is a perfect differential so you simply write phi when you integrate along the stream lines a perfect differential. So, this is a central result.

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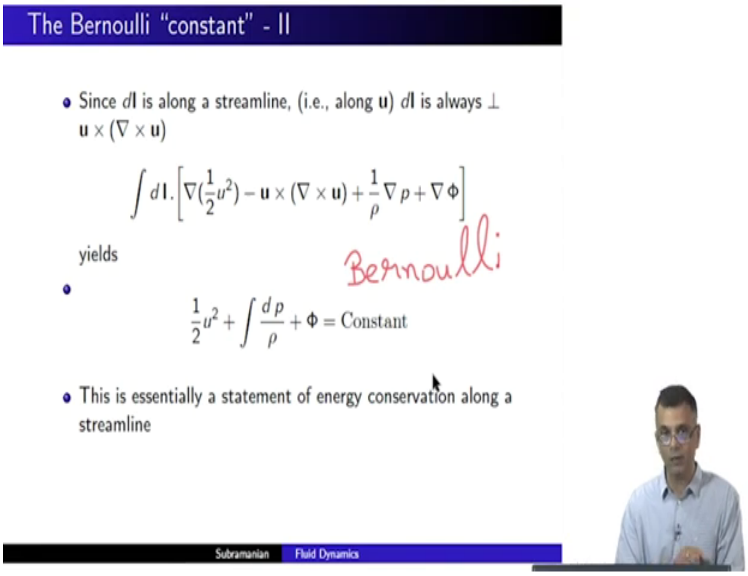

And because it is a constant so this is what is called many times people write this as the you guessed it is the Bernoulli constant the Bernoulli constant ok. This is essentially a statement of energy conservation if this is not immediately evident to you, you can see you know half u squared is kinetic energy per unit mass.

So, the rest of them had better have units of energy and you can check that this is the gravitational potential has the units of energy. And this also if you know put down the units properly you will find that it has a units of energy per unit mass ok. So, this is essentially a statement of energy conservation along a stream line.

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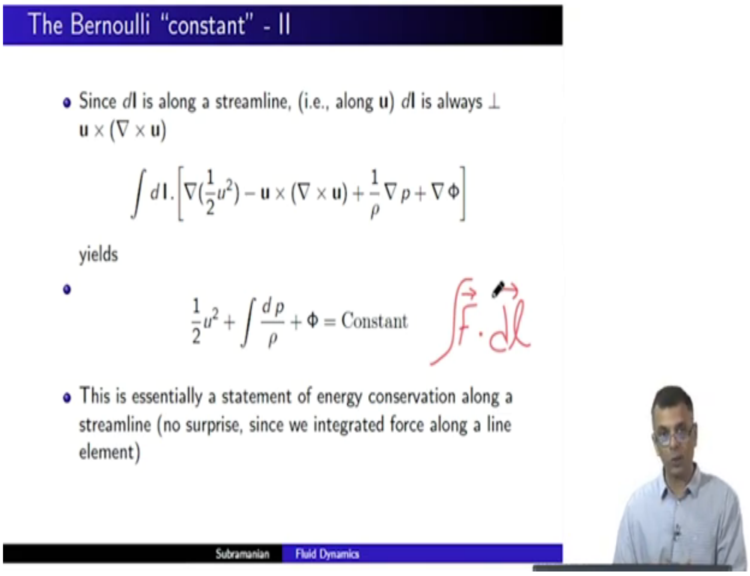

No surprise right since we integrated force along a line element, what we did? You recall, we started from the Euler equation right. Euler equation this is m a and this is f right. So, we essentially integrated f we essentially did you know, we essentially did f dot this is what we did and this is exactly how you get work right this is you integrate force a long distance you get work and that is it.

So, it is no surprise, so that we get energy that we get a quantity with the units of energy the energy per unit mass to be precise, but the remarkable thing about this is that it is the energy it turns out to be a constant and this is called the Bernoulli constant.

This is something this is yet another label for a streamline the stream function was one. And this is another very important practical very important label from a practical point of view there is yet another label for a streamline right.

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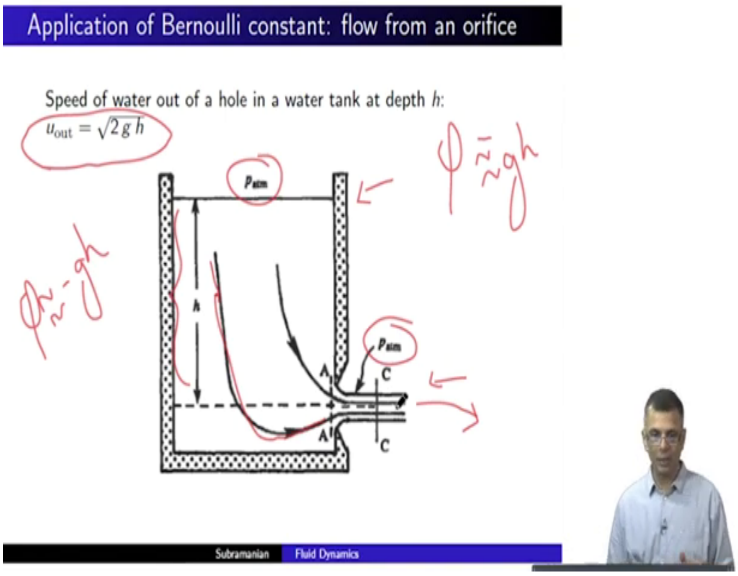

So, here is an application of the Bernoulli constant the application is that of flow from an orifice. So, what you have is a you know a container of water with an orifice here. And you must have you know encountered this many times perhaps specifically from the point of view of a Bernoulli constant must have encountered this problem several times.

And you have and the entire container is subjected to atmospheric pressure and. So, and you have a water receding from the top under the influence of gravity pressure and you have a water flowing out from here like. So, the question is, what is the speed at which water flows? And the height of this is h right here this is h ok.

So, turns out that you can you can solve this problem quite simply, the assumption now is that the atmospheric pressure the container is so small pretty much any container in real life satisfies this condition. The container is so small that there is really no difference in pressure between the top of the container and the orifice. So, its P atmospheric here as well as here ok.

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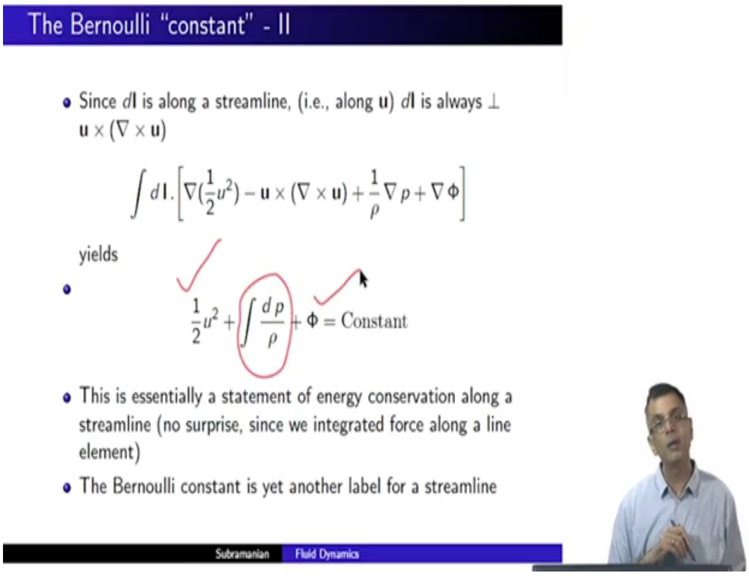

Therefore this term does not matter anymore only this and this matter ok, and what is phi? Phi is simply in this case phi is simply something like gh, where g is a local acceleration due to gravity yeah. So, you have minus gh, let me write this again ok. So, if you plug this in here.

If you plug this in here phi is minus gh you can immediately see that you get half u squared is equal to you know gh. And therefore, u squared is equal to I mean u out is equal to square root of 2 gh, what did we do?

We applied Bernoulli's constant to a particular streamline to one stream line like this ok. So, we invoked energy conservation along a particular stream line this is what we did and that immediately gives you this familiar result.

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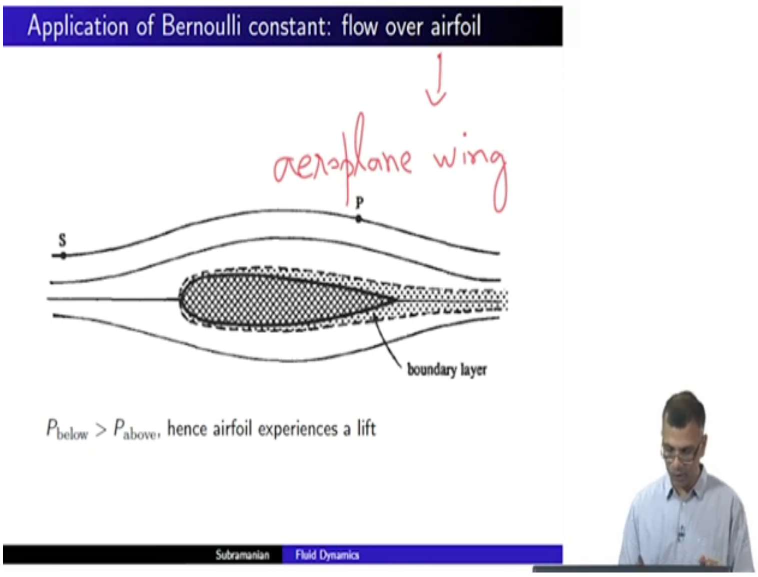

Let us go to another nifty result and that is this is flow over an airfoil or essentially an aeroplane wing, why does an aeroplane wing experience a lift? Ok. This is how this is the fundamental principle around which aeroplanes are designed and this is a rather simplistic way of explaining.

If you really want to explain it properly you have to take resort to, what is called Kutta Jokowski theorem. But we would not bother about those details in a minute and it turns out that we can invoke the Bernoulli constant to show this, ok right.

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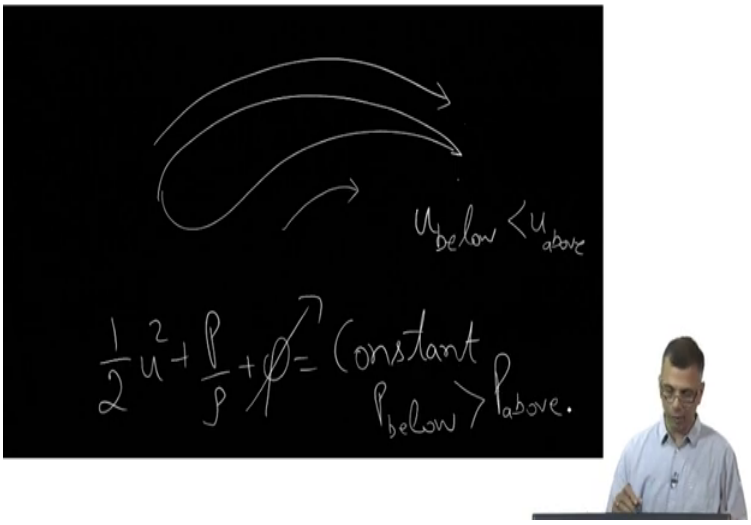

So, turns out that now this figure does not emphasize this enough. So, I will go to this and I will draw an airfoil that really, this is really how an airfoil looks. It is exaggerated of course, that the main thing is it is curved a little more here than here it is curved a little more.

And the Bernoulli constant recall the Bernoulli constant is half u squared plus p over rho plus phi ok this is constant, so you write this down ok. So, consider a stream line that is enveloping this kind of a circulation ok. Now what happens is because that the airfoil is designed this way, it turns out that you know the flow flows faster over its design in a way that I should probably erase this, is designed in a way you know the velocity here is slower than the velocity out there ok.

So, the u here I will draw this with the smaller arrow and the u here I will draw this with a larger arrow ok. You can intuitively think of it like the palm of your hand and in velocity and fluid flowing like this. You can intuitively understand that that the curvature the cusp is such that it slows down fluid at the bottom as opposed to the top ok.

As a result in order and the phi is pretty much the same as in this is a very thin airfoil there is no difference in phi between here and here the gravitational potential is pretty much the same. So, we can essentially throw this away, we do not have to worry about this ok. Now, the u below is smaller than the u above right.

So, and in order for this to be constant because you are considering one stream line that is going around like this. Therefore, P below has to be greater than P above right and there you have it because P below is larger than P above the airfoil experiences a lift. So, this is how airplanes are able to fly this is how the wing of an aero plane experiences a lift.

Simply because, it is designed in such a manner that the you force the air to flow its cupped like this and therefore, the air at the bottom flows a little slower than the air at the top. And therefore, because if you consider one streamline you know all throughout and the Bernoulli constant is constant all throughout the streamline, you are making many assumptions here of course, the flow is inviscid and so on so forth.

And not necessarily true in real world situations in particular you know you are neglecting effects like the boundary layer and things like this, but nonetheless it is a very useful thing to its a very useful rough way of understanding this thing yeah. So, because the velocity above is larger than the velocity below the pressure in order for the Bernoulli constant to be constant, the pressure above is smaller than the pressure below.

In other words there is a lift the gradi ent in pressure gives you a force, remember force goes as gradient of pressure and which direction is that force in, the force is in the upward direction yeah. So, this is another neat illustration of the utility of the Bernoulli constant.

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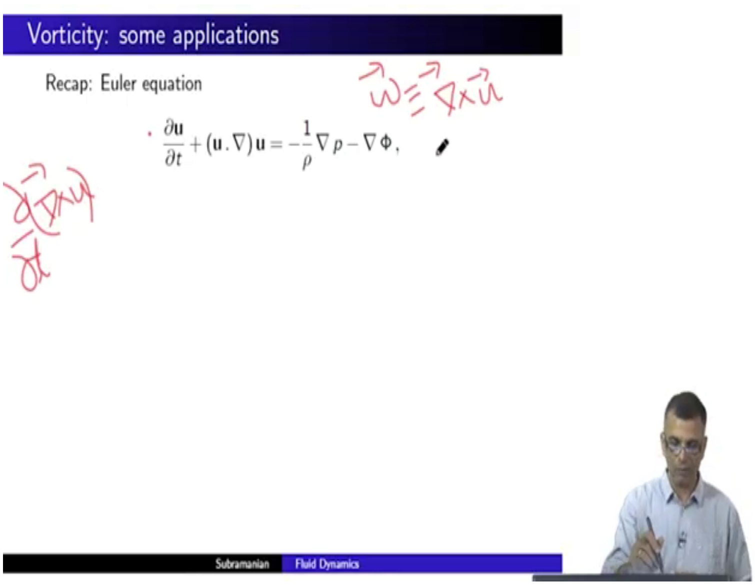

And the next thing we will talk about is the vorticity equation and maybe I will very quickly go through it and we will we will make a stop. So, what you really do in talking about vorticity recall is this quantity is the vorticity, this is what the vorticity is defined as.

You start with the Euler equation and you take the curl of everything ok. So, what you do is you do d over t and you assume that the space and time derivatives can be interchanged. So, you do a d over d t curl of u, that is how you do this and you take the curl of this you take the curl of this yeah.

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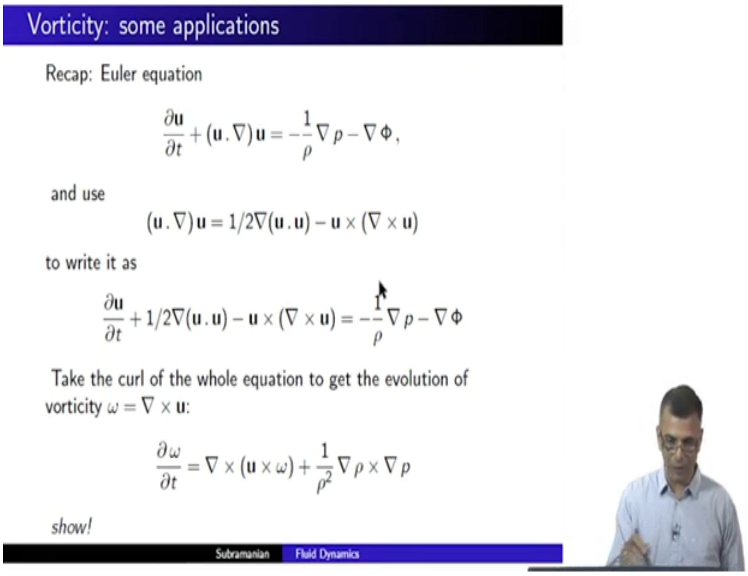

So, you curl everything and you use this identity same thing as what we encountered earlier while discussing the Bernoulli equation and you write it as this yeah. And you take the curl of the whole equation to get the time evolution of this quantity which is the vorticity.

And you end up with this equation which we will discuss a little further and we in particular we will specialize to what are called barotropic fluids where the pressure is simply a function of the density you have already encountered these things before p equals nkt.

For instance yeah many fluids in real life are barotropic. And in doing so we will get an evolution equation for this quantity called the vorticity and that has very nice applications. It is a dynamical equation for the conservation of for the evolution of vorticity. And we will use that equation to derive to illustrate some other real life examples such as the lift on a spinning ball this is something that people who play tennis topspin are familiar with.

We will discuss this when we come to it and in order to understand that its we need to understand vorticity and the evolution of vorticity. So, as we said we are interested in getting an evolution equation for the vorticity, how the vorticity omega evolves as a function of time? Ok and there will be. So, the left hand side has time derivatives the right hand side is going to have space derivatives. And the way we proceed is essentially you take the curl of this entire equation here this entire equation right.

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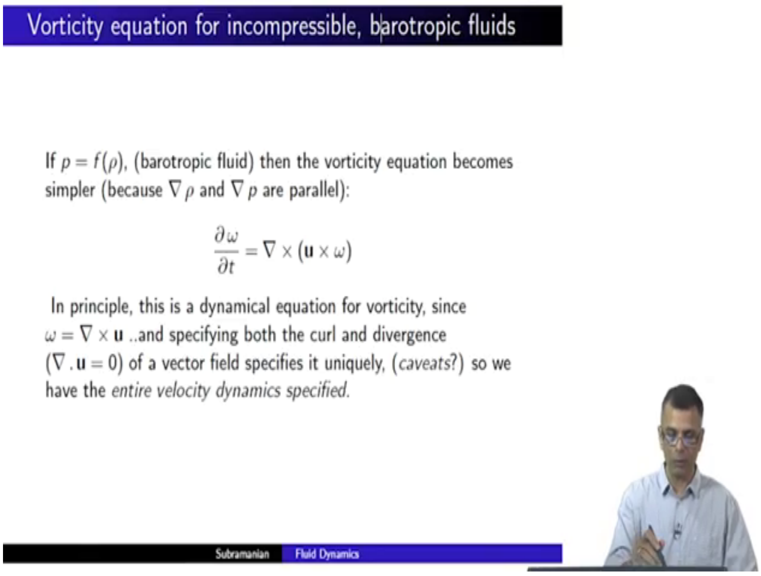

So, we specialize to barotropic fluids where you know the pressure is simply a function of the density. So, if the pressure is a scalar function of the density and this is something that you are familiar with; obviously, you know p equals mkt or rho kt for instance.

If this is the case then you know the gradient of the pressure you see you have a situation where you have a gradient of the density cross the gradient of the pressure, if the pressure is simply a function of density then these two vectors are parallel to each other and the cross product vanishes right.

So, you have a much simpler equation comprising of just these two terms and yeah so there you go. So, this is a dynamical equation for vorticity ok. You might wonder you know omega is a curl of u and you have u itself here. So, if it is really you know a dynamical equation not exclusively in terms of vorticity, but all in terms of say the uncurl of the vorticity which is the velocity itself, but that is ok, it is still a dynamical equation for the vorticity yeah.

So, in principle there is a since omega is that is what I just said since omega is curl of u right. Specifying both the curl and divergence of a vector field specifies the entire vector field uniquely ok. And there are some caveats that the field has to be well behaved in other words it you either at infinity you assume that the field dies away smoothly you have to make that assumption or you have to make sure that if you are imposing a boundary the boundary conditions are well behaved.

So, this is simply by way of amplifying the statement that specifying both the curl and divergence. So, the curl is the vorticity and we are talking about you know an incompressible fluid in other words divergence of u equals 0 is always satisfied. And of course, we are also talking about barotropic fluids, which is how we were able to throw away this last term right. We were able to throw away this last term because we were talking about barotropic fluids.

But the point is you are looking for an evolution of the curl and you are looking at a situation where the divergence of u is 0.

So, you are in effect specifying both the both the curl and the divergence and so it specifies the entire velocity field uniquely with some caveats about the well-behaved nature of the of u at infinity or if you are specifying a boundary you have to specify the boundaries properly. But essentially this is a full dynamical equation for the vorticity or equivalently the velocity fields ok.

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So, we will use this dynamical equation to define something called a circulation and show that for an inviscid barotropic fluid the circulation is a quantity called this. It is a line integral of the fluid over it is a closed line integral and this is something that is conserved and the conservation of circulation will enable us to derive some nifty results that we talked about. So, we will do this when we meet next.

Thank you.