Introduction to Classical Mechanics
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Lecture No. 09
Properties of Lagrangian

Some time back we wrote down Euler Lagrange equations and last time we were looking at some of simple examples where we got to get some practice with using Euler Lagrange equations. Since we have been making Lagrangian as a central thing, let us look at some of its properties now.

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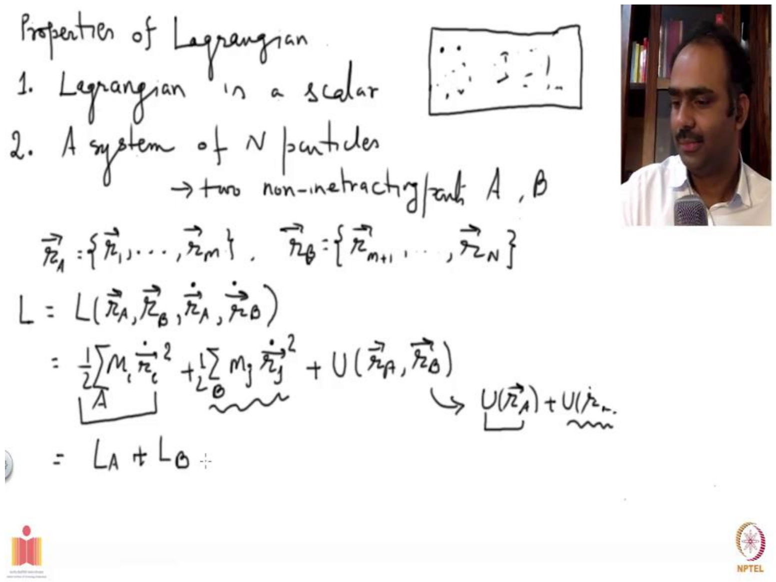

So properties of Euler, properties of Lagrangian, first thing is that Lagrangian is a scalar quantity, unlike where you when you use Newton's equations of motion, you have to know forces which are which are vector quantities on all the particles to find out how the system is going to evolve. But when you are using Euler Lagrange equations, what you need to know is the Lagrangian which is made up of the kinetic energy and potential energy. And both of these are scalar quantities. So there is one thing Lagrangian is a scalar. So, for our present context, it is scalar under rotations.

Also, let us say you are looking at a system which has several particles, but you can group the particles into 2 parts. The particles belonging to Part A, let us say one of the parts is called Part A, those particles do not interact with the particles that belong to the part B, so I am imagining a system with 2 parts; Part A and part B, there are also particles in there, the particles within A, they interact among themselves, particles within within B, they interact among themselves, but the particles from this group do not interact with particles from the other group.

Let us say that is a situation. And can I say something about the Lagrangian of this entire system. And what I want to show is that in such a case the Lagrangian of the entire system can be written as a sum of Lagrangian of individual parts. That is what I want to say and it is quite easy. So, what I am trying to convey is the following. So we have a system of, let us say N particles and I am saying it has two or more than two, it does not matter. You can take more than two also, two or more, let us say two two non-interacting parts, so here is your system, it is isolated from everything else. And let us say it has also particles here and there.

And let us say I can classify these particles into 2 sets. I am not dividing a line; I am not putting a line here to say that this is part A, this is part B because I am imagining the particles are all moving around. But I am saying this guy is belonging to Part A, this guy is going to part B, A, this guy, maybe part B, they do not talk to each other. So, I have two non-interacting parts, A and B by which I mean I mean what I said. So, let me represent by r a all the particles or the coordinates of the particles which are belonging to Part A. So, let us say r 1 to r m belong to that set.

And then r B denotes all the Cartesian coordinates of the particles which belong to set B. And I write them r m plus 1 to r n, so that makes a total of, and if you look at both the both the parts, that is good. Now, if I look at the system as a whole, then the Lagrangian of the system, which let me say, let us call it Lagrangian of the system would be a function of r A, r B, it depends on all the coordinates of all the particles in in in general, that is a general statement, r A dot, it also depends on all the velocities of each of the particles that is what our system is.

Now, this you know what form it has, it has the form of kinetic energy minus the potential energy. So, I will write it as half m i r i dot square and I have to sum over all the particles, but I will split this sum into two parts, I will submit because I can this is an ordinary sum. So, first you have sum over all the particles belonging to the class to the to the part A, then again you have half and let me put j, it does not matter you can put i as well, but I will just put j, there is no special reason for that.

And here are some over all the particles which belong to set B and then you have the potential energy which depends on all the particles in the system. Let, so in shorthand I write it like this. That is good. Now, if the potential if the parts are not interacting, then the potential energy of the entire system can be written as the potential energy of the Part A plus the potential energy of the Part B, that is what it means by non-interacting.

So, that is what I will do. Because they are not interacting, this U can be written as U of r A plus U of r B, it can be split into two such parts. Maybe I should use a different symbol for U, but because it is clear from here that these are energies corresponding to part A and Part B so, I will not bother with writing the changing the symbol for you for these individual parts.

If that is the case, then this Lagrangian as you can see, I can combine this piece with this and that piece with this. So, I get L of A that is the Kinetic plus potential energy of that part plus L of B that is one thing which you wanted to show. So, this splitting can be done only if these parts are not interacting with each other. Otherwise, it makes no sense, you cannot we cannot do this. Also, let us go to next page.

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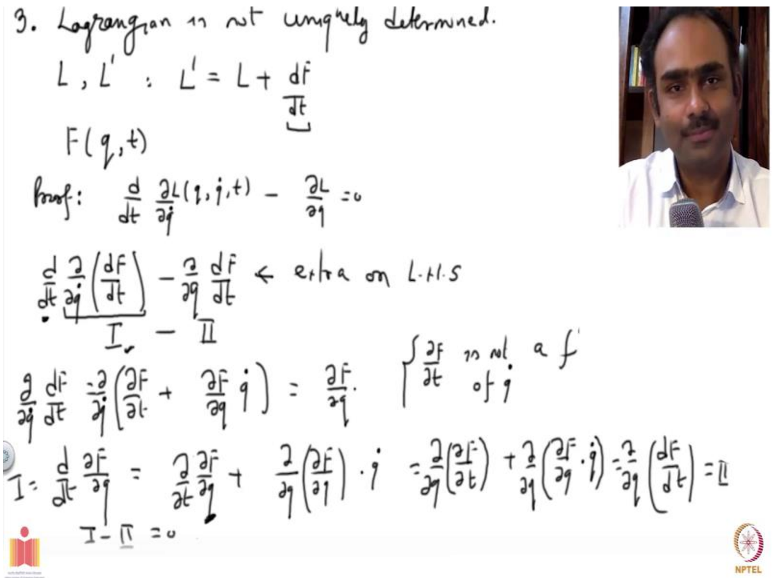

So, I have written down 2 properties, property number 3. It is useful to notice that that the Lagrangian which we are writing, it is not unique, so it is not that if you give a system to me, there is a unique Lagrangian that I can write for it, we will come to this the reason for this also in our later video, but for now I will show that you can add a total time derivative to the Lagrangian and it will still satisfy the equations of motion, it will still satisfy Euler Lagrange equations. So, let me write it down.

So, Lagrangian is not uniquely determined, which means that if I give you 2 Lagrangian L and L prime, they will describe the same system meaning you will get the same Euler Lagrange equations from them provided the L and L prime differ by a total time derivative.

So, more precisely if L prime is L plus d f over dt, where F is a function of all the coordinates, generalized coordinates q q 1, q 2 and so forth, and also possibly of time, if this is a situation then L prime and L they describe the same system. So, how do I check this? Well, what I should do is look at the dF over dt term and plug into their Euler Lagrange equation.

And if the contribution arising from dF over dt, this piece goes to 0 then we can conclude that indeed L prime and L satisfy the same equations of motion, they they give the same equations of motion. So, that is what I will do. So proof, I will not bother to give proof for general system with several degrees of freedom, I will just give proof for one degree of freedom, so, I have only one q, but you should try to repeat the proof for a system which has more than one q, but the proof will not be very different, it will follow the same lines which I am going to write on here.

So, let us look at this d over d t. Let us say I start with L, del L over del q dot as I said, I am looking at only one coordinate now, I mean only one degree of freedom q dot and t minus del L over del q equal to 0. I am saying if I replace this L by L plus dF over dt, the term coming from the f will go to 0 which means if I plug it in plug in this piece also, then in addition to this, what you have on the left hand side, you will also get the following piece.

You will get d over dt del over del q dot d f over dt that is correct no mistakes, minus del over del q and d f over d t. This is what will get extra on the left hand side. And I would like to show that this is 0, how do I do that? So, let me call this term 1 minus term 2. Let us look at term 1 first. So, I look at this total derivative df over dt . The term one I am looking only df over dt first. So, df over dt is remember the F is function of only q and t, not q dots.

So, when I take the total time derivative, I get delta f over delta t, because I have allowed for an explicit time dependence in F. So, there will be this term plus delta f over delta q, q dot that is correct. Now, if I take the derivative with respect to q dot on this, let me take del over del q dot here, over del q dot what will that be? Now, look at this piece, this piece is only a function of q and t because taking a time partial time derivative will not generate any velocity terms.

And this piece will be only a function of q and t this piece, del f over del q, and the time the velocities are only in here. So, when this derivative partial derivative with respect to the generalized velocity X on this piece it gives 0. Here I should use a chain rule but this piece because it does not depend on velocity, this partial derivative is going to act only on this q dot. So, I get delta f over delta q and del q dot over del q dot, which is one so I get del f over del Q. And as I said before, maybe I should write down explicitly delta f over delta t is not a function of q dot, it is not a function generalized velocities that is good.

Now, I should take the total time derivative, I have taken care of up to here now I should look at the total time derivative of this entire piece. So, I take d over d over dt, now I am taking the total time derivative on the left which means I should let me let me repeat it. Now, I take the total time derivative d over dt, this piece, this will become a total time derivative acting on del F over del q del F over del q, this will be your term number 1 what is that?

Again I do the same thing, I write del f over del q, del over del t so I have first del over del t acting on this piece, then I have del over del q, del f over del q times q dot. Good, it is not a dot product; this is just a dot, perfect. Now what can I do? That is good, that is good. You see here I can interchange q and t, they are independent variables so I can interchange them, which means I can write this as del f over del t, del over del q plus del f over del q, del over del q q dot. If you were looking at more than one degrees of freedom, here your i and j will appear and you will have to take care of those things.

Now this is, so you have del over del q here, del over del q here. So I can write this, I wanted to do one more thing. So, you have del over del q and then I can bring in the q dot in the bracket, I can do so because when you take a derivative of, partial derivative of velocity with respect to coordinate that is 0. So I can I can bring this piece in, which means I can write this as del over del Q, dF over dt and let us see what our second term was.

Second term was del over del q del f, df over dt which is the same here. So, this is your equal to second term, which means first term minus second term equals 0 and that is what we wanted to prove that the extra term will not give any contribution. That is another very important property that you should keep in mind that you can always add a function to the Lagrangian which can be written as a total time derivative of another function, which depends only on the generalized coordinates and time and not the velocities, that is nice. Let us see what else I can say.

Next, I want to look at the kinetic energy term, when we are using generalized coordinates. So, you know typically when you are using Cartesian coordinates, you have just sum over half mv squares. And when you write V squares you are really summing over velocities of individual particles and there is no cross term. So, you do not have any term which is a product of velocity of this particle and that particle and the kinetic energy. But this situation changes when you go to generalize coordinates and that is what we want to look at in the next video. So, see you there in the next video.