Introduction to Classical Mechanics
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Lecture No. 08
Euler Lagrange Equations. Examples continued

Last time we gave an example of a free particle that is moving in a three dimensional space and we used Cartesian coordinates as the generalized coordinates and looked at the Euler Euler Lagrange Equations and from there we derives our or let us say we obtained our Newton's equations of motion. That was the simplest thing we could ask a free particle in Cartesian space is the simplest thing that you can ask.

To gain more experience let us take the example of again a free particle moving in, let us say not 3, but 2 dimensions. In two dimensions, X and Y plane, let us say. But this time, we do not want to use Cartesian coordinates as the generalized coordinates, rather, polar coordinates, so I will use r n theta as the generalized coordinates, and let us look at what we get for equations of motion from other Lagrange equations. That is one example and if possible I will take 1 more example after this one. So let us get started.

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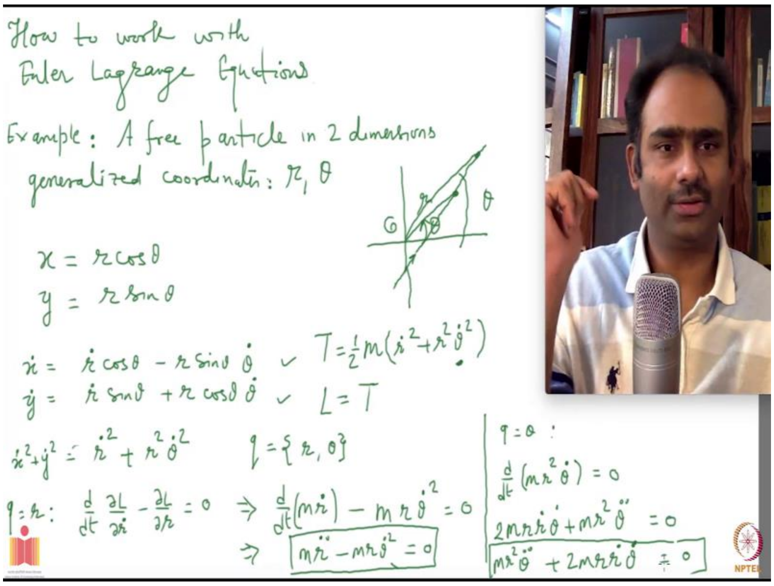

Here, so I will take the example of, I am trying to use green colour for examples. I do not know whether I will remember this later in other videos but today. So as I said, a free particle in 2 dimension. So what do I want to use for the generalized coordinates? I think it is already clear that it is a 2 dimensional problem, there are only 2 degrees of freedom and instead of x and y, I can use r n theta.

And this will typically be the case, most of the time you will be able to guess what the generalized coordinates are appropriate for the problem. In this case, r and theta and x and y they are all equally good. Actually, x and y will be even nicer, the Cartesian ones, but generally the symmetry of the problem will dictate to you what should be the generalized coordinates.

So let us take this. Now before I look at Euler Lagrange, let us have some expectation of what we are going to get. So imagine, this is the origin of the coordinate system and this let us say let us say you are going to fire the particle in in this way in this direction. So, you know it will continue along the same straight line because there are no forces. If you take any point P on this, this will have some radial distance R, and some polar angle theta, theta is measured from the x axis.

And what we want to know is how the coordinates r and theta are evolving with time. If you had fired it readily outwards, if it was going like this through the origin, then clearly the angle will not change. And it will have only the radial velocity. But because you have fired it not from here, it is not passing through the origin, it is going from some distance away from the origin, it will have an angular velocity because you see the angle will keep changing, when it is here this is the angle, when it is there, the angle has changed, this is the new angle now.

So it will have both angular velocities and radial velocities and also, you will see it will have it is also you can just, if you think for a moment you will realize that it will have angular acceleration and also radial accelerations, but we will see it explicitly. So that is what we are expecting in general. So how how do we go about writing down the equations of motion?

So, first I do write down the relation between r and the generalized coordinates. And by r I mean x and y vector I meant, so x is r cos of theta, and y is r sin of theta. Now, because I want to construct first the Lagrangian of the system, which involves T and U kinetic energy and the potential energy, potential energy is 0, I need to construct only the the kinetic energy which involves velocities x dot square plus y dot square.

So, let me calculate x dot and y dot, x dot is r dot cos of theta minus r sin theta, theta dot, so I am taking the derivative of cos theta in here. Then your y dot would be r dot sine of theta plus r cos theta, theta dot. Now I want to do x dot square plus y dot square, so I have to square this and this and add, so when I square them, you will have a square of this term, and a square of this term, which will add and cos square theta and sin square theta will give you one. Let me write down x dot square plus y dot square equals.

So, I am adding the squares of this and that right now. So, I will get r dot square. Let us say add squares of these two as well, you will get r square theta dot square and sin square theta. And this one will give the same terms tan cos square theta. And again, I use the identity sin square theta plus cos square theta is 1, it goes away. So, I get r square theta dot square, then there will be the cross terms.

So, you have from this one, you will get r dot r sin theta cos theta and theta dot with the minus 2. And here again, you get r dot r sin theta cos theta, theta dot, so all the terms are identical only the difference is in the sign, so they will cancel when you add them up. So, it is clear that this answer is correct and your kinetic energy T. Let me write it down here, the Kinetic Energy T is half m r dot square plus r square theta dot square that is good, that is correct, no mistakes and your L is T in this case so the same thing is the Lagrangian.

Now, I have to take the derivative with respect to both the coordinates. So, my q is the generalized coordinate r and theta as I said several times already. So, let us look at the equation corresponding to r. Then you have d over dt del L over del r dot minus del L over r equals 0, the particle is free, which means d over dt del L over del r dot, so you get half m and 2 r dots, so half and 2 cancels so you get m r dot; m r dot. del L over del r, the only second term has r, the first term does not have.

So, the partial derivative of this first term with respect to r will be 0 and this one will contribute and you will get minus m r theta dot square equals 0,, which implies m r double dot minus m r theta dot square is equal to 0 that is good. Let us look at the q equal to theta equation. Now, del L over del theta is 0 because kinetic energy does not involve theta. So that term is gone, I am only left with d over dt del L over del theta dot.

So, your del over dt and del L over del theta dot will come only from here and it will be m r square. It does not do anything to r square, it is independent coordinate theta dot square will give you 2 theta dot which has the 2 has cancelled the half, so you have theta dot that is good. Perfect, no mistakes. So, now I can write this down as m, I take the derivative of r square total time derivative, I am just doing the chain rule. So, you get 2 r r dot theta dot, next I differentiate the theta dot term so I get m r square theta double dot equal to 0.

Let me write it first the theta double dot terms and then the theta dot terms, so I get theta double dot m r square plus 2 m r r dot theta dot equals 0. Let us check, m r square theta double dot 2 m r r dot theta dot equals to 0, perfect everything is correct. So, your equations of motion are this one and this one. There is a theta dot here, there is a dot, one dot only this one. If you recall this is a familiar result.

So if you take acceleration which is a time derivative of velocity and write it in polar coordinates for the radial component, I mean the component of acceleration along r hat, the unit vector along the radial direction will be what you have on the left hand side here. And for the component, if you look at the component corresponding to theta hat, the the tangential direction, you will get this piece, this term here on which is on the left hand side here.

So these are our familiar results and which we have derived using Euler Lagrangian equations. And I wanted to say, so just note that this is your linear acceleration along the radial direction, this is your centripetal term. This is your linear acceleration along the theta hat. Tangential direction that is your Coriolis term. And because there are no forces, your right hand sides are 0.

Now, we will assume that there are forces also present that look at the same problem, a particle a single particle in two dimensions, when their force is present, and let us see what it will look like. So as far as the left hand side goes in the Euler Lagrange equations, what you have got here is already correct. All you have to do is look at the generalized forces on the right hand side q alpha, so that is what we are going to now look at.

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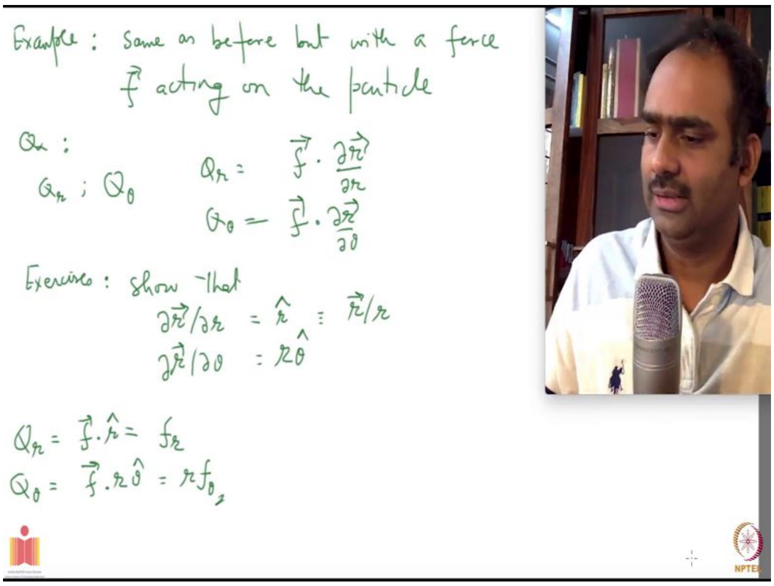

So your example did I call it example one? Example, same as before, but with forces present that is about with a force, with a force affecting on the particle. And this I am assuming that this force is also working only in the 2D in the plane. So, let us see. So, I have to look at Q alpha, where in our present case, it becomes Q r and Q theta. And what are Q r and Q theta? So, Q r is f there is only one particle, so, there is no label on the particle dot del r over del r. Remember del r over del Q so that Q is r here. There is no submission involved because only one particle. And similarly your q theta will be F dot del r over del theta.

So, you do these two simple exercises now. Show that if you take the derivative del r over this r, I will tell you the result, but even before that, you look at on the left hand side, this is a dimensionless quantity, you have the dimensions of length in the numerator here and dimensions of length in the denominator. So whatever you get on the right hand side has to be dimensionless. So, clearly it cannot involve r, then it has to be a vector quantity, because the left hand side is a vector in the numerator.

So, whatever you get on the right hand side has to be a dimensionless vector. And it is no surprising it is not surprising that your, it is not obvious from what I am saying, you have your unit vector r hat here. And let us look at another part for you to show us is del r over del theta, this guy, the left hand side should have the dimensions of r and should be a vector again, because theta is dimensionless.

So, the dimensions have to match and you get r theta hat. That is correct. And this R hat is if you are not sure, so r hat is, what is r hat, r hat is vector r over r so that is why it is dimensionless because the dimension cancel. If you do those two exercises, and plug it in here, you get Q of r, f dot r hat which is just the component of force in the radial direction, so I put a subscript r, and your Q of theta will be f dot r theta hat, which is r, this r and f of theta, so I am taking the component of the force along the tangential direction, and I call it f theta.

And as you can see, this is nothing but the torque, and this is the radial component of the force. So, your equations of motion in this case would become you will just substitute Q r and Q theta in here, so I will put here Q r, which is f r and here I will put Q theta which is r times f theta, which is the torque, those will be your equations of motion when you have forces present and you are using not the Cartesian coordinates, but polar coordinates, that is good, I think I can take one more simple example to get some practice let us do that one as well.

So, till now, I have not taken any constraints in the system. Both both the examples which I took in this video and the previous video, they will work without any constraints, but now imagine you have a single particle which is moving along a circle, it is constrained to move along a circle. So, imagine some wire which is put in that form and think of a bead which is sliding along it without any frictional forces, and then there are also no forces.

Now, this is a problem with the constraint so the particle is though it is in let us say x y plane, but it is It has only one degree of freedom in just angle theta with respect to x axis for example. So let us write down the equation of motion for this. And because we have taken care of the constraints already, we do not have to worry about it. They are built in the equations of motion, they were already done away with the (())(20:20) were already removed. So let us look at this example. Where is it?

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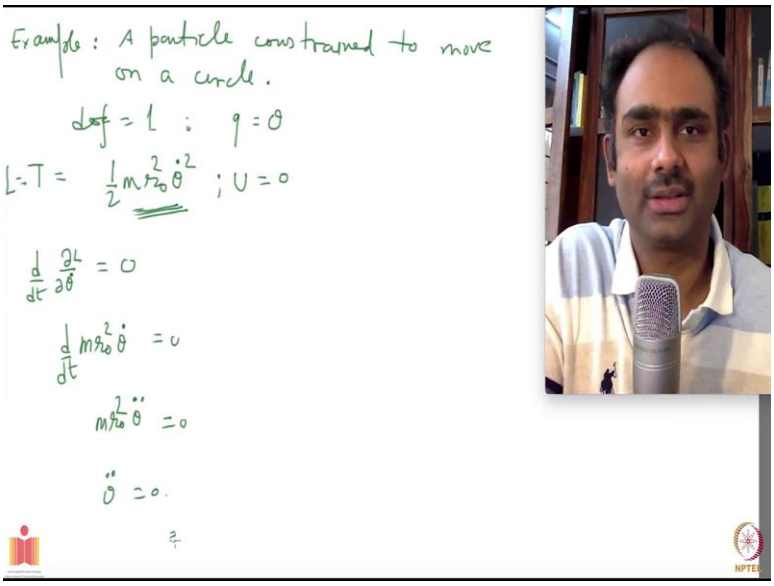

Example: A particle constrained to move on a circle. So, clearly there is a force of constraint and there are no other forces on it. So, this problem has degree of freedom 1 and for the generalized coordinate Q, I will use theta and note that theta is dimensionless which is fine, no problem. We already obtained the kinetic energy in polar coordinates for a particle in the previous example and I will utilize that one. So, remember that one had where is it let us go there.

Here this this place, now r is fixed so r dot will be 0. And let us say the value of r is some r nought, so I will get kinetic energy to be half m r nought square theta dot square. Half m r not square theta dot square that is the kinetic energy and as we said no forces, so potential is 0, which means the same thing is the Lagrangian for us in this case. So, let us write down the equations of motion. It is only one there is no r involved in this so only theta. So, del L over del theta is 0, there is no theta in the Lagrangian and there is only theta dot. So, I have del L over del theta dot d over dt equals 0 theta dot, which means let us look del L over del theta dot m r nought square theta dot and d over dt, which implies m r nought square theta double dot is 0 or theta double dot is 0.

Meaning there will be no angular acceleration, so the particle will keep moving in the circle at the same angular velocity. That is what this equation is saying which is also consistent with what you expect. And these are some simple examples of how to use Euler Lagrange equations. There are several examples which you can find in different books and I will encourage you to have a look at them and make sure that you are comfortable using Euler Lagrange equations. This is where we will stop today and continue next time. See you then, bye.