Introduction to Classical Mechanics
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Lecture 07
Euler Lagrange Equations - Examples

Last time we wrote down the Euler Lagrange Equation and we will take up from there today. So, the equation that I wrote last time was the following.

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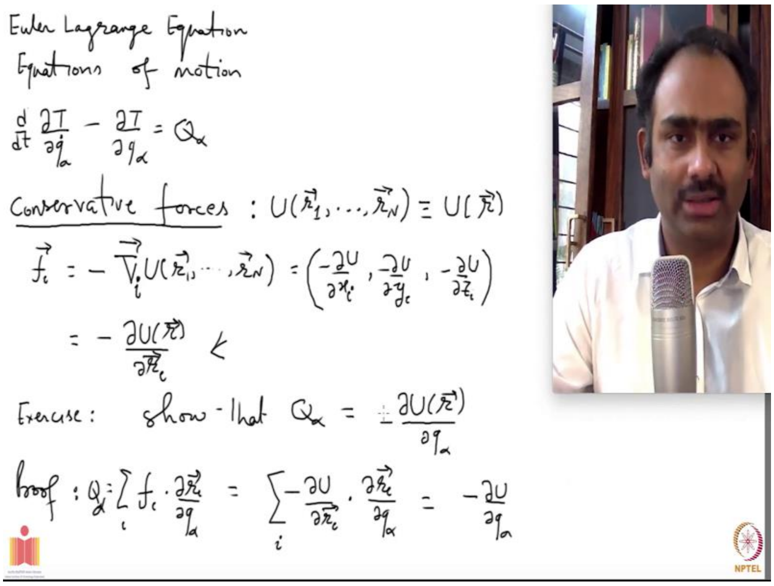

So, what we wrote was d over dt the total time derivative of del T over del q alpha dot minus del T over del q alpha equals q alpha, equals q alpha. Where T is the kinetic energy, the total kinetic energy of the system. So, you sum up over the kinetic energy of each individual particle of the system and that is what gives you the total kinetic energy and because here you have derivatives involved with respect to q alpha and q alpha dot, you clearly have to express your T not in terms of Cartesian coordinates person, but in terms of generalized coordinates, that is one thing.

Second is you have q alpha on the right, which we wrote down explicitly last time as fi dot del ri over del q. del ri over del q alpha that is correct. So, let us start from here, let us assume now that the forces that are present in the which are acting on your system, they are all conservative forces, meaning you can express those forces as the gradient of some scalar potential.

So, you can write down a potential energy which will be a function of all the coordinates r 1 r 2 and so forth r n and you will be able to express the f is as gradients of them. So, that is what we want to write down now. So, let us say we have conservative forces operating on the system.

So, as I said I can, if this is the case, then I will be able to assign a potential energy to the system which will be function of all these coordinates and then if I want to know the force on the ith particle then I should take the gradient of the U or rather minus. Minus gradient of U r1 rn. Because I am going to write it down several times I think I would prefer to define U of r.

So by this r, I mean the entire set r1, r2 so and so forth rn. So, the force on the ith particle is the gradient of U. So, here you have to take the derivatives and derivatives with respect to which coordinates? The coordinates of the ith particle, that is what it is and explicitly what I mean to say is this.

So, you take minus del over del xi of U have U minus del over del yi have U minus del by del zi of U that is what is the meaning of this grid in which I have written down here and we can also write this as our, in our notation which we use as a replacement for gradients sometimes is I write it down as minus delta U, again I write r instead of all the rs over delta ri, this is just a notation. So, that is what your forces would be if your system was conservative.

Now a simple exercise, show that q alpha your generalized force q alpha is, I mean instead of taking the, see here you have gradients or derivatives involved with respect to ri and I am saying that q alpha can be expressed as del U r over del alpha, that is a nice result, minus. So, you can get a generalized force, just by taking the derivative with respect to the respective coordinate alpha, q alpha. So, let us see the proof is quite simple. So, you have fi dot del ri over del q alpha and if you sum over all the particles, that is what your q alpha is, q of alpha.

Now, because your forces are conservative I can write down this fi by substituting this here. So, I put minus del U r or even better I just do away with the arguments of U, you understand what that is. Minus del U over del ri dot del ri over del q alpha and you have sum over all of I and what is that? There is just the derivative, partial derivative of U with respect to q alpha, because if you do a partial derivative of U with respect to q alpha that is what you are going to get.

You will run the chain rule and this is will because your U is expressed as functions of r. So, you will take the part partial derivative with respect to r and dot with del ri over del q alpha. So, that is what is minus delta U over del q alpha and this is what we are going to utilize in the next step.

So, what I will do is now is, take this result, which I have just shown and plug it in here. Let us see what will happen if I do that. So, I will have a derivative of U with respect to q alpha. What I will do is, I will take to the left hand side and combine with this, this also has a derivative with respect to q alpha and this will become T minus U, and you will be left with a 0 on the right hand side. So, that is what you will get, so let us see.

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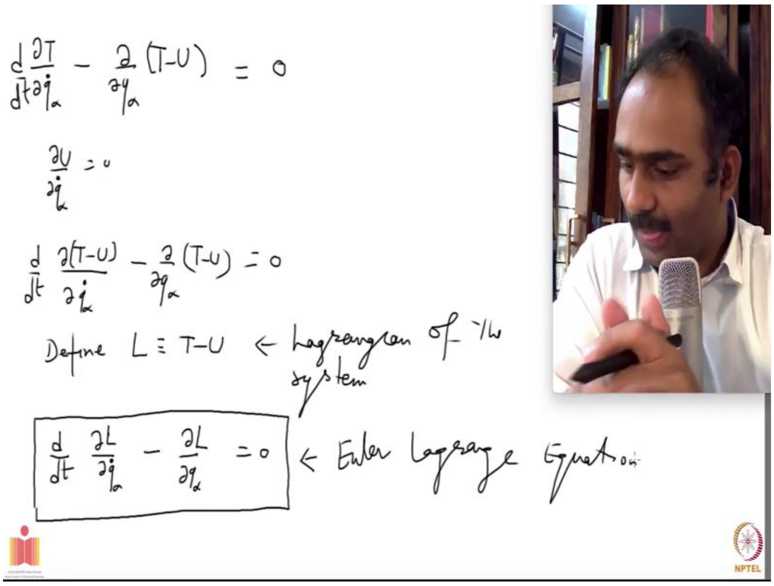

So, I get delta T over delta q alpha dot and you had d over dt minus delta over delta q alpha. As I said I will bring T, I mean T was already here, I will bring the U from the right hand side to the left hand side and that is what you get. Now I define, before I define let me say something else.

Here U is a function of rs and your rs are going to be functions of the qs, the generalized coordinates. Which means, when you go from r to q your U becomes a function of U of q alphas, qs, generalized co-ordinates, it does not become a function of generalized velocities. In this transformation velocities are not (())(08:49) from going from r to qs. Which means derivative of U with respect to the generalized velocities is 0, because this transformation from Cartesian to generalize will not bring in the generalized velocities.

So, what I am saying is delta U over delta q alpha dot is 0, which means I can take the 0 and put in here, because I can always add a 0 to the equation and nothing will change, so I can write the above equation as d over dt delta of T minus U over del q alpha dot. So, because this piece delta U over delta q alpha dot is 0, I can do this, minus I bring this here again, delta over delta q alpha T minus u, this is equal to 0.

Let us define this quantity T minus U by L. This is called the Lagrangian of the system, of the system because you have summed over all the particles, so L is the difference of kinetic energy of the entire system minus the potential energy of the entire system and that quantity L is for the system. So, this is called the Lagrangian of the system.

So, I can write my Euler-Lagrange equations as del L over del q alpha dot minus del over del q alpha L equals 0. Let us see if I can bring in some colour. Colour puck. I do not know where it is, never mind, no colour for us. Usually people call this as Euler-Lagrange equation, but we will call the previous one also which we were, which had a q alpha on the right hand side also as the Euler Lagrange equation.

That is a nice result, this gives the equations of motion of your system using generalized coordinates. So, here you have to express T and U both in terms of generalized coordinates because the derivatives involved are with respect to the generalized coordinates and velocities.

Let us see if I want to say something else here. What if not all the forces on, which are acting on the particles are conservative. Let us say you have a situation where some of the forces are conservative and some others are not. So, each particle is acted upon by a force, a part of which is conservative, so you can assign a potential energy and a part of which is not conservative. If that is the case, it is clear what we can do, we go back to this equation here at that was our starting point and I split the q alpha into two parts.

One involving only the forces which are conservative and I bring it to the left as I did just now, so that I define a Lagrangian of the system involving the potential energy T minus U and whatever forces are not conservative I leave them behind on the right hand side. So, your equation in that case, would become the following.

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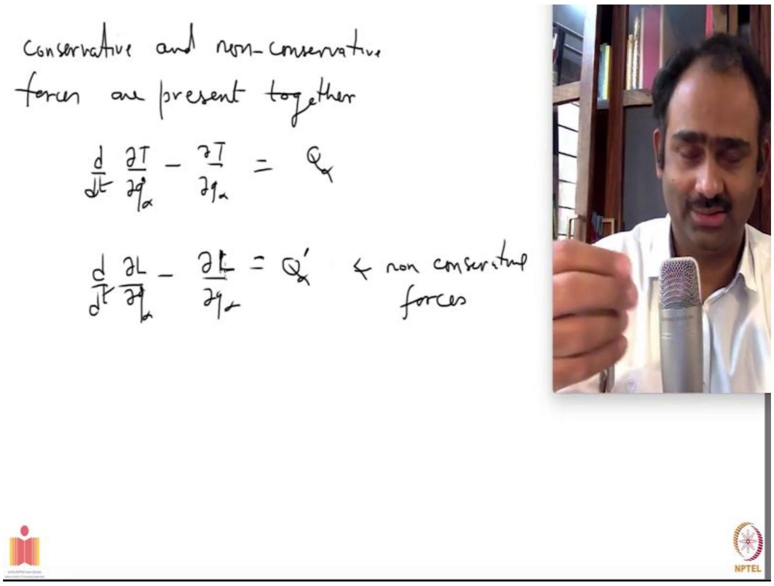

So, if let us say, conservative and non-conservative forces, are present together are present together or simultaneously t o g e t h e r that is correct. In that case as I said, I will take dt, d over dt del T over del q alpha dot minus del T over delta q alpha equals q alpha and as I said I will split the forces into two parts and take all the ones which can be obtained by the gradient of a potential to the left and I will get d over dt, delta L over delta q alpha dot minus delta T over delta q alpha equals q alpha.

This q alpha and that q alpha are not same, they are not same. I am not creating a new notation. So, here by or maybe I should maybe, let us say prime. So, here Q primes are due to non-conservative forces. You can think of various situations in which you will have such a possibility.

This is L, good. So, now that we have equations of motion, let us get some very simple examples to get an understanding of how to work with these equations and the simplest thing you can imagine is always a free particle. So, imagine a free particle that is moving freely in space, no forces are acting and let us see, we get our expected results that is the Newton's laws which will say that the acceleration in any of the directions x, y or z will be 0, so that is what the example we are going to take now. Let us go to a new page.

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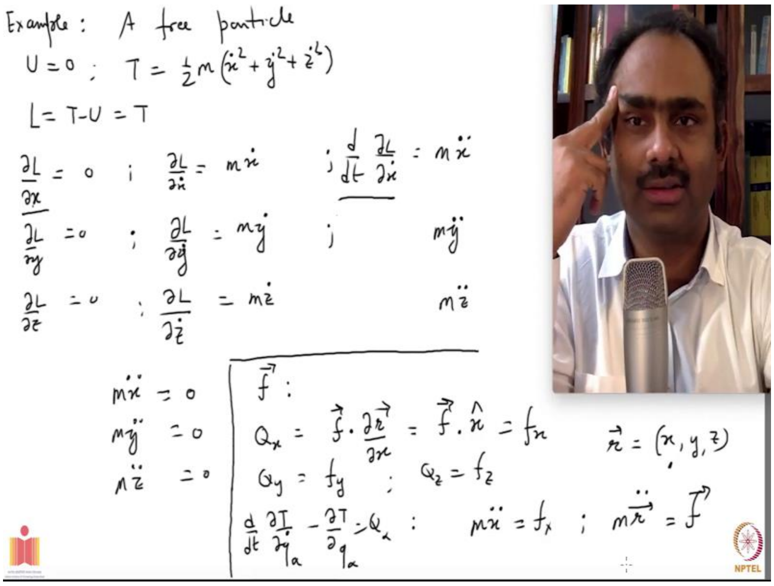

Example of free particle. If the particle is free, then clearly the potential energy involved is 0. There is only one particle and there are no forces on it, it is free. What is the kinetic energy? Kinetic energy is half m v square, which is x dot square plus y dot square plus z dot square, that is your Kinetic energy and to plug in my equations of motion, I should calculate my Lagrangian first, which is T minus U, which is T in this case. So, I take the derivative with respect to the generalized coordinates which I will take the Cartesian coordinates, I will take the generalized coordinates to be the Cartesian coordinates in this example.

Later, we can look at an example where we take some, let us say spherical coordinates. So, what is del L over del x? 0. What is del L over del y? That is also 0 because there is no x or y here, there are only the velocities and del L over del z that is also 0. What is del L over del x dot? del L over del x dot is m x dot. So, if you take derivative of this, you get m x dot. The derivative of this with respect to x is 0, derivative of this is 0, these are partial derivatives. Del L over del y dot, again m y dot and del L over del z dot, this is a dot here is m z dot.

Now, I take the total time derivative of del L over del x dot all the three. So, d over dt delta L over delta x dot, so I have to take the total time derivative of this which will be m x double dot, x double dot is the acceleration in the direction x and similarly you will get m y double dot when you take derivative with respect to the corresponding one here for y and similarly m z double dot.

So, what is your Euler Lagrange equation that is d over dt del L over del x dot minus del L over del x equals 0. So this minus this, so this piece minus that piece is equal to 0. So, I get m x double dot minus 0 because this is 0 is equal to 0, m y double dot equals 0, m z double dot equals 0.

So we get our Newton’s equations for free particle, which is what we should expect. Now imagine if the particle was not free. Let us say it was acted upon by a force. Let me do it here itself, let us say there is a force F. So, let me not invoke potential right now, let me just work with the generalized forces. So, my generalized force Q x would be f i dot, remember what that would be delta r over delta x, remember f i dot delta r over delta q i.

Now q i is x here and I should sum over all the particles, but there is only one particle, so this is what it is and what is this? F dot delta r over delta x so what is your r? Vector r is the position vector. So, there is component of it is x. So, delta x over x is the unit vector in x direction.

So it is let us say x hat, which is f x. So, delta so let us say R is x, y, z. So, if I take the derivative of r with respect to x, this is what you are going to get, you are going to get 1, 0, 0. So that is what I am doing, I am dotting the F with 1, 0, 0 and I get f x. Similarly, your Q y would be f y and your Q z would be f of z.

Now, you see the equations of motion in this case would be, remember your equation of motion when you write in terms of generalized forces does not have Lagrangian on the left but has T on the left. The derivative of T with respect to the Q and Q dot. So, your d over dt of delta T over delta q dot minus d over dt, dt over delta q is q and I should write alpha remember this.

So, from here you get easily m x double dot, this will give you m x double dot, the left hand side which we (())(21:32) and the generalized force is the x component or if you combine all the three equations, write them in the vector form you get, which is your expected Newton's second law.

That is, that is good. Maybe I will take up two more examples in the next video and we will we will stop here. Let me see if I want to say anything else. Yeah, maybe I can make that comment here itself. Let us go to this place.

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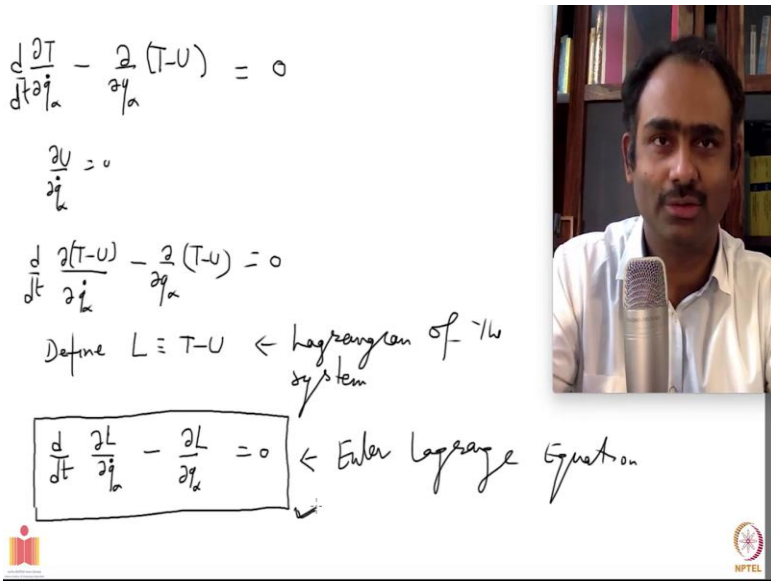

This is Euler Lagrange equation. Now see this is very nice, imagine you have a set of particles. Let us say your system is isolated from everything else and there are a number of particles in there and they are interacting with each other with whatever interacting forces they have, whatever constraints are there.

Now to know what or how this system is going to evolve with time, you do not need to track, you do not need to know all the forces that are acting on each particle. That is what you would have used if you are using Newton's equations, but here you do not have to. What here you have to do is, you have to know a quantity L the Lagrangian, which knows about the entire system, L is for the system. So, once you know the L, you can derive the equations of motion and you can describe the trajectory with time. So, L is a very central very important part of the subject and you are always after finding the L.

In fact later you will see, that when you start doing let us say looking at the fundamental forces and try to make quantum theory of nature, you start trying to guess what the Lagrangian could be, from whatever data is available to you, you start guessing it, you start to make certain assumptions that the Lagrangian has to be this form and there are you impose certain symmetries that you know. You remember we have already talked about some symmetries in the very first video, first or second, I do not remember and that put constraints on what could be the nature of forces.

So, similar constraints you can apply on the Lagrangian itself and that is what the strategy is when people are working in, for example, high energy physics. Anyhow, that is all for now and we will take a few more examples in the next video. See you, bye.