Introduction to Classical Mechanics
Professor Anurag Tripathi
Assistant Professor
Indian Institute of Technology Hyderabad
Lecture 66
Invariance of Poisson Brackets

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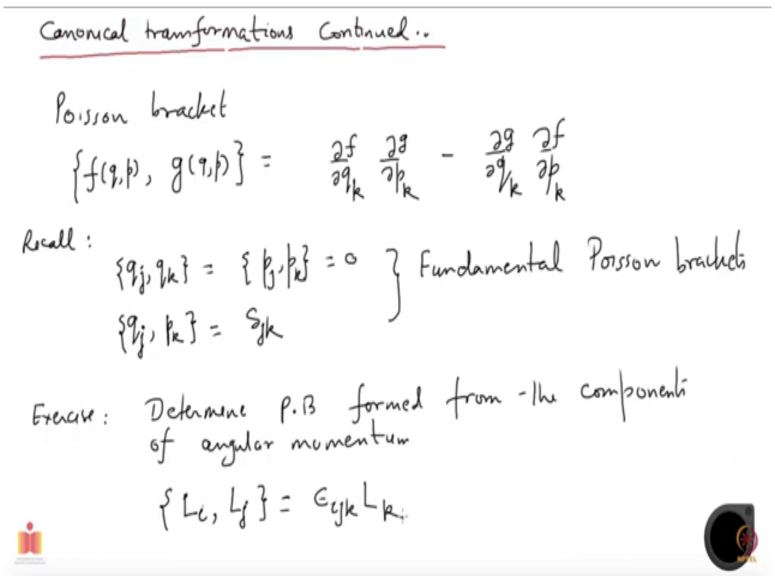

We will continue our discussion on canonical transformations in this video and I want to return to Poisson Brackets that we were discussing sometime back. So, let me remind you what Poisson bracket is, so if you are given two functions f and g which are functions of the canonical coordinates q and p, I am not assuming that it is only one dimensional system, you could have n dimensional. So, your q will run from q1, q2 so and so forth to qn.

Let us say, there are two functions f and g then I can construct this quantity which is called Poisson bracket, which is the following. There could, as I said these are, I am talking general of n dimensional systems, so your k runs from 1 to n and there is a summation over k implied here and you have f g, g f that is all Poisson bracket is and it is an anti-symmetric quantity and also recall that if you take q j and q k, two of the coordinates or two of the conjugate momenta which is let us say, pj pk then their Poisson brackets are 0.

However if you take one of them to be a coordinate and the other one to be a conjugate momentum, then this is delta jk, delta jk. So, unless j is equal to k this will be 0 and when j is equal to k then qj, pj will be 1 and these are called fundamental poison brackets. I will give you a small exercise to do, so you determine the poison brackets formed from the components of angular momentum, so determine poison brackets formed from the components of angular momentum. So you can take Cartesian coordinates and corresponding conjugate momenta, construct the angle momentum L is r cross p and then calculate this.

So, let us say by L i denote the angular momentum so Li Lj are this components then show that if you construct the poison bracket you are going to get this, which means for example for i you have 1 j you have 2 then it will be on the left hand side L 1, 2 and on the right hand side it will become epsilon 1, 2, 3 L3 which epsilon 1 2 3 will be 1, so it will become L3 on the right hand side. So, please show this, I will state a theorem I will not prove it, you can prove it yourself or have a look in one of the standard books and here is the theorem.

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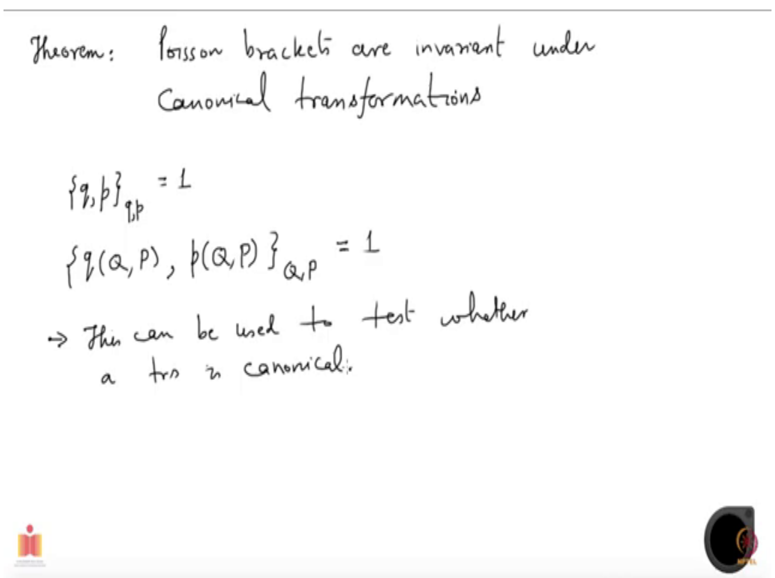

It states that poison brackets are invariant under canonical transformations, so no matter what set of coordinates you are using, a poison bracket value does not change, so here is the theorem; are invariant under canonical transformations. So, one consequence of this is that if you look at the fundamental poison bracket here and here all the partial derivatives in constructing this are with respect to the coordinates small q and small p, let me emphasize that thing by putting a subscript here.

So this is 1, so here I am looking at some one dimensional system, it has only one coordinate q and this we know that it is 1, now what holds true is that if you were to express the function small q or the coordinates small q in terms of capital q and capital p which are obtained by a canonical transformation similarly for small p.

And if were to evaluate this, this time taking the partial derivatives with respect to capital q and capital p you are going to get the same thing because poison brackets are invariant. Now this can be used to test whether a given transformation is canonical, even without knowing what the generative function is so you do not have to start to first figure out what the generative function is, just by doing this test you can figure out whether the transformation or whether the new sets of coordinates that are given to you are canonical, whether they are going to satisfy canonical equations of motion.

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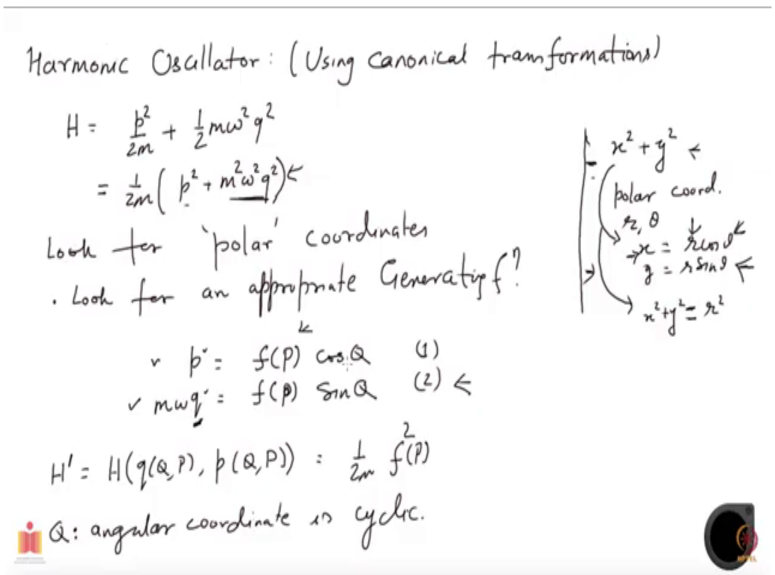

I can give you a simple example and that example will be based on what we did here for the harmonic oscillator. So, let us do the test on the capital Q and capital P here which we had. So here somewhere here, anyway let me write it down here.

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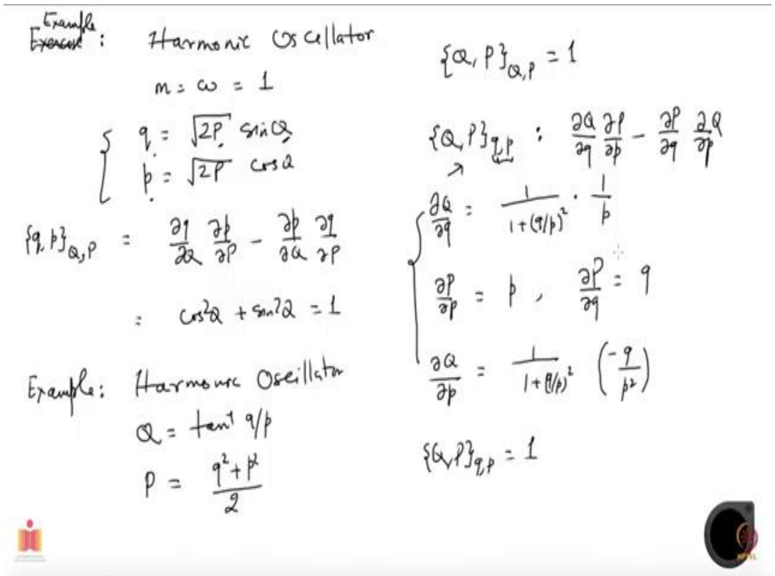

So you can go back and check whether what I am saying is correct, so case of harmonic oscillator, so first of all let me put m is equal to omega is equal to, is equal to 1, I have put everything to be 1. If I do so then my q is 2p sin of q and small p is 2p in a square root cos of q and let us find out the poison bracket q, p using this.

The partial derivatives now will be with respect to capital Q and capital P. So, what is this quantity? This is delta q over delta capital of Q, delta p over delta capital P minus, if you evaluate this you will find that you get cos square q plus sin square q and which you know is identically 1.

So, we have seen that our theorem is indeed working and this also proves that the coordinates which we found were canonical, if we assume that the theorem is valid. Let me give another example again with the harmonic oscillator, this was example. Now this time what I will do is, I will do the opposite thing harmonic oscillator. Now let us invert, let us invert these relations and instead write in terms of capital P, instead express capital P and capital Q in terms of the old coordinates q and p.

Remember this were as I was telling you these were, these are angular and radial coordinates and these are more like your Cartesian x and y, so if you invert you will get the following, you will get q tan inverse q over p and capital P equal to q square plus p square over 2. So, just divide this and this equation you will get a tan q and then you take the inverse, so that you get this relation and you sum the, take the square of this, take the square of this and sum them up you will get this one.

Now again, let us check whether the following holds true. So, if I take start with capital Q and capital P which if I were to evaluate this I know will be 1, but now let us evaluate this using small q and small p. So, now you take the partial derivatives, let me write it down it is fairly trivial. So, this is what we want to calculate, check that I am just writing down, so that we can immediately see that my claim is correct that is what you get by taking the partial derivative.

Similarly for this one you will get p, this one you will get q and this one will give you p square, that is what you are going to get and when you plug it in here the expression here would be delta q over delta q delta p over delta p minus del over del q of capital P del over del p of capital Q. When you plug these derivatives in here you will see that you will immediately get this equal to 1 that is another way of looking at the transformations to be canonical.

So it makes sense to drop these because no matter which set of coordinates you use, you are always going to get the same answer, so I would stop this lecture here. In fact the entire course will end here, nevertheless our lot is still left to be learn in this subject and we cannot cover much more in these 30 hours, so I will strongly recommend that you look at those standard textbooks which I had mentioned in the beginning.

For example, the book by Hand and Finch Goldstein and the book by Arnold (())(14:45), the book by Pars. So, you can find a lot material in the subject and I would also encourage you to solve exercises and problems that are given in this book, so that you develop a stronger understanding of the subject and with this I would like to wish all of you the best for your exams and I hope that you all have enjoyed learning this subject, I definitely enjoyed giving these lectures. So, thank you for attending this course and I wish you all the best for your exams and for you academic future, thank you.