Introduction to Classical Mechanics
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Lecture 65
Harmonic Oscillator (Canonical transformation)

In this video we will take a simple example of harmonic oscillator and how to use canonical transformations in that.

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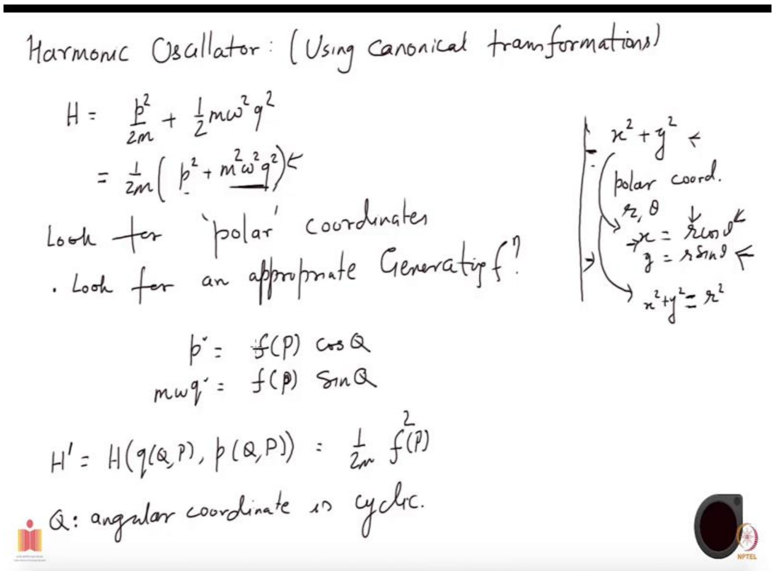

So, let us look at using canonical transformations. So, let me first remove this. So, the Hamiltonian you are well familiar with now, which is h is p square over 2m and the potential term is half m omega square q square. So, this is a sum of 2 squares this is a square, that is a square.

I will just pull out a factor of one over two m, not necessary but I will do it. p square plus m square omega square q square . Now, if you look at this you have the q's and p's as squares and this immediately suggests that we should be using some better coordinates for this better set of conjugate coordinates for this instead of what we are using here.

So, let me tell you why I am saying that. Imagine you were looking at x square plus y square, you are doing some problem and this was something in two dimensions with x and y as the coordinates and this is all your functions that you have in your problem appear to be having composed of these kinds of arguments. So, they always have x square plus y square and such things.

Then you realize that this is not the most convenient choice of coordinates, you could instead go to the polar coordinates, meaning you could go to r and theta. The advantage of doing r and theta is your x square plus y square where for example x is r cos theta and y is r sin theta immediately turns x square plus y square into r square. You see your theta is gone now. So, you have only the radial coordinate.

So, this is what we want to do here in our case also because this is the sum of these two squares and my Hamiltonian has such a nice form that I should be looking for a better coordinate system in the phase space. So, I would like to look for a polar coordinate something which looks like what we have here. So, that is the idea. So, how should we do that?

Well I should be looking for some canonical transformation and because I do not want my equations of motion to get changed, so I am looking for canonical transformations and which means I should be looking for an appropriate generating function. So, I should be looking for an appropriate generating function. So, let us see what we want first. What we want is that my small p just like my x here should become r times cos theta I want here to have some function f of p which is the equivalent of r here times cos of Q.

So, instead of having a p cos theta here as we had in this case here I look for a coordinate q which appears in the argument of cos and similarly my q, small q should be f of capital p you can write here as m omega q. So, this this piece is m omega q whole square. So, I am writing m omega q is sine f (p) times sine of q.

So, this is in the same way the things are written on this side in the margin, that is good. That much I can do. Now, this will be nice because in the new coordinates if this if I could show that this transformation is canonical then that would imply that the new hamiltonian H prime would be same as H with the small q being expressed in capital Q and capital P and small p being expressed in capital Q and capital P.

So, that will be the new Hamiltonian and let us look at what it would be? Well by construction what I am doing is I am choosing polar coordinates so that when I look at the sum of squares it will be left only with the p coordinate capital P and the angular coordinate q would disappear. So, what you will get here is 1 by 2 m and you substitute p and q from here to there and you get f p whole square, that is what you will get.

So, note that the Q which I will call as angular coordinate because it is very analogous to what you have here is cyclic, is cyclic because it does not appear in the Hamiltonian, which is nice, Because if that is cyclic then the conjugate momentum which we have to find out f of the conjugate amount I have to find out that conjugate momentum would be conserved, it will be a constant. So, that is why I want to do this. Now, let us go to figuring out what will be the this transformation function.

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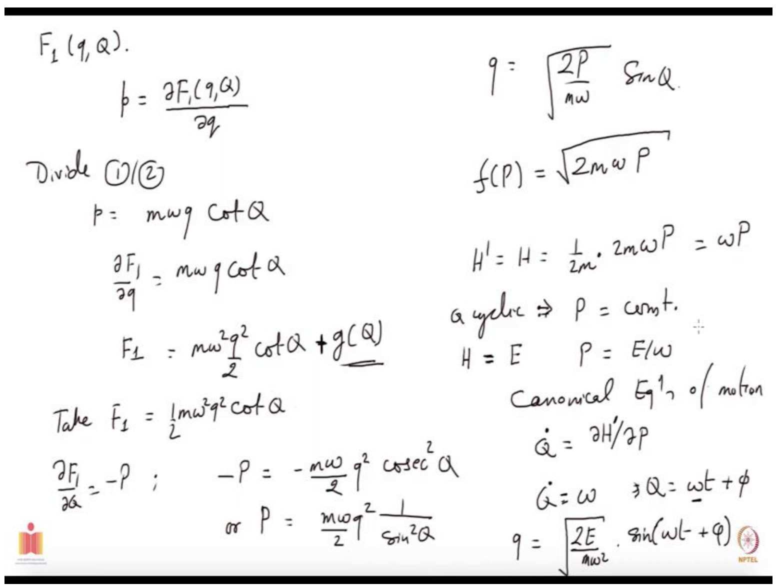

So, let us look at or try to find out what would be the sorry I think I said transformation function I mean generating function what will be the appropriate generating function. So, I am looking at a generating, I am trying to search for a generative function which will do the above transformation and this relation I already know. You remember those two relations which we have already tabulated her, the first one.

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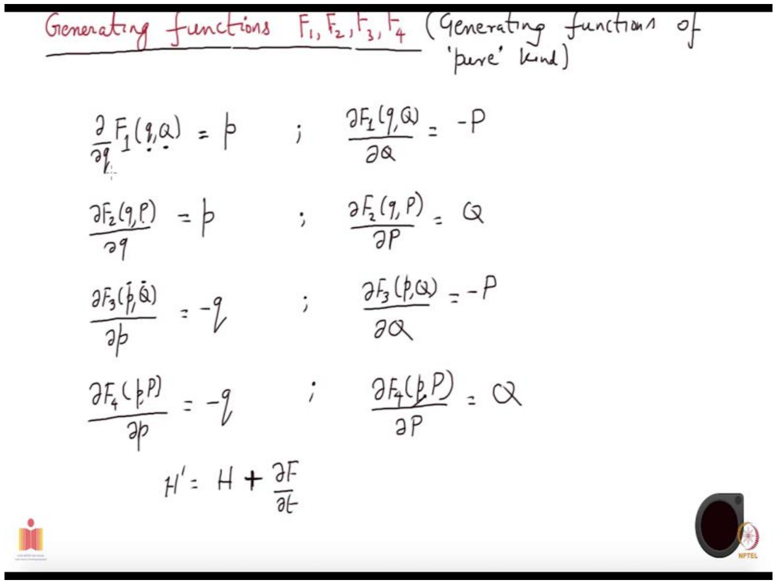

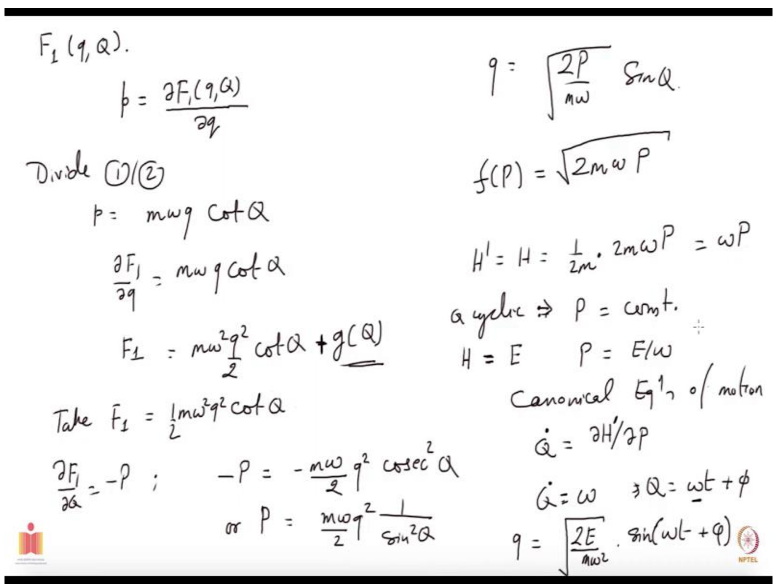

So, derivative with respect to small q gives small p and that is what I am writing here. Small p is what you get from taking a derivative with respect to small q. So, right hand side is a function of small q and capital Q. So, I should express small p also in those functions, in those arguments. So, if I divide these two equations, let me call this 1 and this one 2, if I divide equation 1 by 2 then, I get let me write divide 1 by 2 then I get small p is mwq cot of Q, it is nice. From here I can try to find out what the capital F1 is what the generating function is okay.

Now, the requirement here is that if I take a derivative of F1 I should get mwq cot Q. So, delta F1 over delta q should be mw or omega sorry q cot of Q and that is easy if I integrate both sides with respect to q and remember there is the partial derivative so I can write it as okay. So, taking the derivative with respect to q will generate a 2 q which will cancel the half and that will give this piece.

And of course, if you add to this any term which is only a function of capital Q taking a partial derivative will kill this piece. So, we can drop this I mean this is the general form but in what I am going to do next is I am going to just drop this piece. So, I will take F1 to be simply this, half cot of Q that will be the generating function.

Now, I have used one equation here. I still have the second relation available to me I will use that one now. So, this one I am going to use. Let see, if I do that then I have to take a derivative of F1 with capital Q and that will give minus p. Let me write it down here. For ease this was the equation there.

So, if I do this I get minus P is equal to derivative of F1 with q and that is minus m omega by 2 q square cosecant square of Q. All capital p is m omega over 2 q square one over sine square of q. I can invert it and if i do so i get the coordinate small q the whole coordinate to be 2 p over m omega sine of q.

So, now we see immediately that our f of p is equal to 2 m omega capital P. So, we have figured out now what this piece is. That is good. Now what? So, I know the transformation and let me ask what is the hamiltonian now. My H prime is H which is just 1 over 2 m, let me go back where it is here, 1 over 2 m and you substitute in here p square plus q square term you substitute this thing.

These equations 1 and 2 and remember what we have found for f of p and if you do that you will get not surprisingly 2m omega capital P which is just omega times p. So, that is your new hamiltonian which only has a radial coordinate p and the angular coordinate q has disappeared and that is the reason why we did this transformation.

Now, as I said q is cyclic which means the conjugate momentum p has to be a constant and also let me right because Q is cycling also the system is conservative system which means that the hamiltonian is also a conserved quantity and let us call this constant, the value of energy to be E then I see that P is just E over omega. So, that is nice.

Now, let us look at the canonical equations of motion. So, I want to find out how for example the coordinate q evolves with time. Let me remove this. Now, if you recall the canonical equations of motion then you will remember that Q dot is delta H over delta P and that we can easily calculate our Q dot is.

If I take the partial derivative of I should put a H prime to be more consistent, let me let me put it, H prime is anyway H. So, a Q dot is derivative of this and which is just omega. So, that is the equation of motion and it is of course the first order equation of motion and I can immediately solve it and I get q equals omega t plus some constant phi.

And if you insert this back into the definition or the relation here. You will find the small q and the small q will be this, So I insert this and I get 2E where I have used the value of p to be whatever it is times sine of omega t plus phi. So that's the solution which we are very familiar with.

But we have also derived it now using the yeah using the canonical transformations. Of course, this is not the simplest way of deriving it but it does show you the use of canonical transformations, in doing, in looking at certain problems. So, this is for this video and we will meet in the next video.