Introduction to Classical Mechanics
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Lecture 64
Examples of Generating Functions

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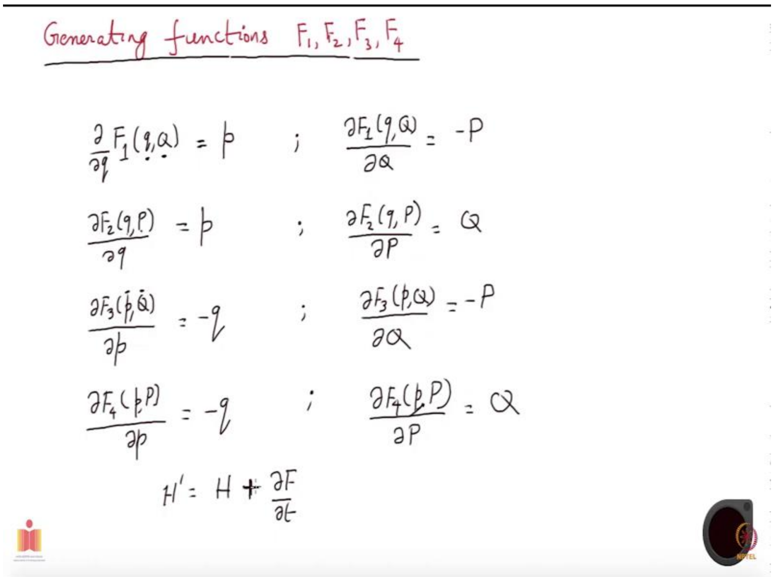

Let us continue, let us continue our discussion on Generating functions. Last time, we had derived certain relations, which will take us from 1 coordinate to another set of coordinates. And let me summarize what we found last time. So, if we take generative function F1, which was a function of the old coordinate q and the new coordinate capital Q, then if I take the partial derivative, I get small p, the small qs and small ps and capital Q and capital P, these are set of coordinate and momenta. So, small p for example is p1 p2 so and so forth to pn.

So, this was the relation for F1, and the second relation for F1 is del F1 small q capital Q or del capital Q is minus new momenta. That is good. Now, delta F2, this is also something we saw last time. Then, delta F2 q comma, so it depends on old coordinate q and new momenta P. And if you take the derivative with the new momentum, its Q. So, that is fine.

Now, note here that this q is the canonical conjugate variable to the p here. Similarly, this Q is conjugate to this capital P. This one is conjugate to this. So, its always, always like this. Let me remove those. And the same will be true for the remaining 2 relations also. Delta F3, and this will be our table which we can refer to later, minus small q.

And delta F3, this is small p and capital Q over delta capital Q is minus p. And also note that every time, you have 1 old variable and 1 new variable, old new, old new. And same will be true for the fourth one. p P p is minus q. And we have delta F4 is equal to Q. And here also, this is conjugate to that, and this is conjugate to that.

Let me remove this. So, that is fine. And also, I wrote that in all the cases, as we observed last time, that the new Hamiltonian H prime will be just the old Hamiltonian written down in new variables plus delta F over delta t, where f could be F1 F2 F3 or F4. And if there is no explicit time dependence in F, then of course, H prime is just H. Lets take a few simple examples. So, simple.

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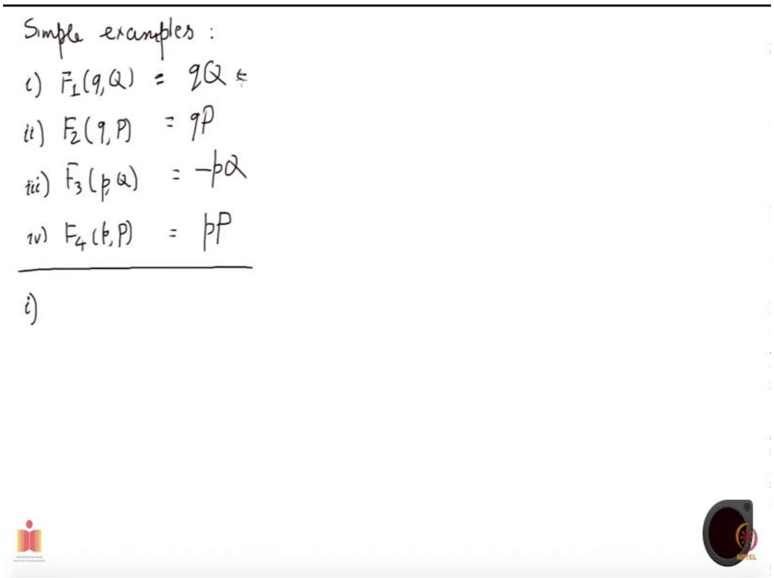

So, first one here, I want to know if I take this as the generating function, where F1 is q times capital Q, how the new and old coordinates are related. Second, if I take this generating function, F2 to be this, again how they are related to the old coordinates. If I take F3 to be this. And the fourth one, F4 to be this. So, these are fairly simple, which we can very quickly check. So, lets see, I will, I will show here explicitly. So, let us look at first case.

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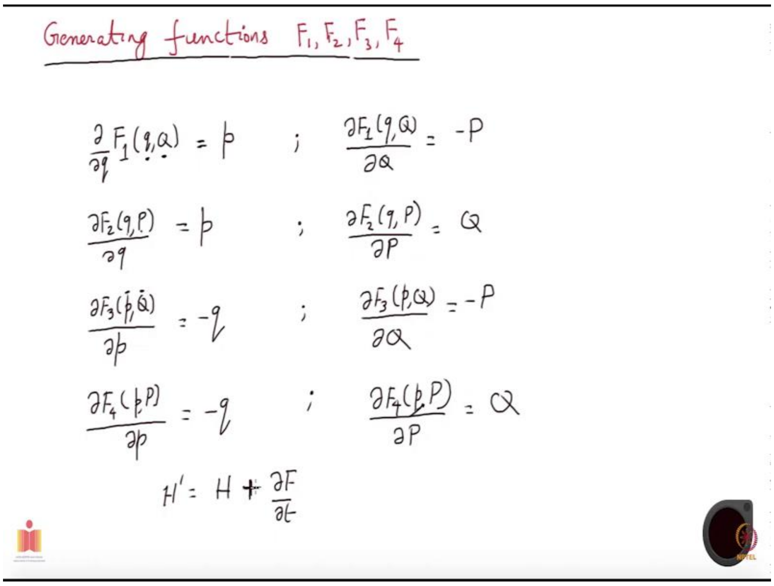

If you do this, if you take the partial derivatives as required by this one, which is here, meaning this.

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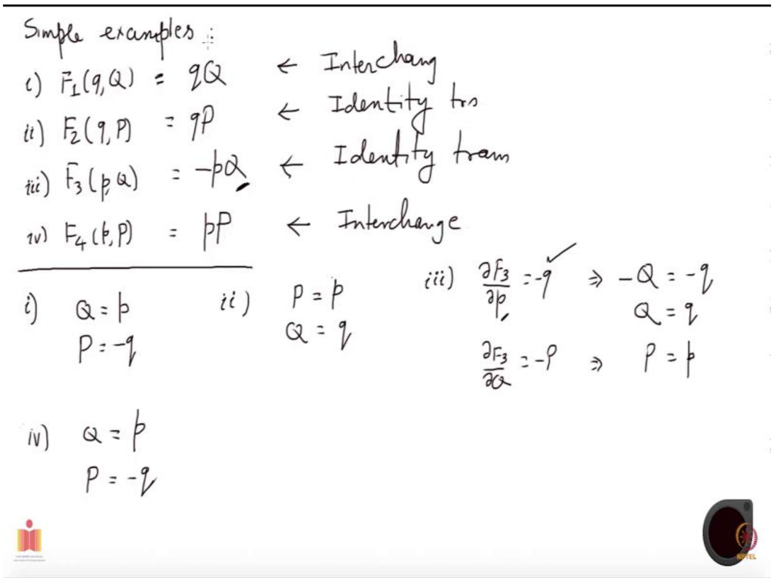

Then, lets go, then what you will get is the following. Doing that calculation will give you that capital Q, the new coordinate will be p and the new momenta would be minus q. Meaning, it will just interchange Q and P and I believe that we saw that earlier also. So, let me write it down here. This just interchanges.

It will just produce an interchange of the coordinates. If you look at the second one, lets put it here, you will find that the new momentum is same as the old momentum and the new coordinate is same as the old coordinate, which means this is an Identity Transformation. This is not changing anything, so this is identity transformation. Let us look at this one. Maybe here. So, third one, if you do, maybe I will just write it down how to get this (())(08:29).

So, you have delta F3 over delta p is equal to minus q, this is what you have on the previous slide. Here. Now, let us see, if I take F3 to be minus p capital Q and I am taking derivative with respect to p, you get a minus q. So, this will give you minus q by taking the derivative of this thing, which is equal to this minus q here, which implies that capital Q is same as old q. And doing the same thing for the other equation, this one, you can again see this here. If you take the derivative of F3, which is p times capital Q and the derivative with respect to capital Q, you are left with this minus p, minus small p.

And minus small p is equal to minus capital P, which means capital P is same as small p. The new coordinate, new momentum is same as the momentum, which means it is again an identity transformation. And if you look at F4, then you will find that the new coordinate is p, the old momentum and the new momentum is minus the old momentum, all coordinate, which means this is again an interchange. So, this is what it is. And these 4 generating functions F1 F2 F3 and F4, these are called Generating Functions of the Pure Kind.

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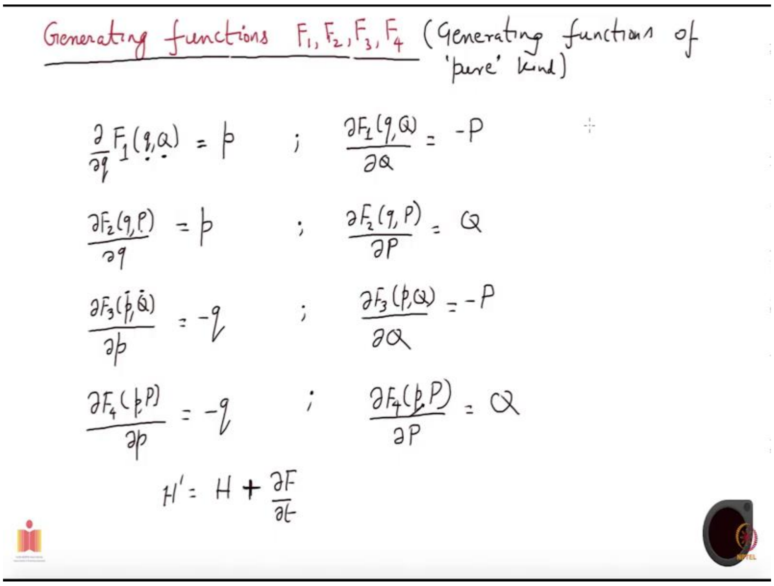

So, let me write it down here. That is fine, colour is okay. Generating functions of pure kind. And there may be transformations, which will be canonical, but are not necessarily generated by generating functions of pure kind. So, they may be of mixed form. They could be a mixed form of generating functions of type F1 and F2 and so forth. Let me give you an example. And such generating functions are called Generating Functions of Mixed Kind.

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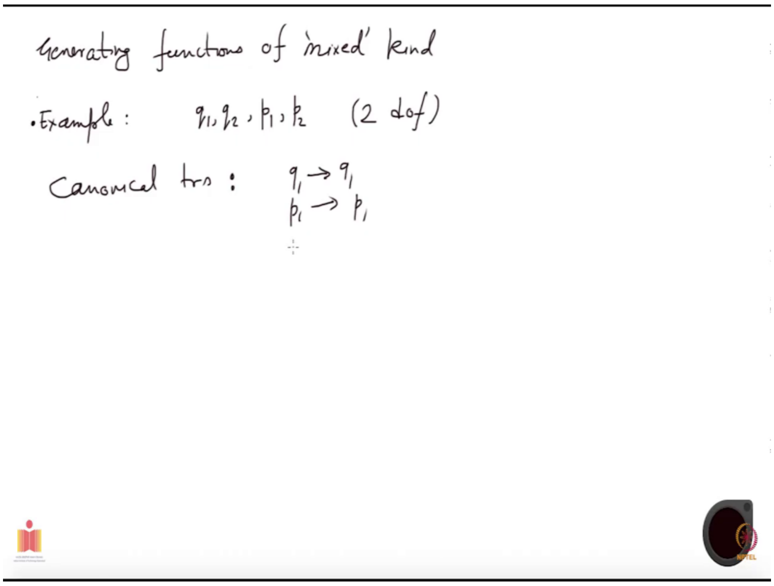

Functions of mixed kind. So, for example, let us take a system of 2 degrees of freedom, meaning, it is described by q1 and q2, these are the 2 degrees of freedom. And the other, let me denote the conjugate momenta by p1 and p2. So, there is the system of 2 degrees of freedom. Now, you may want to think of a canonical transformation, which will keep q1 and p1 unchanged.

So, think of a canonical transformation, which will not touch the coordinate q and p, so it does not touch those, so where q1 remains, remains unchanged. So, the new coordinate, coordinate is still q1, the new moment is still p1, but for the q2 and p2, there is some non-trivial transformation. And let us say, we want to interchange q2 and p2, so, let me write it more nicely, so I am looking at a canonical transformation, which will take my q1 to q1, p1 to p1 and my.

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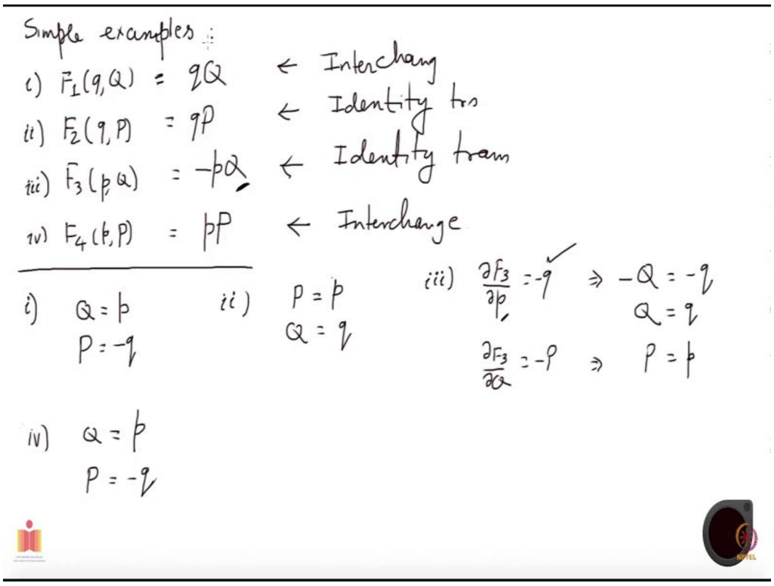

Let me check, capital Q will be p and capital P will be minus q, so coordinate remains, because momenta.

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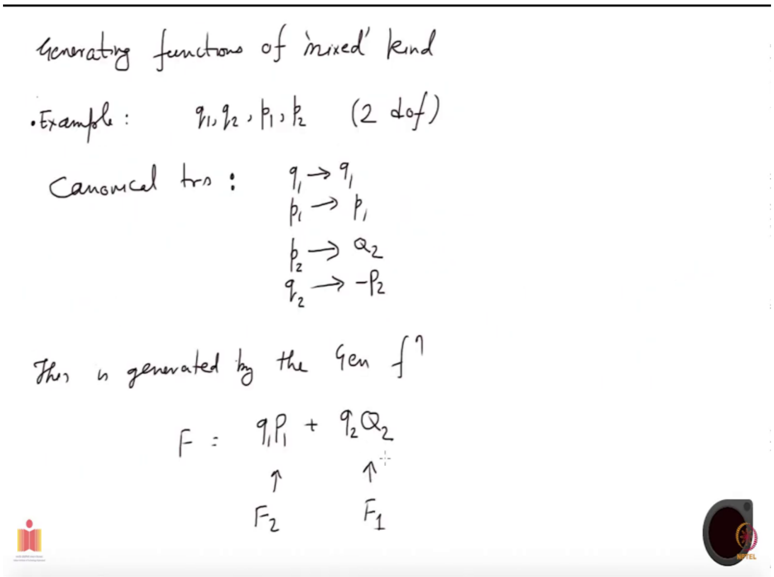

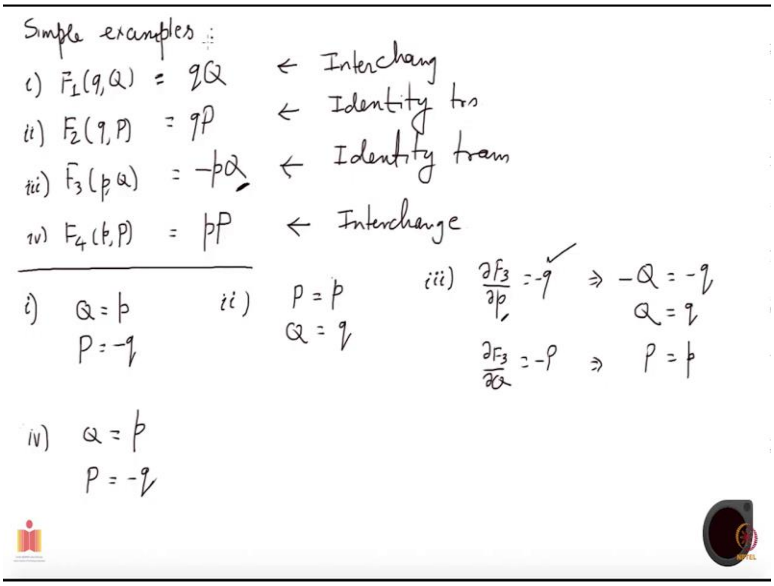

And just a second. So, p becomes capital Q, p becomes, lets call it capital Q2. I am looking here for right now, this is the one interchange. I am just taking care of the sign. And I see that the small q is what is called minus capital P. So, minus capital P2, and I have forgotten putting a 2 here. So, suppose this is the transformation, which I want to do, and then you can check that this is generated by the, by the following, by the generating function F is equal to q1P1 plus q2 capital Q2. So, this one is of kind F2 and this one is of kind F1.

And lets see whether this will work. So here, you see F2 is qP, this is of kind F2. And if you choose F2 to be qP, it does an identity transformation. So, let us see, what we are doing. F2, we have chosen to be q1P1, which means the transformation that will happen will be identity. So, the coordinates q1 and momenta P1 will still remain unchanged. And here, let us go back, sorry, the second one, I am choosing to be F1. And F1 is q times capital, what happened? Let us see. I can fix this. Anyway, this is, this should be fine.

So, where have I, yeah, I was here. So, this one, F, which F1, so I am looking at F1 here. F1, if you chose to be q times capital Q, then it does not interchange. So clearly, this will generate an interchange in the coordinate q2, and its momentum P2. So, this is another kind of generating function that we can use.