Introduction to Classical Mechanics
Professor. Anurag Tripathi
Indian Institute of Technology, Hyderabad
Lecture No. 63
Hamiltonian Mechanics: Generating functions of the 4 kinds

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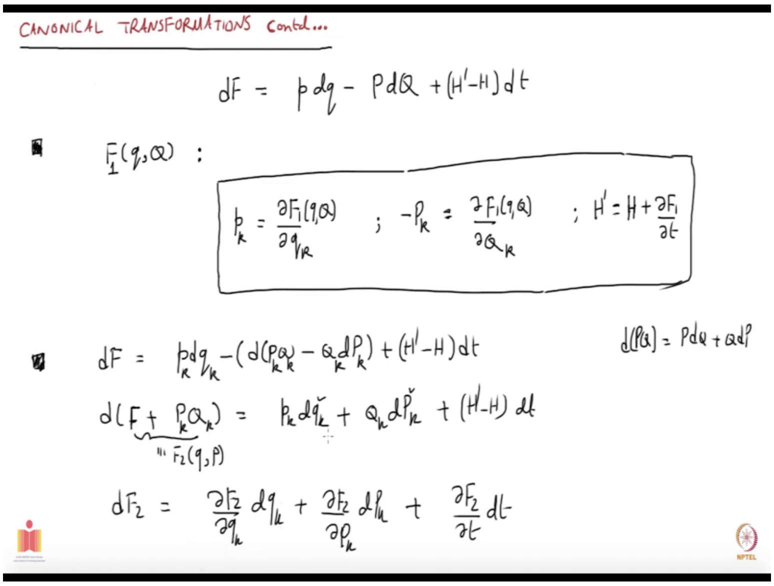

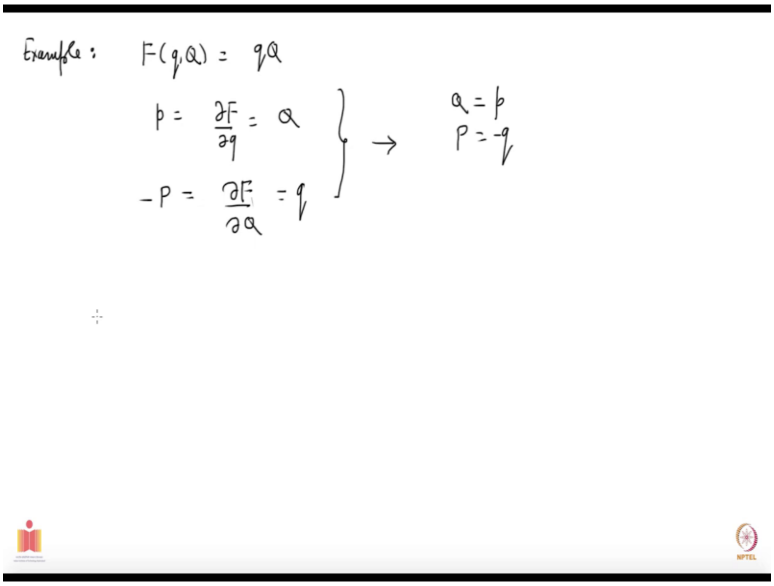

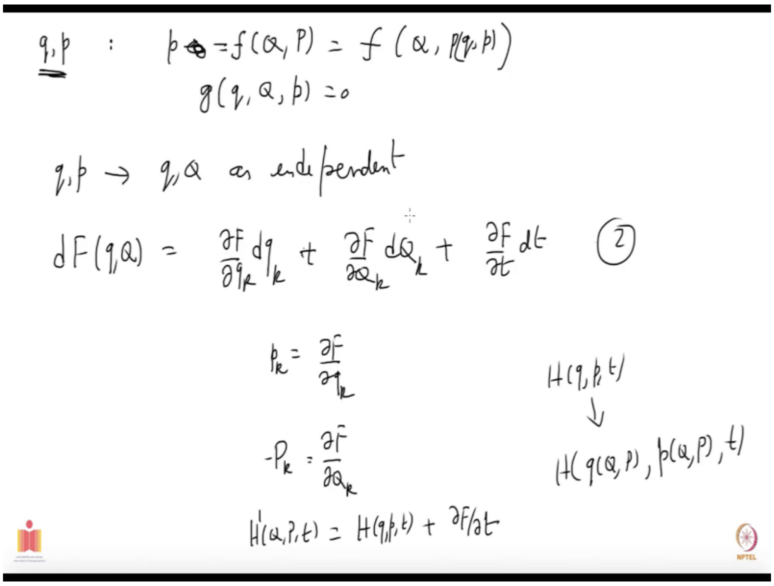

In the last video, we wrote down the Total Differential of the Generating Function, which is going to generate transformations of the coordinates, canonical coordinates, and which was of this form. Let me write here. dF pdq minus capital P dQ and then we had, and then we had H prime minus H dt. I am going to use this one again and again. And I also argued that we can treat F as a function of small q and capital Q.

So, F, we said that we will treat as a function of independent variables q and capital Q. If I do so, then I generate a set of transformations and I will label F with a subscript 1 in this case, if when I make the choice that I have a q, old q and the new Q here. And we saw that here, let us go back, that these are the relations which, see the small p and capital P are given by this. So, both are derivative with respect to, good, so let us write this down here.

If this choice is made, then we saw that we have delta F1 over delta, del F1 over del q. Let me emphasize this. And this gave us p. And then the next was capital P, was delta F1 over delta capital Q, and here we had small q and capital Q and I believe there was a minus sign. Let us go back. Let us correct. And the third was that, my H prime is H plus delta F1 over delta t. So, you can put the indices, I want to put it in.

And we also saw an example of the generating function, and we saw that this was just interchange interchanging what we call coordinates and what we call momenta. Now, let us proceed further. So, let us start again with this expression of the generating function. And now, I want to, let us see, I will take this term PdQ and instead of having a differential dQ, I will turn it into a differential of dP. So, all you need to use is that a differential of P times Q will be PdQ plus QdP.

If you do so, then our expression can be written as this, pdq and you have minus, lets put a minus here, our PdQ is d of pQ minus QdP. So, now I will have a dP instead of a dQ as was the case before and then of course, this term. Now, this total derivative term, total differential term, I can bring to the left and write d of F minus PQ, let me not forget that I have to put these indices and the summation is implied, Pk Qk, and this will be equal to pk dqk, this term is here, this term I have taken, there is a sign issue. Let us see.

So, my PdQ, it looks fine, what has gone wrong? Let us proceed, let us see. I do not see a mistake right now. Minus PdQ minus dPQ is PdQ plus QdP. And I have a PDQ which will make it, this looks correct. So, maybe I made a mistake while writing my note. Anyway, so, dF and there will be a plus here because this guy goes to that side is equal to pk dqk, that is correct. And then we have a plus Qk dPk plus H prime minus H dt.

Now, you see, I have turned this total differential and on the right hand side, you see the differential involved are the Qk and the capital Pk. So, I define a new function F of 2 to be this quantity in the, in the brackets. So, with this, I can write it as F2. And F2 will be a function of q and P, so q, small q and capital P. So, I get dF2 is equal to this quantity. And because it is s a function of q and P, I can write it as, I mean generally, generically, I can write that the dqk dqk plus delta F2 over delta capital Pk d Pk plus if there are any time derivatives, it will have this part, this term also. Now, if you compare these 2, you will get the delta F2 over delta qk as pk.

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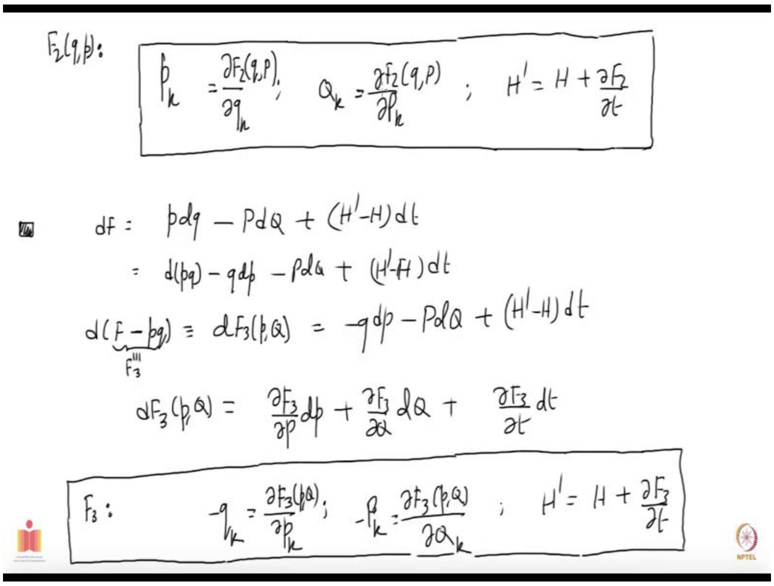

So, we get delta F2 over delta small qk as, let us check, no, small pk. What happened? And delta F2 or delta, del F2 over del Pk (())(09:31) delta Pk is equal to capital Qk. And I should emphasize here, that these are functions of small q and capital P. And as before, H prime is H plus delta F2 over delta t. Delta, del F2 over del Pk, that is qk, all looks good to me. So, this is for F2. And our F2 is a function of q and P. So, this is the second kind of generating function, which involves the function, generating function to be a function of old q and new P, and the first kind involved generative function of old q and new Q.

These are 2 different, they are defunctions. We can again go back and, go back to the original equation here and make a different choice of variables for F. So, let me do that now. This will be a the third. Now, what I do is, I write my, let me write dF again, Pdq minus capital PdQ plus H prime minus H dT. And this time what I will do is, I will look at the first term here, first differential and write it as d of pq. So, this gives d pdq plus qdp. So, I should subtract from this qdp and then you have the remaining terms.

And I will now, as before, take this total (deri) differential to the left hand side and combine with F and I get d of F minus pq, small p small q. Let us see here. And you can supply the indices here, the case, but I will omit right now. So, this you get to be, this we will define as d of F3 of what will, what will be the variables? Variables will be the small p and capital q, because this is gone to the left. So, let me write, small p and capital Q. So, your F3 is this.

And what is dF3, dF3 is minus qdp and minus PdQ plus H prime minus H dT. And as before, I can write my total differential of dF3, which is a function of small p and capital Q as del F3 over del p dp plus del F3 over del capital Q dq plus del F3 over del t dt. And if I compare the coefficients of dp and dQ and dt, I conclude that delta F3 over delta p is minus q. And delta F3, again I should emphasize that these are now functions of small p and q over delta Q is minus p, the new p. And your H prime is same as H plus a total derivative term of F3.

And if your F3 does not depend explicitly on time, then H prime is same as H, expressed in new coordinates. So, let us put the indices. And let me see if it is correct. Minus qk, that is good, minus pk delta F3 over delta capital Qk, that is also good. So, this is the case for your F3. So, this is your third kind of generating function. And now, I will do for one more case. Let me first put a box in here.

So, final fourth variety, let us go here, I will have to now change both the, let me go here, now I will take this dF and change from dq to dp and here from capital dQ to capital dP, that is what I will do.

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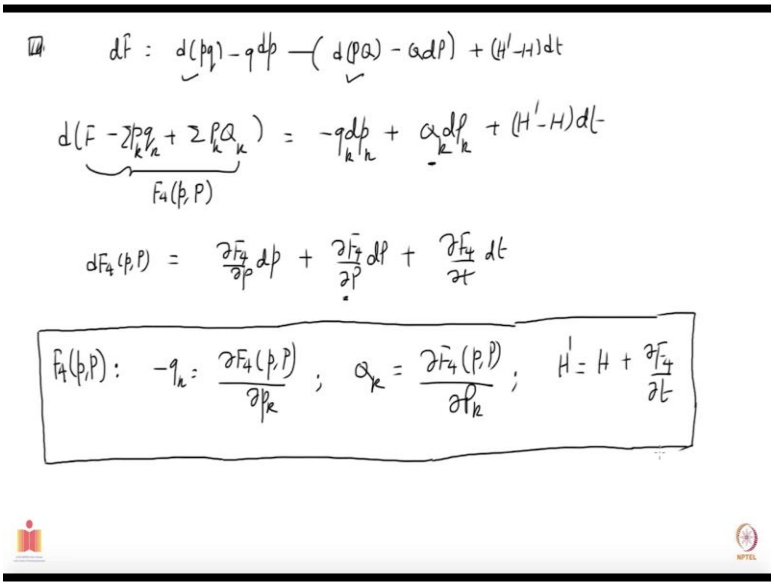

So, go here, and again my dF, if I write, if you go back pdq I had here, I can write it as d of pq minus qdp minus this term capital PdQ I can write as d of PQ minus QdP. And then you have your, these terms anyway. So, if I take this total differential and this total differential to the left and combine with F, I get d of F minus pq. So, you have a summation over all these. And then, you have a plus P, capital P, capital PQ, you have some summation here as well. And this is what minus qdp plus capital QdP, this term, and of course, our term with the differential of time.

Now, we will define this as a F4. And F4, I want to see as a function of small p and capital P. So, it is an function of independent variables, small p and capital, capital P. And of course, there is a summation over all indices here. So, as before, I write dF4 as the left del F4 over del p dp plus del F4 over del capital P dP plus del F4 over del t dt and if you compare, you get that minus qk is del F4 pk, that is one, your capital Qk here is this, so delta F4 small p capital P over delta Pk. And your H prime is again H plus partial derivative with respect to time. F4 as a function of p and P.

So, these are our 4 sets of generative functions and the equations that connect the old momenta and the, old coordinates, old canonical coordinates with new canonical coordinates.

And we will give some examples for these in the next video. But I want to end this video with some nomenclature.

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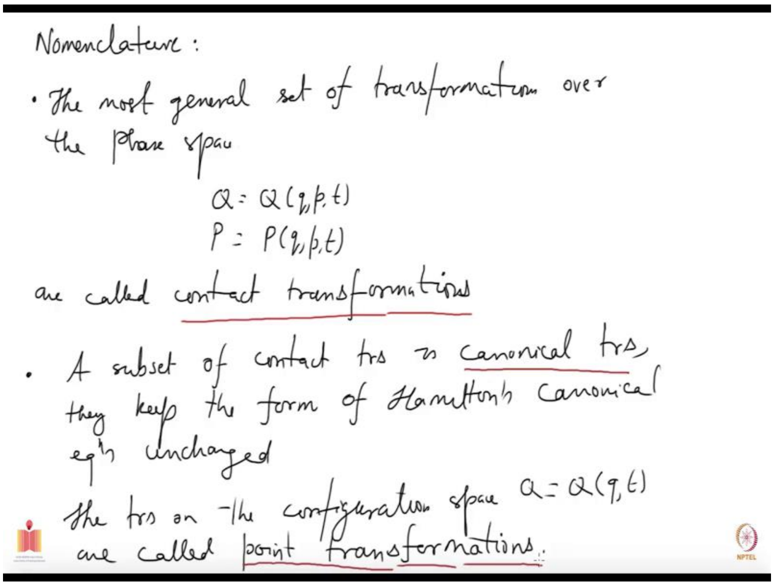

So, let me write down here. So, the most general set of transformations over the phase space, which is possible are called Contact Transformations. The most general set of transformations over the phase space, which we write as Q. So, you go from small q and capital or small p to capital Q and capital P, so you are able to write them as set functions, where q and p, they all denote a set of qs and set of ps. And these are called canonical transformation, these are called contact transformation. Some colour.

And as I mentioned before or as I said before, if you make such transformations, it is not necessary that equations of motion that you get will appear as, will have the form of the Hamilton's equations. So, a subset of these content transformations will be the ones which will preserve the form of Hamilton's equations of motion and those are called Canonical Transformations.

So, let me write down. A subset of contact transformations is canonical transformations, and they keep the form of Hamilton's canonical equations unchanged, equations unchanged. And the transformations on the configuration space, where only the coordinates are changed, they are called Point Transformations. On the configuration space, where your Q is only a function of q and possibly time, these are called point transformations.

We will, some colour again, canonical transformations and then we have point transformations. So, we will continue further our discussion on Hamiltonian Dynamics in the next video.