Introduction to Classical Mechanics
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Lecture 62
Hamiltonian Mechanics: Generating Function of Canonical Transformations

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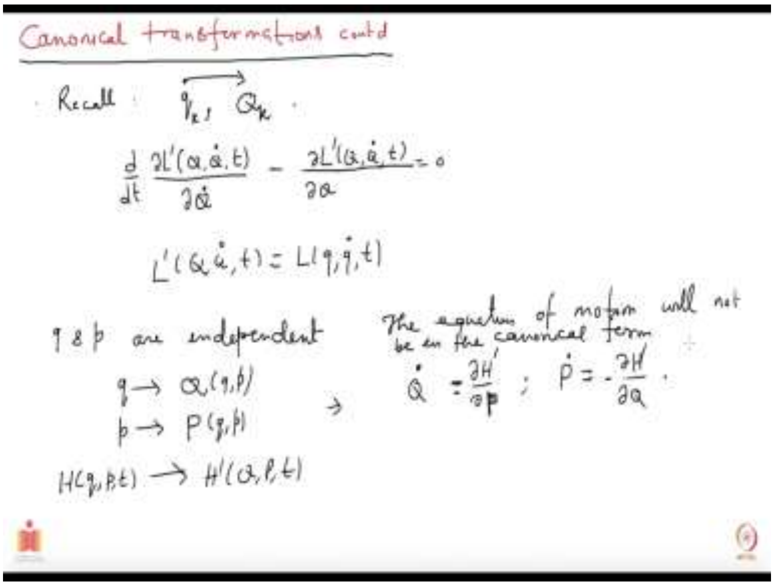

Let us continue our discussion on Dynamics in the Hamiltonian Formulation. We have already introduced phase space which is to 2n dimensional if the system is described by n generalized coordinates. And really the great advantage that we have in Hamiltonian dynamics over Lagrangian formulation is that we get to work in the phase space so because you can treat the coordinates and the conjugate momenta independently of each other.

Also, I mean, recall one thing that when we were talking generalized coordinates we never specified what those generalized coordinates were for a general system. So, we were using qs as the generalized coordinates. And these qs obeyed the Euler Lagrange equations of motion. But if instead of the qs, we made a transformation and went from small q to some other set of qs, let us say capital Q.

Let me put some index here. Even then the equations of motion would look the same. So, irrespective of whether you are using the small qs or capital Q, equations of motion in the Lagrangian formulation would look the same. And I believe I also either showed to you in the class in these videos or maybe gave an exercise, I do not recall, but this statement is true. So, even when you are looking at your system, using the coordinates capital Q, this equation of motion still holds true.

The Lagrangian is denoted by L prime here, but if you recall, L prime is same as L, the only difference is that maybe I should write the equation, it will clear. So, where our L prime Q Q dot and t is same as L q q dot t. So, you take the qs and turn them and rewrite them using capital Q, that is all. So, this is what we have seen already in in our discussion of Lagrangian dynamics.

Now when we are working in the Hamiltonian formulation, I can treat q and p as independent. And so, if I am looking at equivalent of doing what I did here, so I could, if I do not find this set as a good set, and for some reason I find another set capital Q as a good set, then I can go and I can do a transformation from small q to capital Q.

Here, I can do the same thing, of course, but not only that, that I can go from small qs to capital Qs, I can also independently take the ps to some other capital Ps. Just do a general transformation because now I am treating these qs and ps independently and transforming them independently, its not a surprise that our equations of motion, the Hamilton's equations of motion will still be satisfied.

So, what I am saying is, under this transformation from small q to capital Q and small p to capital P, my H will go to some H prime, which will be a function of capital Q, capital P and capital T. And it is not necessary that the Hamilton's equations of motion will be satisfied. Meaning, it is not guaranteed that if you do this transformation, the following will hold true.

What are those? del H prime over del p, no, capital P is equal to Q dot. You can put the indices on Qs and Ps. And p dot is equal to del H prime over del Q and there should be a minus sign here. So, this is not necessarily going to hold true. So, your equations of motion would look different than these.

But we will be interested in those sets of transformations under which the equations of motion are still going to have this form. So, we are interested in those kinds of transformations. Let me write down some here. This this this, will not necessarily be with the equations of motion. Or let me put a different way. The equations of motion will, under a general coordinate transformation will not be in the canonical form.

These are the canonical equations of motion, so they will not be in this form, they will have some different form. And as I said I am look, I am interested in those transformations for which the equations of motion will still look like the same canonical equations. Let me give you one trivial case first.

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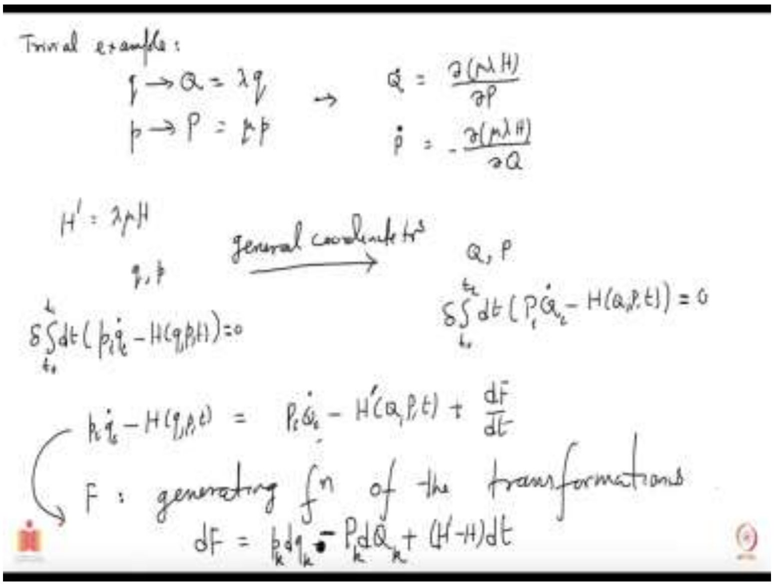

So, a trivial case, trivial example; all you do is, you take your coordinates and scale them up by some factor. So, I am scaling the q by lambda q and scaling small p by some mu of p. If I do this, and I plug these in the equations of motion, these new coordinates, you will see that you get immediately that your Q dot is delta mu lambda H I am just plugging them in here. And P dot, capital P dot is delta mu lambda H over delta Q.

And there will be a minus sign also, which means that if you do this transformation, then your H prime is lambda mu times H. And then you see that if you put it here, these are the canonical equations of motion. So, under these scalings the equations of motion hold true that is simple, you can always scale up your coordinates.

Now, we want to find a wider class of transformations that will give you the equations of motion in the Hamilton in the canonical form. And if that is the case, then we can say that I can write down, I can derive the equations of motion from the variational principle, from the Hamilton's Variational Principle.

So, what I can do is, if my system is described by coordinates q and p, and the equations of motion are canonical, and if I go and use a different set of coordinates, capital Q and capital P, here I have done a general coordinate transformation. If I do so, and if still the equations of motion are in the canonical form, then it means that I can derive equations of motion here, starting from this principle that the action which is pi qi dot, this is summation implied minus H q p t.

If I take variation of this integral, this section then the variations will be 0. And from there, I will get the equations of motion. And similarly, here, because I am already saying that my equations of motion will be in the canonical form, it means that if I start with this section and look at the variations and equate them to 0. So, these are the, so the stationary points are going to, stationary curves are going to give you the equations of motion.

Now, if this is the case, then certainly those transformations from small q, small p to capital Q, capital P those transformations that are related in the following manner will give you the equations of motion, the canonical form. So, let us say this piece, let me see, which sign I want to use. So, if, so the 2 integrands here, you see, this piece and that piece, if they are related by a total derivative, then you immediately see that the equations of motion will be in the canonical form.

Because if you have a total derivative term which depends only on the coordinates, and because we are going to fix the coordinates at the initial times and final times you remember that condition. So, let me put here t0 and t1, these are fixed. You can vary the, the curve at any other point other than this initial and final point of points. So, the total derivative will not give a contribution.

So which means that if my pi qi dot minus H q p t is Pi Qi dot minus H Q P t plus a total derivative of coordinates, if this is the case, then my requirement is satisfied that is good. Now, this F, capital F is called a generating function of the transformation. F, where is it? F, this is called generating function, you will see immediately why, of the transformation.

Now, let us do something, I will write this down as a, this equation as a total differential of f. So, you have dq over dt here. So, I am going to just write the same equation, maybe I should write here. So, if I take this equation and write it as d of F is equal to this piece pdq, i will put the indices later plus this piece is capital PdQ. Then you have H, sorry, H prime plus, something is wrong, this sign should have been minus sign here, plus H prime minus H dt that is what you can write from there and we can put all the indices.

Now, that is good. Now note that here F, I because I am writing on the right hand side, I could regard F as a function of capital P and capital Q, but I can choose any of the small q small p small, capital Q capital P. If I take any of the two, I can treat them as independent variables, so let us see why.

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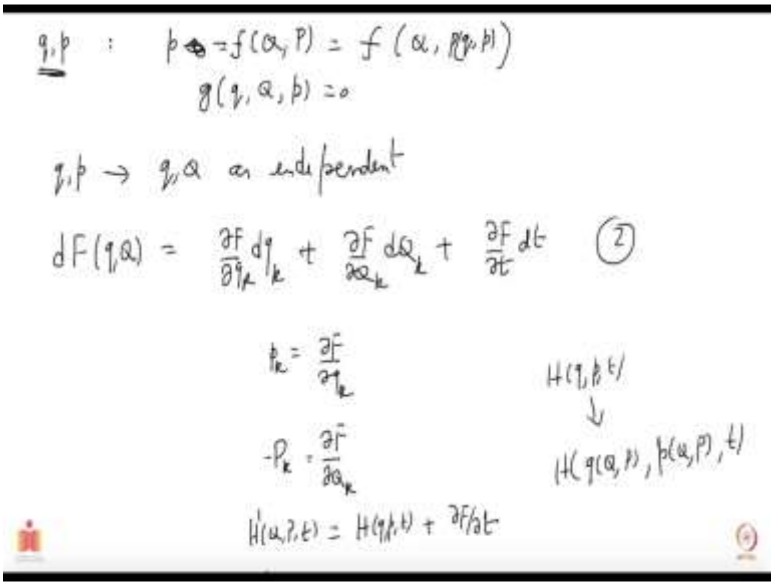

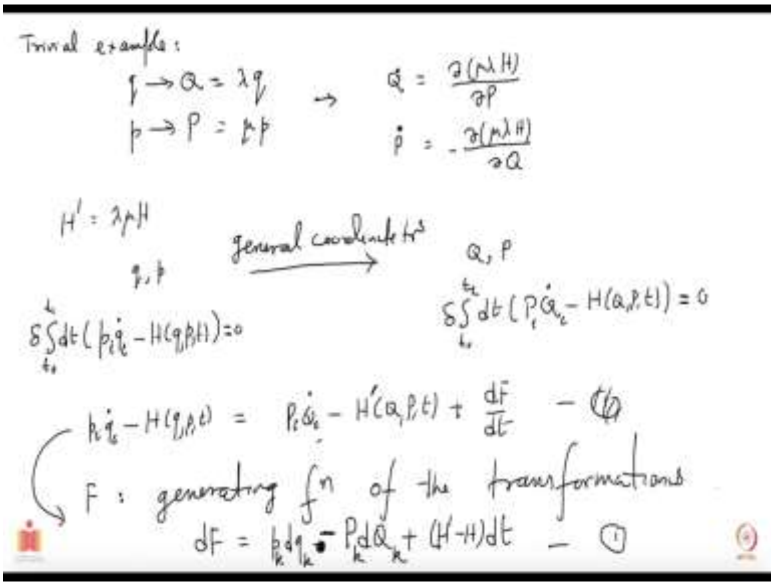

So, suppose you want to, let us say we start with q and p as independent. Now, this capital P, sorry, small p, this small p is a function which we generally denote by p again, but I will for ease of notation or understanding I will write some function f. And this function f will be a function of q and p, that I can do. Now, this capital p, you can again write as a function of small q and small p.

So, some function which I will still write as capital P this is what you can do. So, now what you have is, your p, let me instead of writing p is equal to, so your p is written as a function of capital Q, small q and small p which means this entire thing you can write as some function g of q, capital Q and small p equal to 0.

And from this, you can determine your small p as a function of small q and capital Q, which means that if you start from here, so instead of taking q and p as independent, you can take small q and capital Q as independent and you can run the same argument for any other two coordinates. So, that is what we are going to utilize now.

Here, you see, my total differential dF is written in terms of the differentials of small q, differentials of capital Q and differential of t. So, it is clear that I want to look at F as a function of small q and capital Q. So, that is something I can do. So, if I treat my capital F or the generating function as a function of small q and capital Q, then the total differential of this would be delta F over, del F over del q dq plus del F over del capital Q, d capital Q plus, if there are explicit time dependencies in F, this.

Now, I can take this equation, let me call it equation number 2. And let me call this equation number 1, sorry, this one. And I compare the 2. I can read off these coefficients. So, pk would be equal to del F over del qk. So, pk, the small p del F over del qk. And del F over del capital Qk would be minus Pk. Something wrong, yeah, it is wrong. So, delta F over delta small qk this is small. Then you have delta F over delta Qk which we saw to be minus P, minus capital P.

And then you have this piece is equal to H prime minus H, which means H prime is same as H plus a total time derivative, or sorry, plus a partial time derivative of F. So, if your generating function did not depend explicitly on time, then your new Hamiltonian and the old Hamiltonian will be the same, they will have the same values.

The only difference would be that you will express the new Hamiltonian by turning the small qs into capital Q. So, you will take your H q p t and write it as H of, that is what it means. So now, let us see why this is, this F is going to generate the transformations. Before I do so, let me make a quick remark that your capital F you have chosen to be a function of one old coordinate, which was small q and one new coordinate capital Q. So we will always take it to be having one old and one new.

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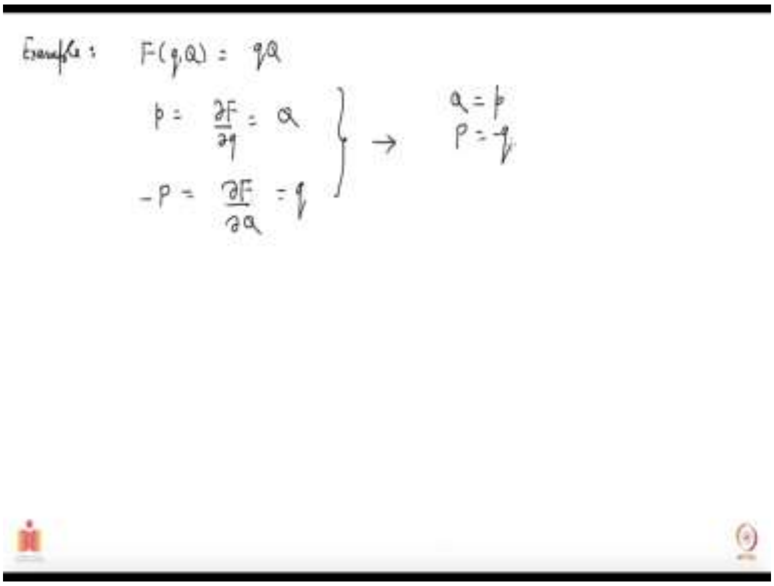

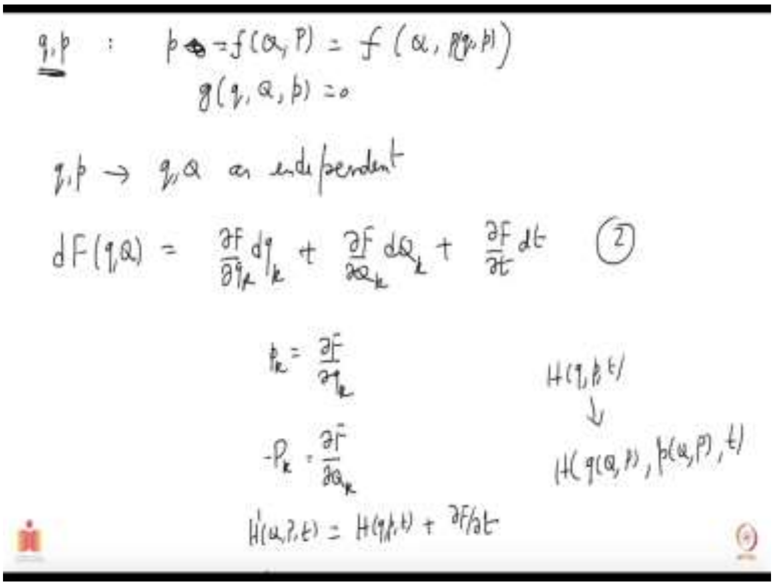

So, let us take a simple example and see how to, how to generate transformations. So, let me take the F to be q times capital Q. And let us say we are looking at a system of one degree of freedom, so there is only one Q. And whatever I write will immediately generalize to a system of n degrees of freedom. Now, what does my generating equation say? So, the small p will be the derivative of capital, derivative generating function with small q.

Small p will be the derivative of generating function with small q, which is capital Q. If I take derivative of this, I am left with only capital Q. And if I look at this equation, it says the capital P or minus capital P would be a derivative of generating function with capital Q. So, minus of the new momentum is the derivative of this with the new coordinate, new coordinate, correct. And what is that? It leaves you with a Q. Let us rewrite the same thing, a little bit neatly. So, your Q is p and your P is minus q. And this is what you have already encountered before in, I think, previous lecture let us see.

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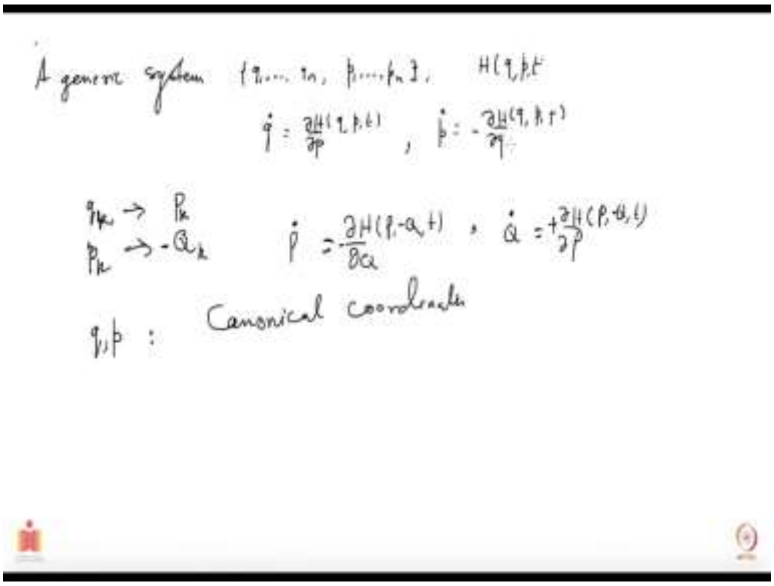

Where was it? Yeah, here you see, we did a transformation and we changed the coordinates, the, what was being called as a coordinate and what was being called as a momentum were interchanged. And ofcourse, there was a relative sign, and that is why we started saying we will always refer to them as canonical coordinates.

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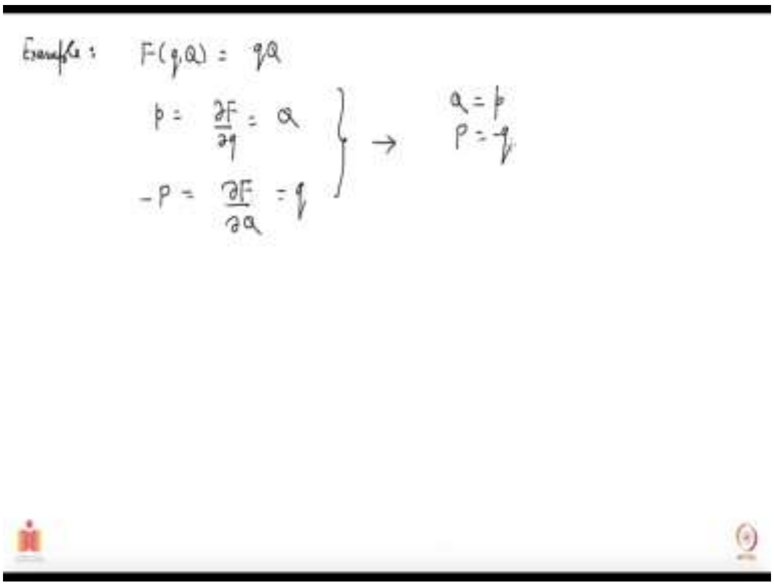

And you can see that what kind of generating function will produce that kind of transformation. So, if you take the generating function to be a product of small q and capital

Q you just interchange the names of coordinates and momenta and ofcourse, the minus sign will be there. We will continue with more on this in the next video.