Introduction to Classical Mechanics
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Lecture 61
Hamiltonian Mechanics: Canonical Coordinates

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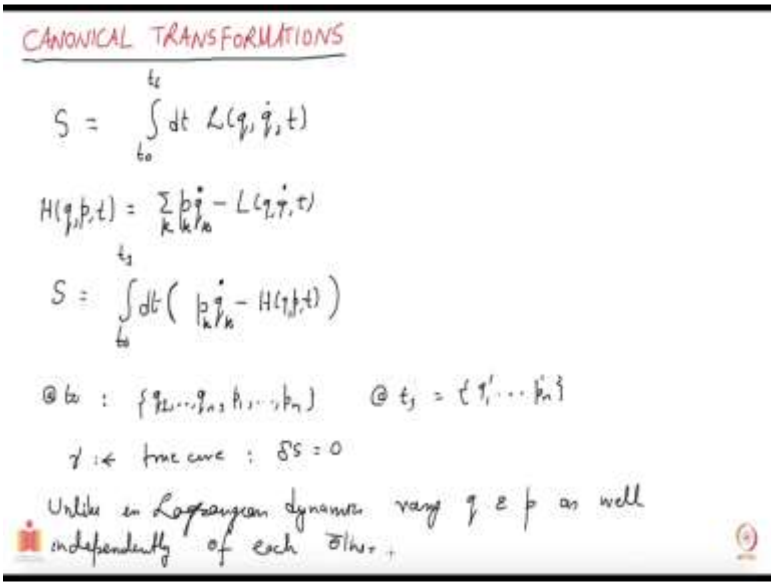

Let us continue our discussion of Hamiltonian Dynamics. In the last video, I introduced Poisson Brackets. And we also wrote down equations of motion using Poisson brackets. And today, we will start building up towards what we call Canonical Transformations. So, that is the thing which we have planned for this one, this video. And the first thing I will do is, look at a variation principle from which I can derive Hamilton's equations of motion.

So, as you have already seen in one of the previous lectures, that we can derive the equations of motion in the Lagrangian formulation by looking at the extremum of the action which is defined as an integral over time of the Lagrangian. By q, I mean all the n independent coordinates. And the limits are specified, the t naught and t1. And also the constraint was that the qs are specified at t0 and t1.

And we were looking for the conditions which will extremize S, and we obtained the Euler Lagrange equations from there. Now, we have already introduced the Hamiltonian through Legendre Transformation. Let me again denote by capital H, the Hamiltonian and the qs and p they are a set of total 2n coordinates, 2n variables qs and ps which is defined as p q dot, there is a summation over k minus L q q dot t.

Can I, if I write the action above using this expression, so I can take the L to the left and bring H to the right, then I will have S as the following, p q dot minus L, sorry, minus H. Now, what we want to do is, derive the equations of motion starting from this expression. Now any, suppose your system at any point of time t naught is located at, so at t naught, the system is located at q p. Let me put vectors to denote or maybe just like this.

Now, I should not use curly brackets, it may confuse with, let us write this way. So, let us say it is here. The system is located at this point in the phase space. And at a later time, t prime or t1, it is at q1 prime so and so forth up to pn prime. And again, let a curve gamma that along which the system is going to evolve, the true path. And let us consider small variations about this curve gamma in the phase space. Just the way we did in the case of Lagrangian dynamics.

So, let us say this is the true curve. Meaning which extremizes the integral S or the action S and the variations of S are 0 about this curve. Now here, unlike the case of Lagrangian dynamics we can treat not only the qs but also the ps independent and vary them independently. So, in the Lagrangian dynamics when we were doing a variation of the action, we were only changing the qs, we were varying only the qs.

But now, in Hamiltonian dynamics in addition to qs, I will take ps also independently and vary them. So, the that is what I am going to do. So, let me write it down. So, unlike in Lagrangian dynamics, vary both, vary q and p as well independently of each other. So, that is what you do. And you run the arguments which we have already used when we were looking at the variations in the case of Lagrangian dynamics and show that this is easy.

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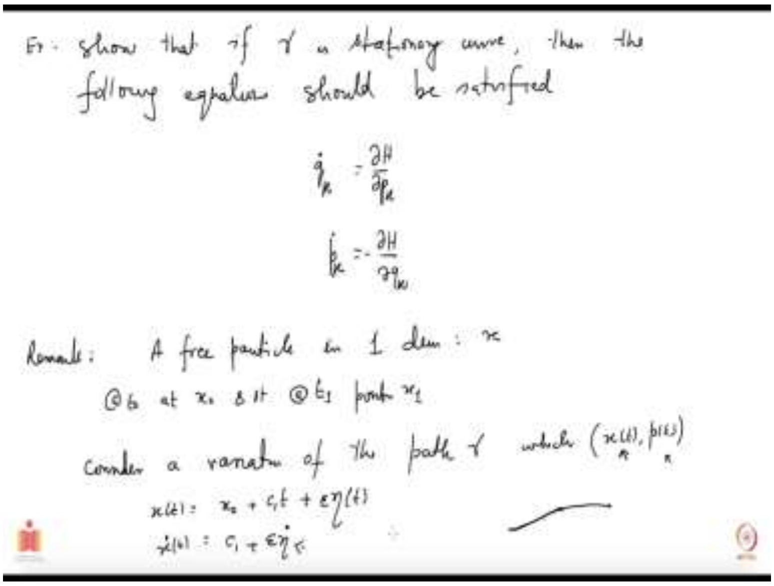

So, I will leave it as an exercise, show that if gamma is stationary curve, meaning, if you vary about the gamma then your variations are vanishing up to first order variations. Then the following relations must be satisfied or following conditions must hold true or following equations must hold true, should be satisfied. And you can guess what I am going to write, that if you take a derivative of H with respect to p, then you get q dots.

And if I take derivatives of H with respect to qs, these are minus sign here, then you get pk dots. And these are your Hamilton's equations. So, this will follow. This is good. Now, I want to make some remarks, which will clarify what we have done here and hopefully will not leave any confusion. So let us take the case of a single free particle, so some clarifications or remark.

So, imagine you have a free particle which is, by free I mean that there are no forces acting on this and it is moving in one dimension. And let us say that coordinate is denoted by x. Now, suppose that this particle is at x naught at time t naught. So, it is at t naught located at x naught. And it reaches at t1 position x1. So, these two are given to you. Now you already know what the actual motion would be. It will just go in a straight line from x naught to x1.

Let us say the (())(09:07) space. So, it goes from x naught to x1 and the velocity will be constant. It will not change or equivalently the momentum will be constant that is what the real trajectory would be. But let us see what, how it goes when we are trying to look at the variations of the action for this case. So, let us consider, then a virtual path, so consider a virtual path or a variation of the path gamma. In this case, gamma is the straight line connecting the 2 points.

And ofcourse, it should also have the velocity should be constant along this line because there are no forces. So, consider variation of the path gamma and which I will parameterize as following. Show the path gamma which will be let us say xt pt we are in the phase space remember. So, your path will be determined by specifying these two quantities for all times between t naught and t1. And when I am considering the variation, let me first write about x, and then we will worry about p.

So, let us say for x I choose the following function. So, I take x equal to x naught plus c1 t plus epsilon eta t. So, when t is equal to this t naught I am right now assuming to be 0 or you can put t minus t naught here it does not matter. So, either you put t minus t naught or you put, originate t naught to be 0. So, let us say t naught is 0. Then, this is a varied curve and I have written only the xt part of it. And this is how you will construct a path. And now, let us look at what dx over dt is.

So, along this path, this path, if you look at what dx over dt or x dot would be at any point of time, it will be given by taking a derivative of the above expression. So, it will be c1 plus epsilon eta dot. But now, I have been saying that we can vary the p, the conjugate momentum to x which for the case of Cartesian it is really the momentum of the particle we can vary that independently in here.

So, if I do so, then it is clear that the relation between the velocity and the momentum, which you know which is there will not hold true on an arbitrary path which you consider in the phase space, I hope this point is clear? So, x of t is given, I take the derivative, I get x dot. But as I am saying that I can treat x and p independently, so which means that the deviations of, the deviated path which I take here, the, if you go along this path, the momentum at different points of time will not be related to the velocity in a manner in which they have to be on the actual path.

So, that relation does not hold true because p is being varied independently. That is, that might be surprising or startling, but let us see, let us just proceed without worrying about it, and we will see, how it gets resolved. So, that is fine. Now let us look at what equations of motion you get.

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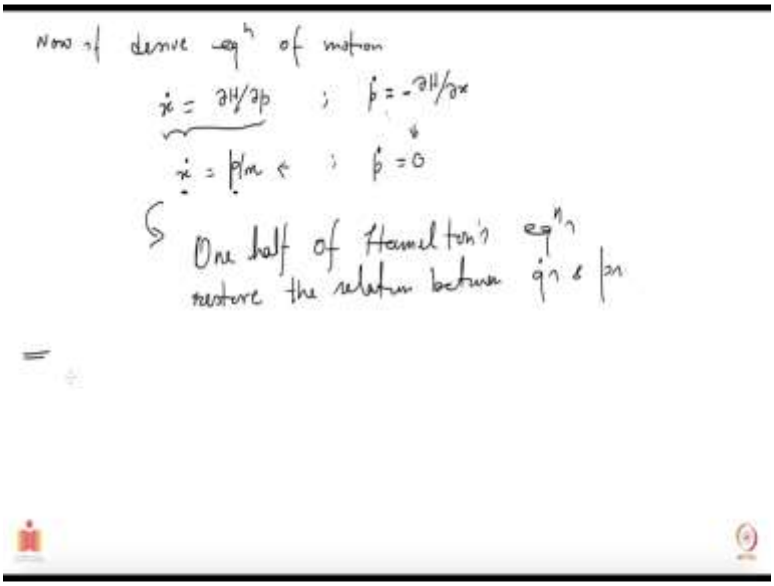

So, if I said, if I derive the equations of motion, now if I derive equations of motion, what you are going to get is the following. So first is your x dot will be delta H over delta p. And the second equation will be your p dot will be delta H over delta q or x with a minus sign. So, now you, here you see.

If you look at the, this first set, first equation or corresponding the first set of equations in any general, for any general system, then here you will get, for example, in this case, you will get x dot is equal to p over m, which is just saying that the velocity times the moment, mass is the momentum. And this is anyway your equation of motion, this one.

So, you see that even though on varied paths, your relation between velocity and momentum this relation was not holding true. But when you are looking at the actual path, actual trajectory taken, then one set of the equations of motion, the first set which involves the q dots that ensures that the relation between the velocities and the conjugate momenta that is restored. So, this is the purpose of the first set of equations.

And the second set is truly the equations of motion for the system. It gives you the rate of change of momenta. But anyhow, as far as the Hamiltonian dynamics is concerned, for us both are, both sets are truly the equations of motion and we do not distinguish between them but this is one thing which I want to, wanted to remark. So, let me write it down.

Here, so what I have concluded is that the first or one half of Hamilton's equations since restore or ensure the relation between q dots and ps, and the other half gives you equations of motion looking from the Lagrangian point of view. But as I said, for us all the equations are equations of motion here. And just to complete this, if you, you already know what Hamiltonian will be for this case. It will be just p square over 2m which means that it will not depend on x and your p dot will be 0 which is just saying that the momentum will be constant, that is good.

Now, let me say, now let us start progressing towards canonical transformations. So, before I do that, I will talk about a very simple transformation.

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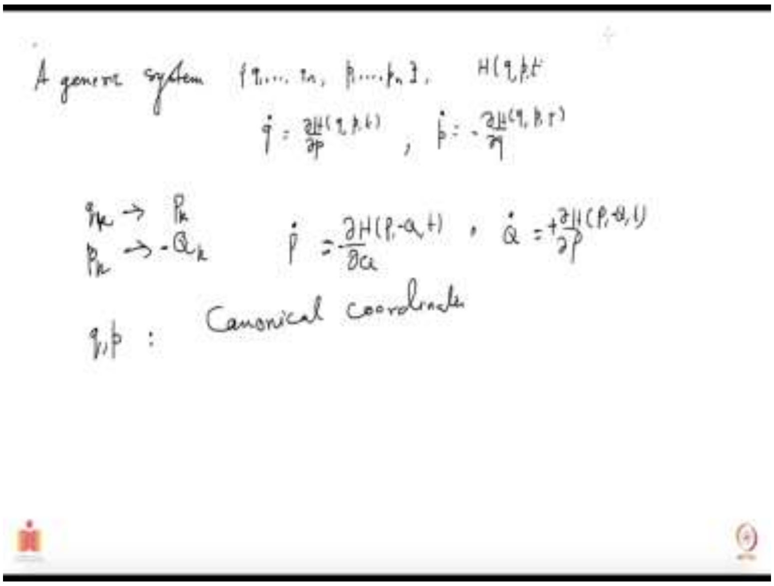

So, let us say you have a generic system which is described by these coordinates and momenta. And your Hamiltonian is a function of all these qs and ps and possibly time. And your equation of motion, as we have written several times is delta H over delta p. And I should emphasize here q, p and t and your p dot is delta H over delta q; q p t, there is a minus sign. I am not putting indices right now, but you can put them there.

Now, let us say, I decide to call the qs to be capital Ps and ps, the small ps to be capital Qs. When I can do that, if I do this and I try to look at the equations of motion how they appear, you will see that you get immediately delta H over, so the small p is being called Q now. This thing is being called Q, P Qt. And this will be the q is P, so it becomes P dot.

And the second one becomes delta H over, the q becomes P, there is a minus sign, q becomes P Qt and this will be Q dot. And you see that the signs are off, so clearly this is not working. What I can do is, instead of this, I can take, put a minus sign here. And if I do so, and put a minus Q here which will bring a minus here and minus Q which will, because of this minus, this will go away, the minus on this side. And then, you get your familiar canonical equations or familiar Hamilton's equations.

So, you see what I have done is, I have gone from q p to a new set of variables where I have labelled the coordinates as momenta, let me call the Ps as momenta and the small ps as Q, and still the equations of motion hold true in the canonical form which means that they, which just emphasizes our point that I can treat them all at equal footing instead of distinguishing them as coordinates and these as conjugate momenta, I treat them all at the same footing.

And that is why these are more simply just call canonical coordinates instead of saying these are coordinates and those are momenta you say canonical coordinates. So, the qs and ps, henceforth, I will be mostly referring to them as the canonical coordinates. Let us see, we will talk more about, not more, I mean, I will start talking about canonical transformations in the next video.