Introduction to Classical Mechanics
Professor Dr. Anurag Tripathi
Assistant Professor
Indian Institute of Technology, Hyderabad
Lecture 60
Hamiltonian Mechanics: Poisson Bracket

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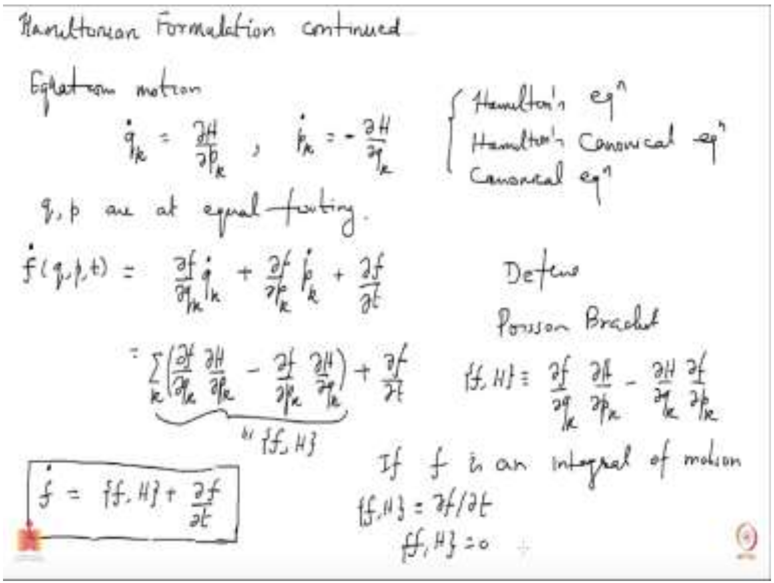

So, let us continue with the Hamiltonian Formulation. Last time, we wrote down the Hamilton's Equations which are the equations of motion in this formulation let me write it down again. So our equations of motion, so if you have coordinate qk and you are looking at the time derivative of this, you have to take the Hamiltonian and differentiate it with respect to the momentum Pk which is conjugate to qk.

And then you have pk dot as delta H over delta qk and this should be a minus sign. Let me check, which one has minus pk dot should have a minus sign. So, these are your Hamiltons equations. They are also called Hamilton's Canonical Equations, and sometimes, just canonical equations. So, they go by all these 3 names. And you remember that our q and p are at equal footing now. And that is why we were working in the phase space, so this is given.

Let us say, I take some function f, which is a function of these coordinates q and conjugate momenta p and possibly time, sorry, let us say some function f is given which is a function of q p and t, and we ask, how this function will evolve with time. And its time evolution will be governed by the time evolution of q and p and ofcourse the explicit time dependence and these are in turn in, determined by the canonical equations.

So, let us find this out. So, we are looking at df over dt which is f dot. And this is, so df over dt is delta f over delta q or del f over del q q dot. And let us put this, this is a summation over k implied. Then, you have del f over del p p dot summation over all the conjugate momenta. And because of the explicit dependence on time which I have shown here, you will have the partial time derivative. Now, in this I can substitute what you have here in the equations of motion.

So, qk dot will become del H over del pk, and pk dot will become minus del H over del qk. Let me write that down. So, you have delta f, del f over del qk, qk dot is del H over del pk minus because of this one del f over del pk and pk dot is del H over del qk. And then you have the partial derivative with respect to time. Let me make the summation explicit here for once. Now, I define what is called a Poisson Bracket.

So, I will define Poisson bracket between f and H. This is denoted by these 2 curly brackets. So, the definition is you, whatever you have in this first term, not the first one, but these 2 terms together form the Poisson bracket. So, the definition is, I think, instead of writing it again, let me just show it here. So, this is defined to be the Poisson bracket of f and the Hamiltonian H. So, with this, I get f dot to be equal to the Poisson bracket of f and H plus the partial derivative of f with respect to time.

And slowly, it will emerge that this Poisson bracket is a quantity of fundamental importance. But for now, at this juncture, it is just a definition and a simple way of writing. But we will see eventually, it is quite an important quantity. That is good and note that this definition, this thing is anti-symmetric in f and H. So, if you interchange f and H this will be anti-symmetric. So, that is why I am going to write it in a slightly different, not different, but the same thing I will rearrange and put it this way.

So, I will write del f over del qk del H over del pk minus del H over del qk del f over del pk. There is a summation over k implied. So, all I have done is, the order of derivatives is kept the same del, del over del q, del over del p. And in here also, you have del over del q and del over p. And only the order of f and H is interchanged in these 2 terms. So, it is clear it is anti-symmetric if you interchange f and H here, that is good.

Now, if your f is a conserved quantity, meaning if it is an integral of motion, then your f dot is 0. So, if f is an integral of motion that is df over dt is 0, then you get f Poisson bracket H to be delta f, partial derivative of f with respect to time. And if further this quantity does not depend explicitly on time, then you get the condition to be this for conservation of f. So, let me write this down again.

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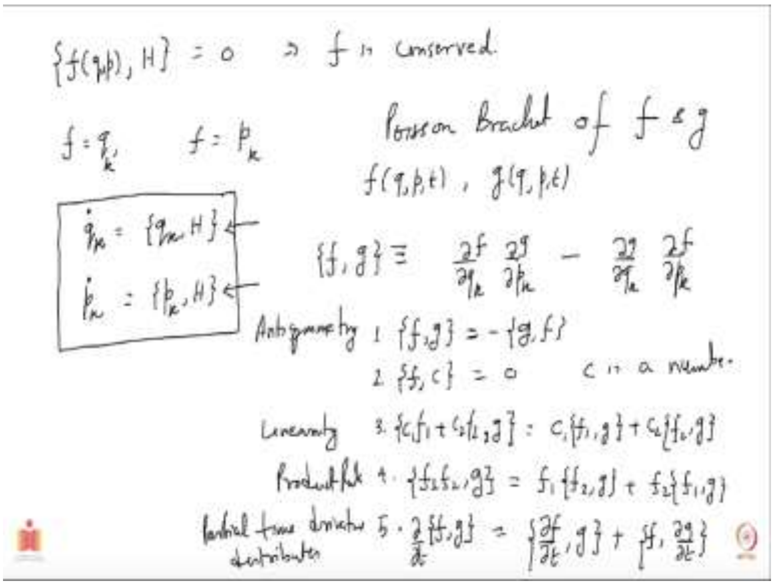

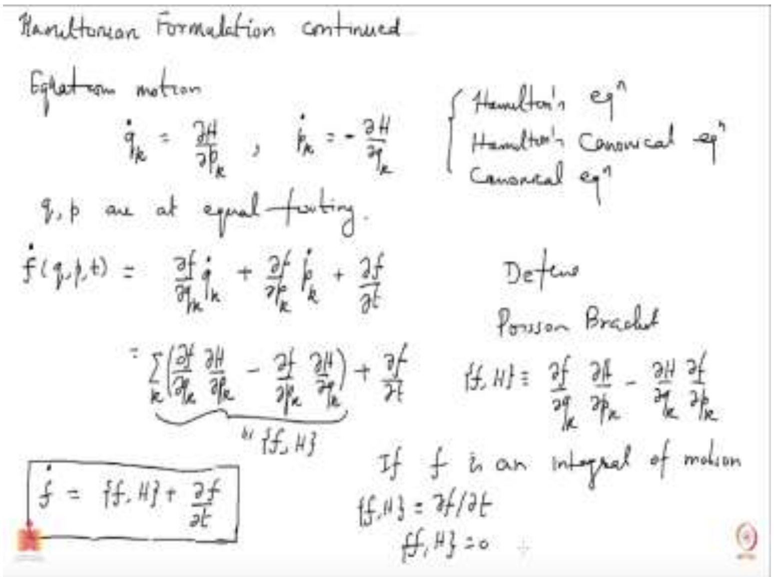

If you have f, which is a conserved quantity, and it depends only on the q and p and not time explicitly, then if you take the Poisson bracket of f with H, you get 0. So, this implies that f is conserved now it is good. Now let us take in turn f to be q and then p. So, I take once f to be equal to one of the coordinates, and then I will take f to be one of the momentum, conjugate momentum.

So, f is one of qks and this, and substitute in the definition of the Poisson bracket. So, let me put it here. And you can check that you are going to get the following. So, if you look at q dot of k, then you are going to get qk H, this Poisson bracket. And if you look at p dot of k, then you get pk H, this Poisson bracket and if you look at p dot of k then you get pk H. So, your equations of motion which are here, the q dot and p dot, which you write in terms of the derivatives of H, now become the following.

Let me put a border around this. Now, see the, these relations, these equations of motion appear very symmetrically for q and H, the annoying minus sign which was there is not here now but that is a minor thing. But you see, it looks, the 2 equations here, they look identical. The only thing is qn, q is replaced by p in this one. So, this also emphasizes the fact that q and p are being treated equally in the Hamiltonian formulation.

Now, what I will do is, I will take this definition of Poisson bracket of any function f with H, and put it and use it to define a more general quantity, not more general, but just Poisson bracket of any two quantities f and H. So, if I am, so here is the definition, Poisson bracket of f and g. So, you are given two functions f and g, where f is a function of q, p and t all the coordinates and their conjugate momenta and possibly time and the same for g.

And then I define f and g, the Poisson bracket of these to be del over del qk, del over del pk minus del over del qk del over del pk and then I put here f and g here and g and f here. So, that is the definition of this Poisson bracket. We can immediately see certain properties, which I will list down. So, one property is ofcourse anti-symmetry which you can see immediately from the definition. So, if you interchange g and f, their positions within the bracket, you pick up a minus sign.

So next, if you take the Poisson bracket and, take the Poisson bracket of one function f with a constant, you are going to get 0, because the derivatives of the constant will give you 0. So, let me put it down, c is a number here. Also, the Poisson bracket is linear. So, if you have c1 f1 plus c2 f2 and you take a bracket with g, it will give you c1 f1 g and c2 f2 g. So, this is your linearity. Then, the product rule is the following.

So, if you take the product, a product of 2 functions f1 and f2 and find out the Poisson bracket with another function g, it will turn out to be f1 f2 g plus f2 f1 g. So, that is the product rule, which you can easily verify starting from the definition of this Poisson bracket. Then if you take a partial derivative of the entire bracket, then you will see that the partial derivative with respect to time, that partial derivative distributes.

So, you take del over del t of this Poisson bracket and you can verify that the following holds true. Let me write down partial time derivative distributes.

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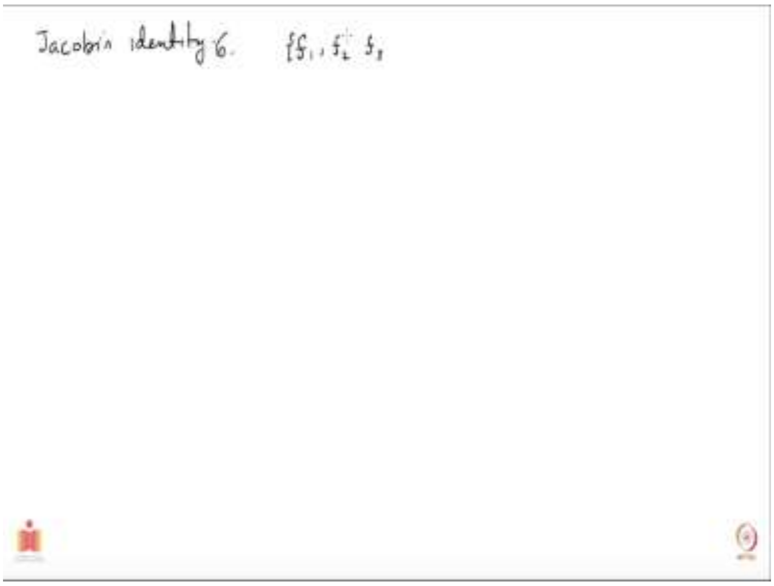

One more thing, you can verify, we have listed down 5, sixth one, you can check that how was…yes, that Jacoby's Identity is also satisfied. Because it is an identity it holds true for any f1, f2 and f3. So, let us say, you are given 3 functions f1, f2 and f3. So, you construct a Poisson bracket of. And finally, we have the Jacoby's identity. So, let us say you have 3 functions which are, this is the 6^{th} property right and this is 5.

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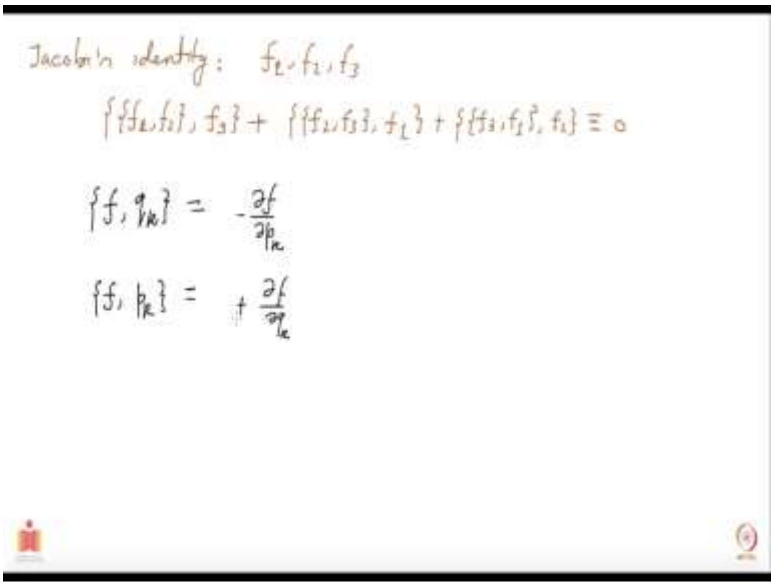

Somehow it has become brown, anyway. So, if you are given 3 functions f1, f2 and f3 then you can take the first f1 and f2 and make a Poisson bracket, and then you can take the Poisson bracket and create a new Poisson bracket with the third function. Now, you cyclically permute the 3, so I shift f3 to here, f2 to there and f1 to the first position.

So, it will become f2, f3 with f1 plus let us do it again, so f3 comes here, then f3, f1 and f2 becomes the last one. So, if you add these 3 up, these 3 will be identically 0. And the reason, it is called an identity is because it does not depend on what f1 f2 and f3 are. That holds true for any 3 functions of q p and t. So, this also you should verify by using the Hamilton's, by using the definition of the Poisson bracket.

So, these are the properties that I wanted to tell. Now also, let me try to see how to change this colour, yeah. So now, if let us say I take the Poisson bracket of a function f with either q or P, and you will see that it will generate the partial derivative with respect to momentum or the coordinate, so please verify the following that if you take a function f and you take a coordinate qk and construct the Poisson bracket, you are going to get the partial derivative of that function with the conjugate momentum q, Pk.

In this case there will be minus sign. And if you take f and Pk then you are going to get del f or del qk. Something nice has happened and it is no more pixelated which is very nice, I would last few videos, I was getting a pixelated writing, so that is nice, that is plus. So, that is nice, you can generate the partial derivatives of these functions by constructing the Poisson brackets. Now, I am going to tell you a nice theorem and this is called Poisson’s theorem. Let me go to the next page.

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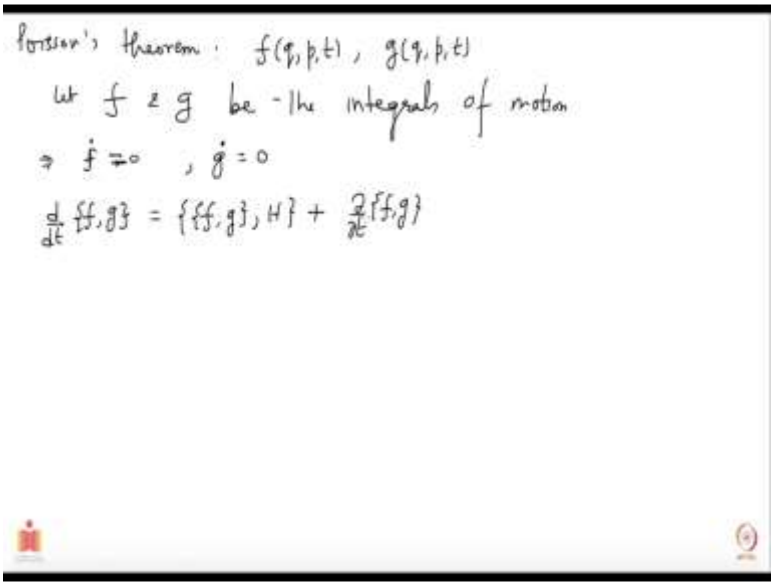

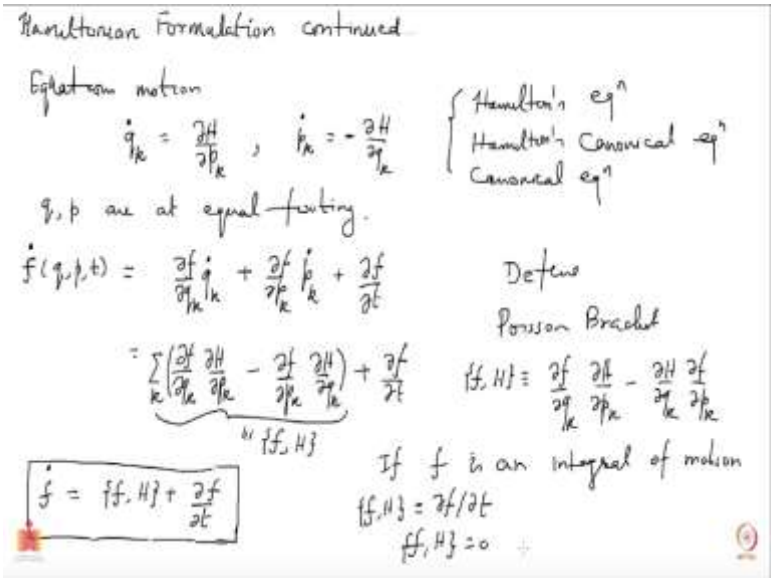

Let me write down Poisson's theorem. So, suppose you are giving, given two integrals of motion f and g. So, you are given f which is a function of coordinates all the momenta and possibly time. And another function g, again depends on q, p and t, where q and p are from 1 to n, I mean, q1 q2 so and so forth. And we say that let us say these two are integrals of motion meaning the total time derivatives of both of these is equal to 0.

So, let f and g b the integrals of motion that is there conserved, if that is so, then f dot is 0. And remember, when I put a dot, it is a total derivative f dot is 0 and your g dot is also 0. Now, let us construct Poisson bracket of f and g and ask, what is the total time derivative of this new quantity? That is what we want to do. So, if you remember, let us go back here, somewhere in the beginning.

So, if you take total time derivative of function, it is equal to the partial time derivative plus the Poisson bracket of that f with the Hamiltonian of that system. So, that is what I am going to do. So, you are going to have f g. This is what you have right now. So, you have to take a Poisson bracket with the Hamiltonian of the system plus the partial derivative of f and g, this Poisson bracket with time that is what you had, and the, on the, that slide. So, let us evaluate this. This one is easy.

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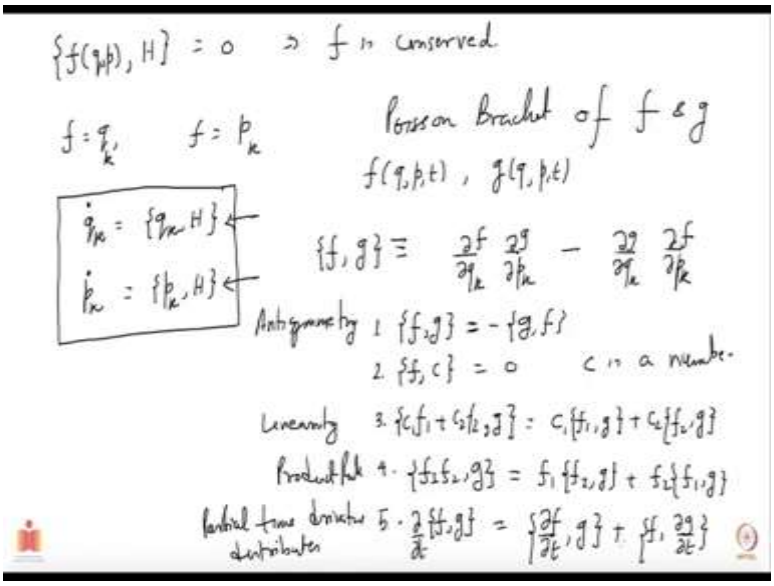

You, we will use this property that it distributes. I, so you please show that this is true. So, I am going to distribute the time derivative, partial time derivative over these two.

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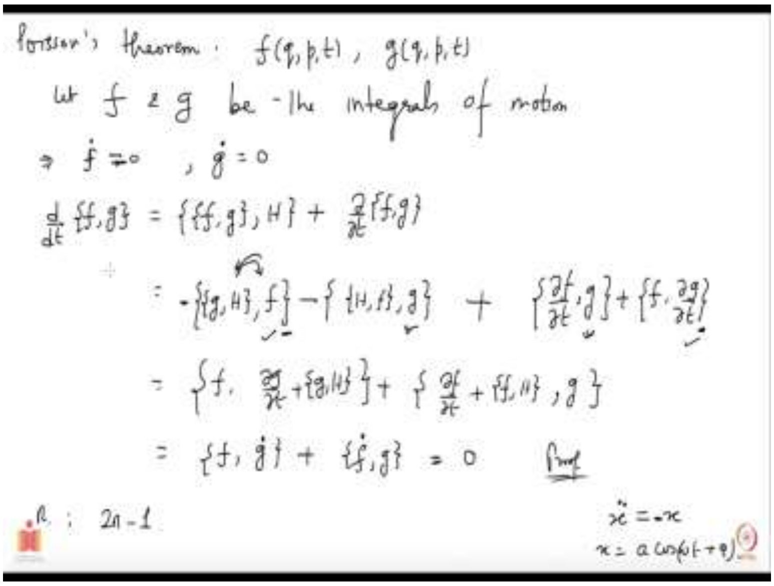

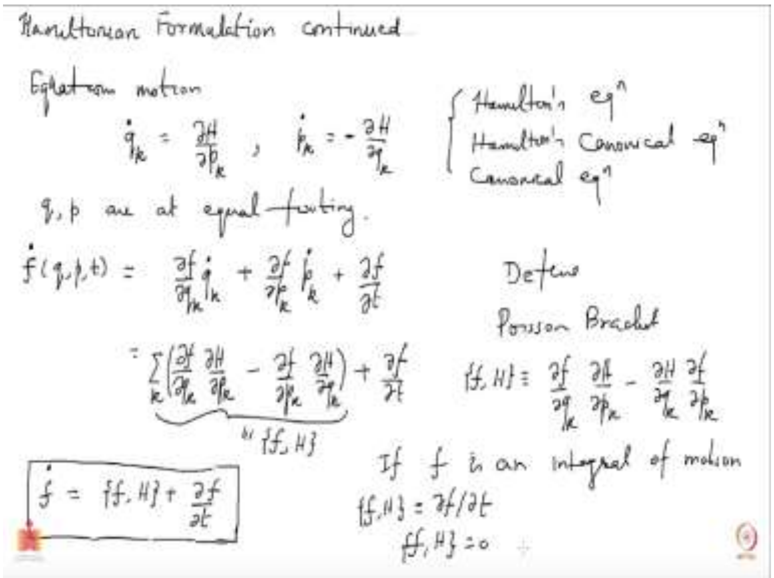

So here, let me write delta f over delta t with g and f delta g over del g over del t that is good. Now this one, it is three functions, so I can use Jacoby's identity which will just cyclically permute but the advantage is that it is going to generate those two terms which you have in Jacoby’s identity, but H will be in that case sitting with f in one term, and it will be sitting with g in the other term.

So, you will be having Poisson brackets with involving f and H and h and H that is why I am doing this. So, you put here, so let us permute. So ofcourse, you are going to get minus signs with these terms. So, yeah, so let me take f here, so it gives you g H f that is one term. The second term will be generated by shifting further. One more shift, so it will be h, f and this entire thing with g. So, this is what you get.

Now, also remember, go back here, this is what I am trying to do. So, I am going to combine, so that I get df over dt. So, this is what the task is in front of me. Where is it? Yeah. So, let us take this term and this term first. So, this is f as the first term, so I will, if I do interchange these two positions, this will because of anti-symmetry, this minus sign will be gone. And then I can use the linearity to combine these two terms.

So, it will become f Poisson bracket delta g over delta t plus g H and this is same thing for the other term. Something is off, g H, yeah, nothing is everything is fine. So here, you have H first, so when I interchange these two, it picks up a minus sign. And, but then minus sign will come out, so you will get, and you will get f and H, so first will be f and the second H. And you can combine with this one.

So, you get del f over del t plus Poisson bracket of f and h and this entire thing you combine with g to make a Poisson bracket, but now this is f comma, this is what? dg over dt, g dot that is what you have seen already. Let us go back. Again, that is your g dot, partial time derivative and Poisson bracket. And this one will be del; no not del, f dot with g. And because g dot and f dot are 0 because we have assumed them to be conserved quantities, this is going to be 0. And that is the proof, proof of Poisson's theorem.

So, you see that if you start with 2 integrals of motion, f and g you can in principle construct another integral of motion which will be f Poisson bracket of f with g. But that will not necessarily be a new integral of motion, because they have only a fixed number of integrals of motion for a given system.

So, if your system is n dimensional, let us say, n generalized coordinates describe the system then you know that you will have a total of 2n minus 1 integrals of motion. So, you can imagine that the total number of constants which will, you will get by doing the integrals will be 2n and minus 1 you can count it this way. So, you know that to describe the evolution of your system uniquely with time, for each coordinate, you need to tell where it is at a specific time, initial time and also the velocity at that time.

So, if they are n coordinates, you get total 2n. And one of these can be chosen as time and you can choose, I mean you can remove that time for, I mean, you can shift the origin of time, so that will be gone and this will this you can do for any closed system. If this is what you are hearing for the first time, I do not recall whether I have told this previously in the course. If not, then you can, this simple example will help you understand. So, imagine you are looking at a harmonic oscillator, 1-dimensional harmonic oscillator.

I have put the constants to be all the masses and the spring constants to be unity. So, this is 1-dimensional. And for this, you know the solution, that solution is x is a cos omega t plus phi. So, you have two constants a and phi which is what you expect. But then you can remove the phi by shifting the origin of time, it can be just removed. So, instead of 2, you get 1.

But if you imagine you had several harmonic oscillators, they all will have some different phases but you cannot remove all the phases. You can shift the time and remove one of the phases, all the other phases will just get shifted but they will be there. So, you see, if you have an oscillator you will have 2n minus 1 constants and the same thing is true in general, you can convince yourself.

So, if they are 2n minus 1, such constants, such integrals of motion. So clearly, you are not going to generate new constants of motion by just taking Poisson brackets, you may or you may not, depends on what you start with.

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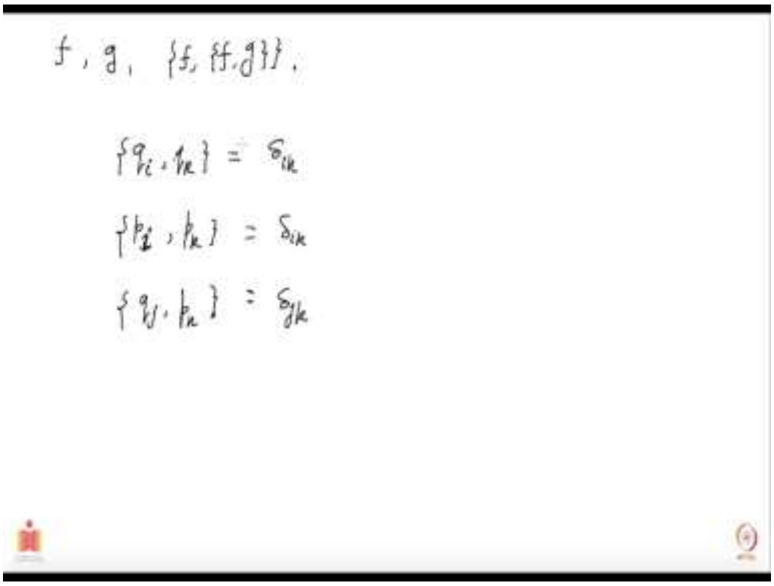

So, it is not necessarily true that if your f is a constant of motion, your g is a constant of motion then you will get a new independent constant of motion this, this may again be a function of f and g. This also for example, may be just a function of f and g. So, you do not get anything new, but you may also get new constants of motion. So, this is a useful theorem. And before I stop this video, I want to give one small exercise and derive this nice result.

So show that, if you take coordinates qi and qk and construct the Poisson bracket, you get delta ik. Similarly, if you construct pi with pk you get again delta ik. So, unless the two coordinates in this poison bracket are the same, you get a 0. If they are same, you get 1. The same is true for momenta, which is nice, I mean, look, again looks completely identical with respect to q and p, there is no distinction between these 2.

And also, you will see that if you take qj with pk, you get delta jk. So, I will stop here and we will continue in the next video.

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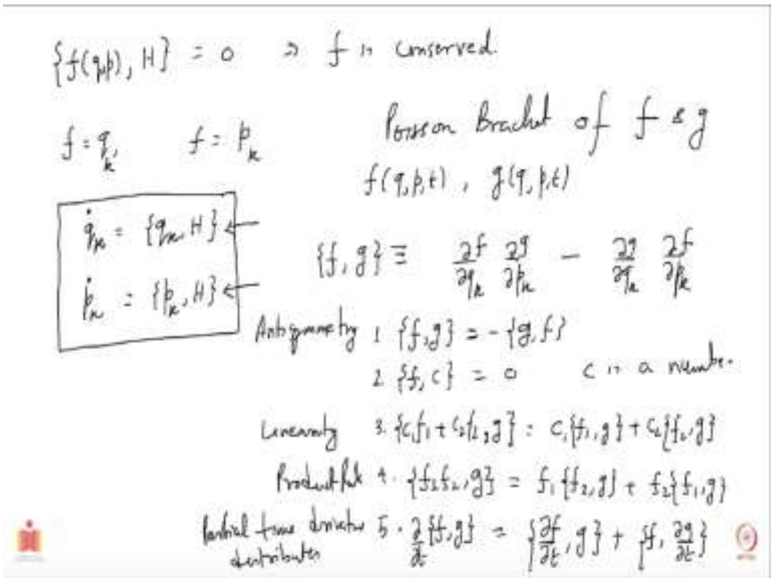

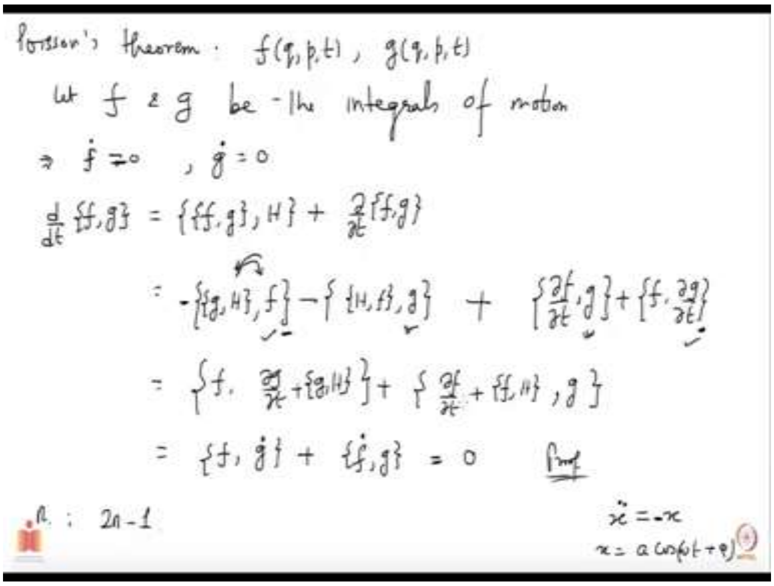

Please do all these exercises and also show that these results are true. These are trivial to show. So, please do so and lets meet in the next video.