Introduction to Classical Mechanics
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Lecture-6
Euler Lagrange Equation for a Holonomic System

So, in the last video we wrote down d'Alambert's principle and I promised that next time I will derive equations of motion involving the generalized coordinates. And that is what we are going to do now. So, let us remind ourselves what the d'Alembert principle is. And from there we will start changing the coordinates from R to Q, R being the Cartesian and Q are our generalized coordinates.

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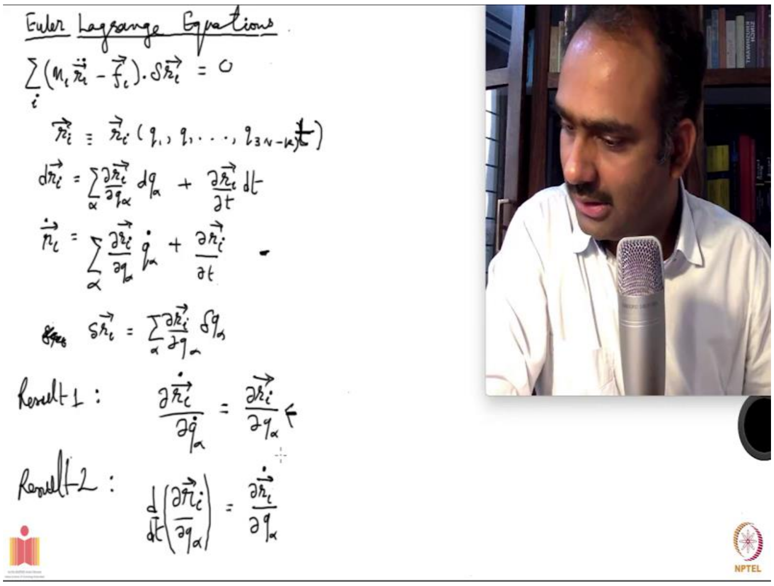

So, my relation was mi ri double dot minus fi and dotted with the virtual displacements. And if you sum over all the particles, this is 0, that is your d'Alambert's principle. Now, instead of the Cartesian coordinates R, I will convert everything to the generalized coordinates. And here is the relation, so I have ri, they are function of all the coordinates q's and they are 3 n minus k, where k are the total number of independent constraints. And we have already assumed that we have holonomic constraints. That is good.

Now, I can take a differential of this and write d ri, I should put vector symbols here d ri is what, del r i over del q, let me call alpha, I am using alpha, dq alpha, and I should sum over all the generalized coordinates and I will also have a time derivative term. Which means I missed writing in the previous line something, so here we should include time as well. Put a comma and make it a t, okay, this is time t and dt, dt is missing. Perfect.

Now, if I want to write down the derivatives then I should write down ri dot, I am dividing everything by the time dt, delta r i over delta q alpha q alpha dot, so this dq alpha divided over dt becomes q alpha dot and you still have a sum over alpha plus delta ri over delta t. And what will the virtual displacement satisfy? Virtual displacements would satisfy delta r I, sorry, that is correct, that is correct delta r i equals delta r delta q alpha, delta q alpha. As you remember I have to drop the dt term. That is good.

Now, before I proceed further, I should state two results which we are going to use in massaging the d'Alembert principle written here and those results are the following. Result, result 1. And the result 1 is that if you take delta r i delta q alpha and if you put dots both on ri and q alpha then this equals that relation.

So, if I can take this and put dots on both ri and q alpha. So, delta r i dot over delta q alpha dot, that is result number 1 that we are going to use. And then let us look at result number 2, result, which says if you take delta ri over delta q alpha and you put a dot on this, then this is same as delta ri alpha with a total time derivative. That is correct, very good. That is correct. So, I am going to utilize these two results. Okay. Now, let us look at, okay, maybe first I should prove them okay, that is that will be better.

So, let us go here, first one is quite trivial. You see this one, this one is easy this I can see from here. So, you take ri dot and this relation is a linear combination of, is a linear sum of q alphas. So, this is del r i over del q 1, q 1 dot plus del ri or del q 2, q 2 dot and so forth. And this piece is a function of q and q dot dependence is only here. So, if I take a partial derivative with respect to q alpha dot, only this piece will survive, this will drop out because there is no q dot in here. So, first one is quite trivial as you can see. Nevertheless, I think I want to do it for you. Okay, so let us do that. I wish I knew how to copy this. Let us see. Is there a way to copy the frame, no. Okay, anyhow.

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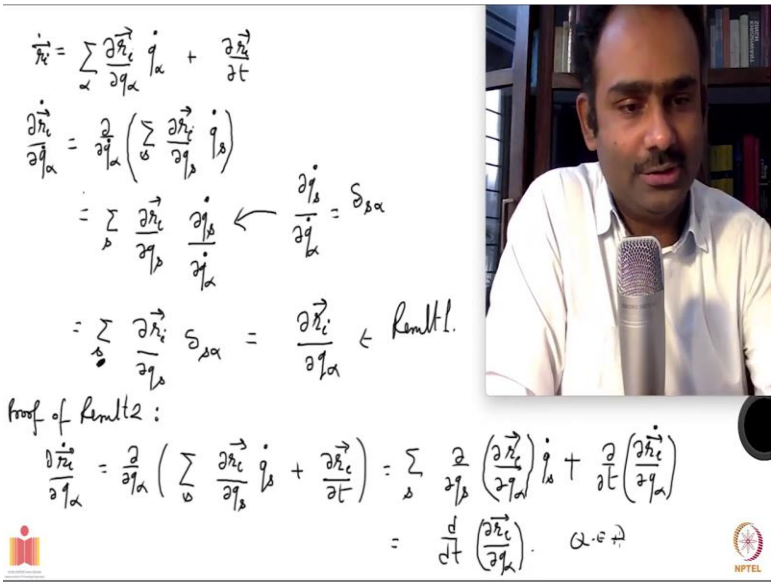

So, what I wrote was ri dot, that is what we had on the previous slide. Let us check and you have a summation of what alpha, that is correct. Now, I want to take a derivative with respect to q alpha, so that is what I do alpha. Now, if I do so, I should not use the same alpha here. Because this is dummy, this is summed over, so it makes no sense to keep alpha there. So, let us use some other symbol, let us say s to denote the sum. So, this will become del over del q alpha summation over s, delta r.

Okay, there is the first term, so I have changed the domain index alpha to s, okay, and this term will go away because this does not involve, I am sorry, there was a qi alpha dot here and this does not involve a q alpha dot. Now, this is summation over S, I can take this partial derivative and pull through the summation and it will not act on this because this is a function of q's and not alphas, so it goes directly to act on this one.

So, I get del ri del q s and del q s alpha dot. This is the right way of doing it. So, sometimes students will just read from here that I am taking derivative respect to q alpha dot and this is what will be left over, which is correct answer but you should know how to proceed and how you should get it. Otherwise there is chances that you will start making mistakes and elsewhere. Now look at this piece, what is this. So, delta q s dot over delta q alpha dot is what is your understanding.

See when s and alpha are same, then the answer is 1. So, delta S q, delta S delta q s dot and over delta q s dot is 1. And when S and alpha are different answer is 0, because these are independent variables. So, clearly unless s and alpha are same, the answer is 0. And when s and alpha are same, sorry if s and alpha are not same, the answer is 0 and if s and alpha are same, the answer is 1 and that is what a Kronecker delta is. So, this thing is delta s alpha. Okay, so I substitute this thing here and I get was delta S alpha.

Now, what is that? So, here there is a sum over s. And unless s and alpha equal, it always gives 0, so it multiplies 0 to this. So, it gives us a 0 contribution. And only when there this s hits alpha, this gives a nonzero contribution, this entire thing because this is one then and the result will be delta r i alpha. So, that is how you show this simple result. So anyway, we have proved our result number 1, which was quite trivial I just spent some time in showing you the algebra explicitly, but the proof was fairly trivial. Now, let me prove the result number 2.

Okay, what I want to show is, Yeah, so, I start with the with the right-hand side here in the result 2. So, del r i dot over del q alpha. So, I start with del r delta q alpha. Now, I will write this as q alpha and I was just I down what ri dot is and you know what that is, again I should use a symbol, a different symbol. So, it should be delta r not dot, delta ri over delta q s, q s dot plus delta r, correct that is absolutely correct.

Now, you see this partial derivative I can pull through the sigma of course, and it will act only this part because q s dot is independent of qs, So, it does not work on this one. So, it will work on this and the q s will also go through the delta over delta t and work on this one. So, when I take this and put it here, because the delta q alpha and delta q s I can interchange, I can do the interchange, they are independent coordinates for different alpha and different S. And when they are the same, it anyway does not matter which whichever you write.

So, the upshot is that I can write this as summation over s, delta over delta q s delta q s dot plus over delta t delta q alpha. And that is what the expression for a total time derivative is. Del over del t and differentiate with respect to coordinate times the velocities. So, this is nothing but d over dt del r. Why, I am putting the dots here. Q (())(15:01) that is correct, proved. The (())(15:04), let me check. Perfect, that is perfect. So, we will utilize these two results in dealing with our d'Alambert's principle. So, let us proceed then.

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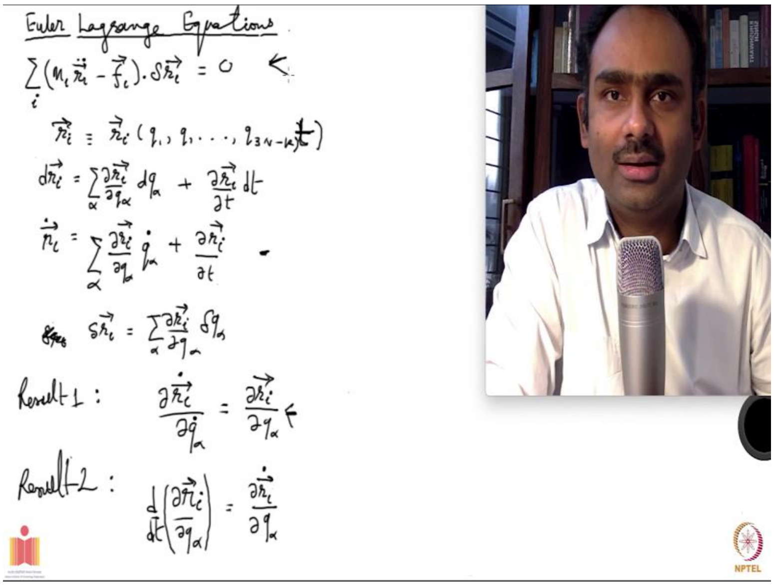

Now, here in this there are two pieces. So, the first term is this, which includes the dot d ri as well and this is the second term fi dot d ri. So, let us look at the second term first fi dot delta ri. There is a minus sign as well.

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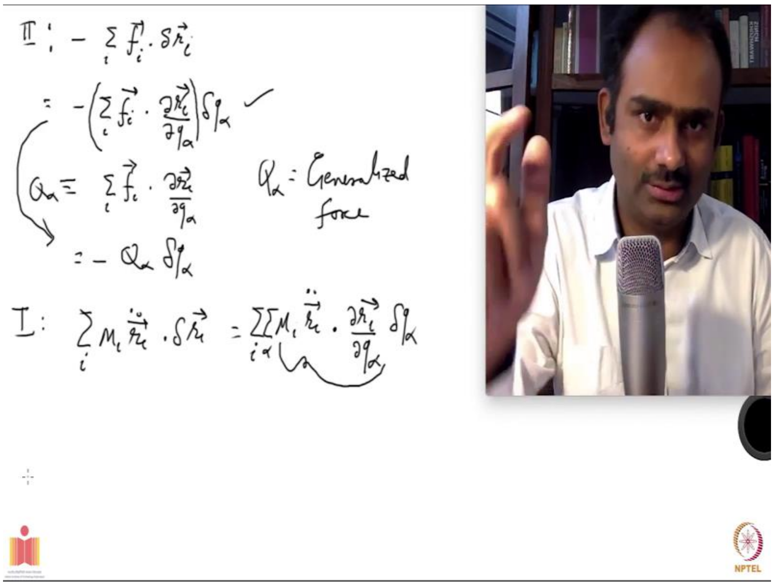

So, second term was minus summation over all the particles we had f i dot delta ri, that is correct. Which is same as minus summation over i fi dot delta ri's del r i over del q alpha dq alpha delta q alpha, okay these are virtual displacements that is why the time part is gone. Now, look at this, I can define something called a generalized force. So, this part I will call q alpha. You see the index is summed over so that is not a free index, the only the alpha index is free. So, I define this quantity to be q alpha.

So, q alpha is summation over i, that is the definition of q alpha and with that, the second term becomes minus q alpha, delta q alpha. And it is quite natural to call q alphas to be the generalized forces. You see, here the forces were multiplying the virtual displacements. Here you again have virtual displacements, but those of generalized coordinates. So, whatever here is, you will call generalized force. Force, and note that the fi dot delta ri is the work done under virtual displacement and so is q alpha times delta q alpha.

That is also the work done under virtual displacements, so we have written down the second term in d'Alembert principle in this manner. Now, let us look at the first term which was here, mri, mi ri double dot dot delta ri. So, that is what you want to write now. So, let us see term number 1, here was term number 2, mi. Now, this is going to be very interesting. We are almost there. And mi ri double dot dot delta ri summation over all i, that is correct.

Now, this thing I will write as mi, again for delta ri I will do what I did here. I will write dot delta r q alpha. That is fine. This is perfectly, that is good, that is good. Everything is okay. Now, I think I can leave around dark, okay, there is also sum over alpha now because I am summing over alpha here. So, what I will do is I will just pick up this piece and leave the delta q for the moment and just work on this part. So, let us take this, this is, I will leave some space here or maybe I should go to. So, I have a mi double dot, okay let me go there.

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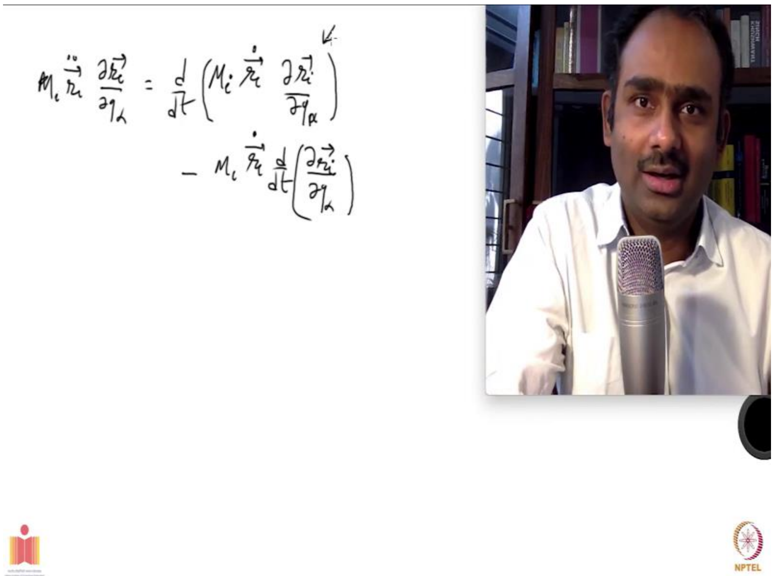

mi ri double dot and then we had del r i over del q alpha. So, what we can do is I will write this in the following form. See the r double dot involves 2-time derivatives, so, I am going to pull out one time derivative and leave the velocity that is r dot behind so I write this as d over dt. Even in fact, I should have left behind the mi's also. So, let us leave your mi also. No, no, no, that is fine, okay. It is okay. So, what I do is d over dt mi ri dot q alpha.

What is this, now this will give you r double dot term, but it will generate extra term that I will subtract out. So, it will have mr mi ri dot d over dt acting on this piece right, that is what is getting generated extra from the first term alpha, that is nice. Now, this is the place where I want to utilize those two results which I showed you. You remember the first one, I can use in here. First one was that that is a time derivative of ri with q alpha I can replace by a time, the partial derivatives of ri with q alpha I can replace by a partial derivative of ri dot with q alpha dot. Let us see here.

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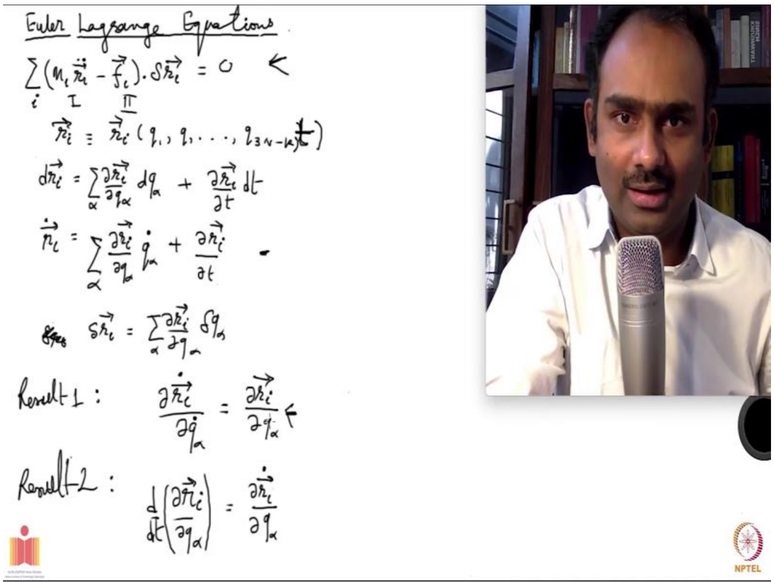

Okay, ri q alpha turn to ri dot q alpha dot, that is what I will do here.

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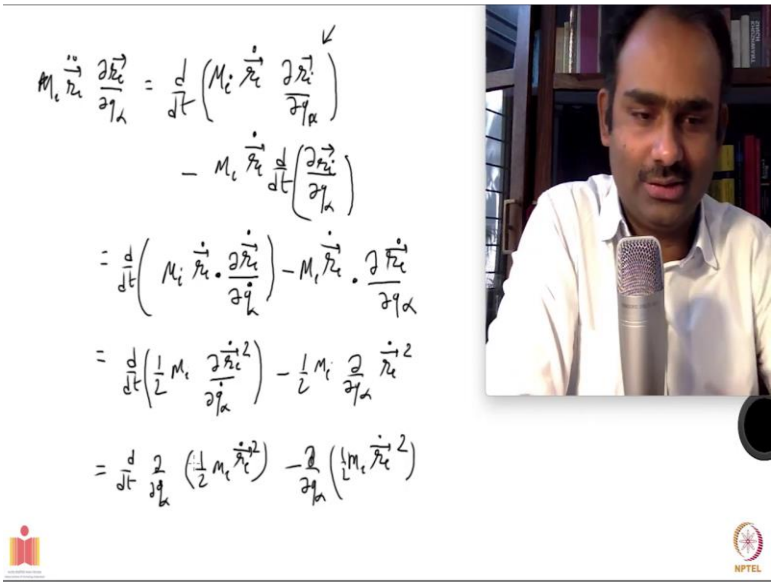

Equals d over the dt mi ri dot and del ri dot over del q alpha dot, you will realize soon why all this is going to be useful, minus in the second term, I will use the second result. And what was that, if you take a time derivative of this quantity you can replace it by putting a dot over ri. So, that is what I will do minus m i, r i dot del ri dot over del q alpha. That is good. And it is good because of the following reason.

This I can write as d over dt of now, what is this? This says I can put the ri dot in here inside, inside the derivative term here. So, it will become ri dot square. But then I will have an extra factor of 2, because if you take the derivative of r square, r dot square it gives 2 r dot del r dot. So, I get an extra piece of 2, so I remove that one by writing half and then del r i dot square over q alpha dot, minus the same thing here half mi del over del q alpha ri dot square.

Now, these half mi pieces, these pieces, I can put them next to the ri dot square, same in both and you know that half m ri dot square is the kinetic energy of the ith particle. So, this will become d over dt half del over del q alpha dot half mi ri dot square, minus again del over del q alpha, and mi ri dot square and there is a half. These are your kinetic energy terms. Okay? Now, if you go back and see here the term, that was the term number one, so I substitute here.

You see this about; this is the thing which I have calculated. So, I plug it in here. What is this? This is summation over alpha, and I am bringing in the summation over i, and what is that, d over dt del over del q alpha dot here, and then you have summation over all the kinetic energies of individual particles. And that will give you the sum of the kinetic energy of the entire, that will give you the kinetic energy of the entire system. So, that is what we get here.

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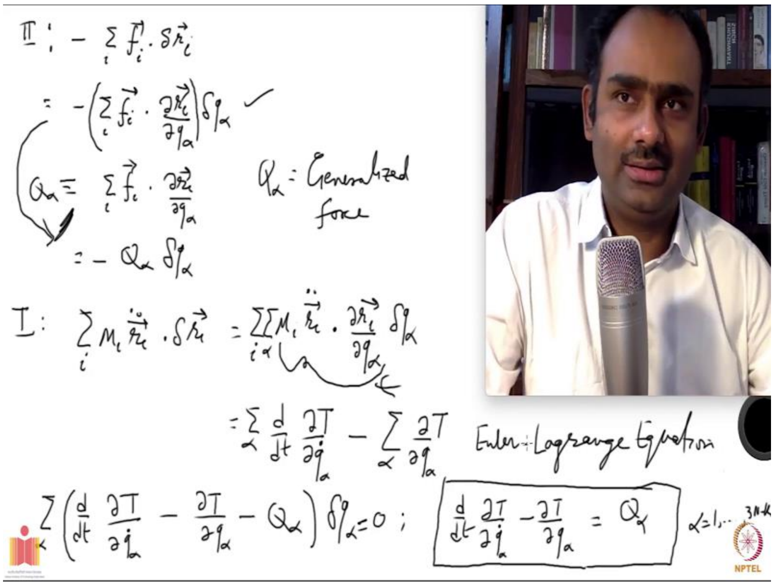

Here, del, d over dt del over delta q alpha dot times t. See again carefully d over dt del over del q alpha dot and the sigma i, this thing is the total kinetic energy of the system which I denote by t that is good and then you again have another one more term, summation over alpha del over del q alpha t. Check it again, this is del over del q alpha, the minus sign here, and T summation over all the particles. So, that is the total kinetic energy of the system.

Okay, that is nice. Now, if I combine the term one and two, I get the d'Alambert's principle in the following form. So, I get summation over alpha d over dt del t over del q alpha dot minus del t over del q alpha. And all this remember was dotted with the q alpha, this equal to 0. So, that is the form of the limit principle in generalized coordinates. And this is very nice. One already, when we wrote down the d'Alembert principle the constant forces are gone.

But now what we have achieved is we have written down a relation which involves only independent variations of generalized coordinates. Remember, we were using holonomic constraints. So, now I can put the coefficient of d q alpha to be 0 for each alpha, because this sum is 0, but because the variations delta q alpha are all independent of each other, this can hold true only if the individual coefficients also vanish.

So, from here, I get my desired result, d over dt delta t over delta q alpha dot. I think I went a little fast, I have included only the second time, the first time I have missed here. So, let me let me just write it down here. Minus, remember this is what you get from the first term. And then whatever I said just now about independent variations and being able to pick out this coefficient to be 0, which they are correct and I get my final result to be d over dt del t over del q alpha dot minus del t over del q, alpha equals q alpha.

This is nice. This is very nice. This is nice for several reasons. This is the equation of motion that the system has to satisfy. Each coordinate q alpha, will evolve according to these set of differential equations, these are not all and they are not necessarily uncoupled. They are all coupled equations, but that is what you have. And look at the T, the T in the numerator and in these derivatives, they pertain to the entire system.

It is not the kinetic energy of individual particles which is entering, it is the kinetic energy of the entire system as a whole that is entering in here. And if you look at the q alpha, that is also you see here, why it is not working? Yeah, yeah. When you are looking at q alpha, there is a summation over all the particles solid in there. So, it is also something not about individual particles, but the system as a whole which is entering the equation.

So, somehow these equations are sensitive to what the system is doing as a whole and is not so much caring about what individual particles are doing. So, from there we will be able to have a full description of our system by solving these equations. Now, what I will do in the next video is write down this in slightly different form. What I will do is I will take the q alpha, the generalized forces and write down using potential.

So, what I will say is, let us say the forces can be described by scalar potentials. And then I will rewrite this equation in a slightly different form, which is mainly more used. But nevertheless this is the equation and this we will call as, several such sets I will call as Euler Lagrange equations. So, this is Euler Lagrange equation. Okay, so we have done a lot of hard work in arriving at this result and that is quite nice. This will be our, almost our starting point now, except for the fact that we have to do a little bit for, something for the q alpha. Okay, see you then in the next video.