Introduction to Classical Mechanics
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Lecture 59
Hamiltonian Mechanics: Liouville's Theorem

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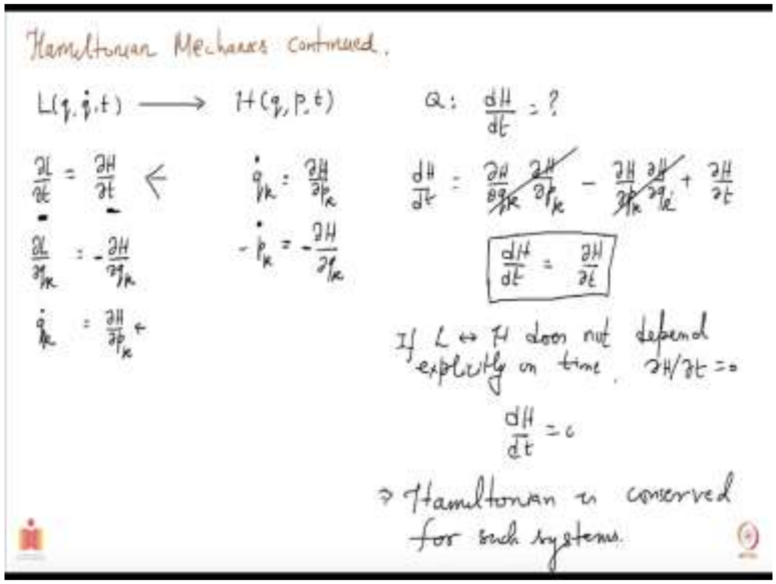

In the last video we saw that we can go from Lagrangian to Hamiltonian through a Legendre transformation. So, I start from L of q and q dot and t and I go to Hamiltonian which is a function of q, p and t. And as you may recall I was saying that we are going to treat q and p at equal footing and when we went from L to H we had the following relations which were true, one is delta L over delta t is same as delta H over delta t.

So, the partial derivatives of L and t with respect to time are equal. And also if you take the time (sorry) the derivative of L with respect to one of the coordinates then it is same as the derivative of the Hamiltonian with respect to the same coordinate except for a minus sign, so these two are identical. And also we had q dot to be equal to the derivative of Hamiltonian with respect to the corresponding momentum, conjugate momentum. Remember the p turned out to be the conjugate momentum that you had seen earlier.

So, this is what you these three you have these equalities you have purely arising from you going from L to H through Legendre transformations, they do not you did not use equations of motion for these to hold true. So, if you also in further demanded the coordinate q satisfies the Euler

Lagrange equations of motion from that you can arrive at the following equation of motion which was the following. So, if you look at p maybe not here, p if you look at the time derivative of one of the momenta, let us say pk then that was same as delta H over delta qk and there has to be a minus sign here.

So, whenever you are taking the derivative of H with respect to qk not whenever I mean these equations of motion there is a minus sign. And we also said that because you are treating q and p at the equal footing we will also treat this one as equation of motion. But remember this this one you did not arrive at by using any Euler Lagrange equations of motion but in the Hamiltonian formalism we are going to treat this at the equal footing to this one.

So, delta H over delta pk and these are your Hamilton's equations of motion that is fine now let us ask what is the total derivative of Hamiltonian with respect to time? So, I am asking what is dH over dT and it is easy to evaluate dH over dT is delta H over delta q and then you have dq over dt which is q dot and q dot is delta H over delta p, delta p, let me put a k, k and that is fine.

So, there is a summation over k implied here plus I will take now take a derivative of delta H over delta p, because it is also function of p and there are several of them, so let me call it pk and sum over all the case and then you have a p dot pk dot here, but pk dot is minus delta H over delta qk, so I put a minus here delta H over delta qk and then the third term will be the partial derivative of as H with respect to time because you are Hamilton could in principle depend explicitly on time.

But now these two terms are equal, you have delta H over delta pk, delta H over delta pk here, delta H over delta qk, delta H over delta qk, so these two cancel, so let me just cancel them and what you are left with is delta H over delta t. So, the result we get is the total derivative of Hamiltonian with respect to time equals the partial derivative of Hamiltonian with respect to the time.

Okay that is nice and it is a nice result you can say more because you also know that the partial derivative of H with respect to time is equal to delta L over delta t, now if your Lagrangian did not depend explicitly on time then your will Hamiltonian will also not depend explicitly on time that is what this equation is saying.

So, let us say the explicit time dependence on explicit time dependence is not there and if that is the case then your total time derivative of H will be 0. So, if L or equivalently H does not depend explicitly on time then your this thing is 0. And if that is the case then dH over dT is equal to 0 which means that Hamiltonian is conserved for such system.

And you may also recall that very early in this course we had encountered this quantity pq dot minus L which is what we are calling Hamiltonian here and this is what we call Jacobi's integral in very early lectures and we also saw that this quantity can be identified with total energy if certain conditions are met, it is not always the total energy, but under certain conditions it can be identified with total energy and let me just remind you what those conditions were, you can go back to those early lectures and see.

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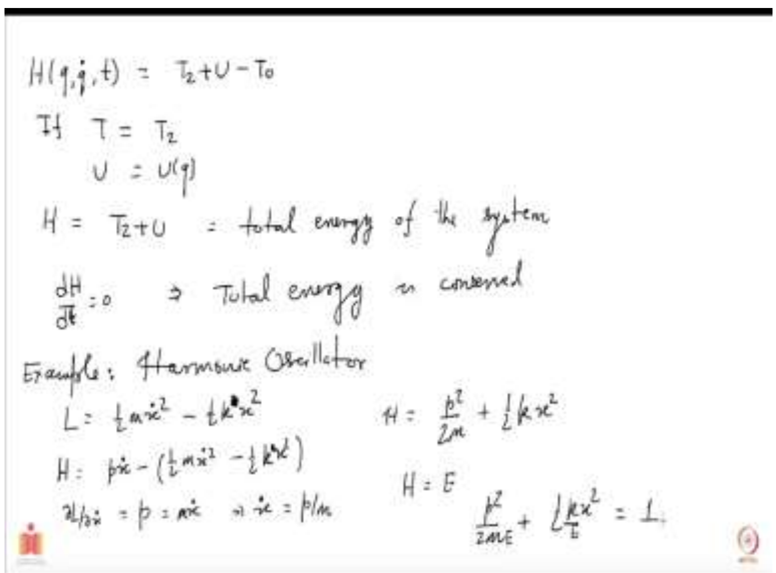

Remember that we showed I think we were calling it small h at that point of time, but the small h and let us call it capital H now but at that time it was being written as q, q dot and t, not as qpt. And we saw that this is T2 plus U minus T naught where the kinetic energy T you can write as T2 plus T1 plus T naught and the terms T1 and T naught are present only if your transformation to generalize coordinates involves time explicitly.

I hope you recall this and if that is the case, if they are explicitly time dependent transformations then you have T0 and T1 as non-zero quantities and this is what you get here then for the Hamiltonian. And in here U is a function of q only the coordinates only. Now, let us suppose that the kinetic energy T only equals to the quadratic piece, so it has q dot square, qk dot ql dot contracted with some ai ak al, only if this piece is there and if U is only a function of the coordinates not of the velocities, then our H becomes T2 plus U and we saw that this is what was equal to the total energy of the system.

So, let us say you have such a system where the Hamiltonian can be identified with total energy and then the above relation dH over dT is equal to 0, if that holds true would imply that total energy is conserved. Let us take a quick simple example of a harmonic oscillator and if you recall the Lagrangian is simply half mx dot square minus half k square x square.

So, if you now do a Legendre transformation then you know that the Hamiltonian will be px dot pq dot is your px dot minus and then you have your Lagrangian which is half mx dot square minus half and you also recall that your del L over del x dot is what is p which if you take the derivative you get mx dot which you can solve to write x dot to be p over m and your (())(12:22) square I think I am there is no square here.

So, that is fine and where were we, so with this I can write the Hamiltonian to be p square over 2m, I am just substituting for x dot p over m, so I get p square over 2m plus half kx square. So, that is the Hamiltonian and which you already knew because this is the total energy for the system. And the system is conservative it depends only, and the potential energy depends only on q, only on the x and this is anyway just has the quadratic piece in velocities, so you expect that the Hamiltonian will come out to be this.

And because this Lagrangian does not depend explicitly on time, remember the coordinates depend on T but that is that time dependence is implicit through the coordinates, but there is no explicit time dependence in here so partial derivative of H with respect to T is vanishing and which implies that the total derivative will also vanish with time, which is just that the total energy is conserved which you already know for harmonic oscillator which means that your H will be a constant.

So, let us call it E that constant energy and which implies that these piece is p square over 2m plus half k x square they will be equal to some constant E and if I divide by energy on both sides I get this that is good. Now, we can ask how things look like in phase space, you remember we introduce phase space, let me go back.

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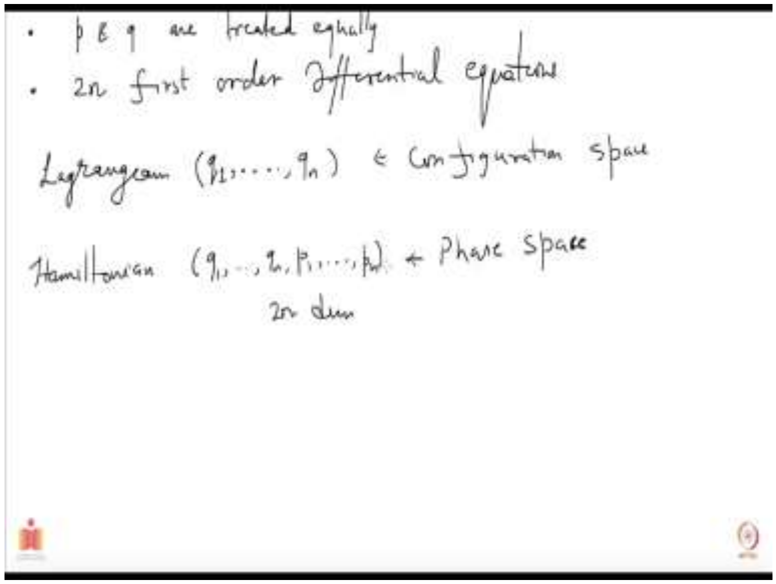

This was the two n-dimensional space in which you have access to be q1 q2 so and so forth and p1 p2 and so forth.

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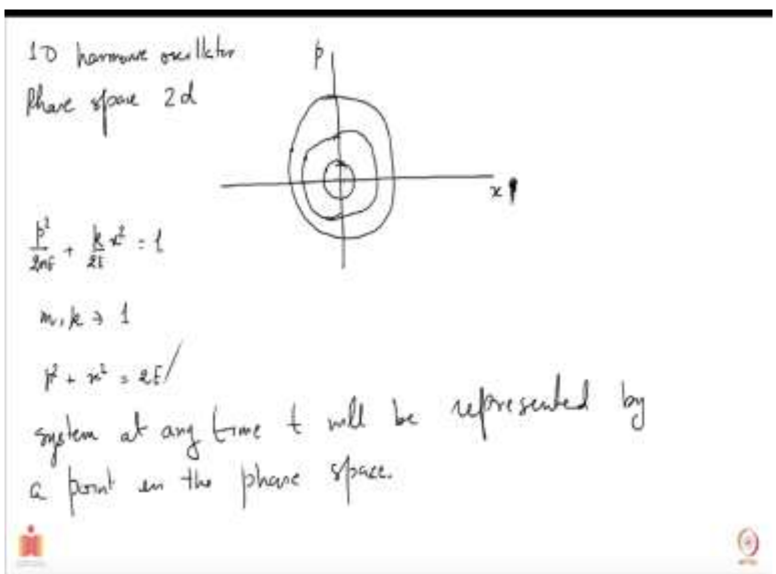

So, for harmonic oscillator 1 dimensional harmonic oscillator which we have just looked at the phase space is how many dimensions? Phase space is two dimensional, one for q and one p, so you have your let us say here q and p or maybe I think I called x and p, so let me write x. Now, we have the equation as p square over 2mE plus k over 2E x square is equal to 1, this is what I think we had.

Now, if you are given one oscillator and then for that oscillator this is the condition that has to be satisfied, so if you look at what this represent, it represents an ellipse in the xp plane but for simplicity let me put all the constants m and k to be 1 then you just have circles. So, you have p square plus x square is equal to 2E.

So, you get concentric circles and radius of the circle depends on your total energy of the system, treat them as circles, treat them as circles these are not very circular thing which I am drawing, let me draw again, so you have 1 circle here which this corresponds to some energy response this (())(17:15) responses to a little higher energy, so your system depending on how much total initial energy has been given to it, it will be moving along one of these circles.

So, your entire system is denoted by one point, at a given time it will be somewhere and as time changes that point is going to move. I hope that is clear because if you specify the coordinates at a given time all the coordinates, in this case it is only one and all the momenta and then that represents one point in your 2 n-dimensional space.

So, your system is going to be represented by a single point in phase space, let me write this down anyway, so system at any time t will be represented by a point in the phase space. That is clear. So, now as time changes that point is going to move with time, so that point is going to flow. And what I have drawn here as these circles is that if you if your energy such that your point is here then it is going to move along this circle for this one dimensional harmonic oscillator.

And if your energy was different and it was such that it is here and it is going to move on this circle. So, that is what our phase space will look like the trajectories of this point or the system will look like in the phase space that is good. Now, I am going to prove a very nice theorem and the theorem is the following, actually before I say the theorem let us ask the following.

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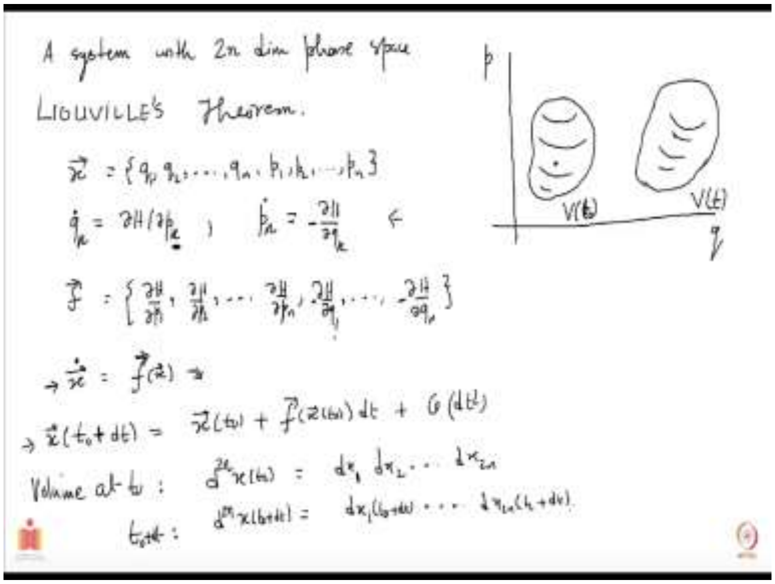

So, let us say I have some let us say we are given a system with two n-dimensional phase space, meaning it has n generalize coordinates and correspondingly and generalized momenta and the particle and this the system will be denoted by a point in that phase space by a point in the phase space, so I do not wish to draw when I cannot not that I wish but I cannot draw a two n-dimensional space but I can draw two dimensional space, so I am going to draw it like this.

So, you can imagine that q stands for all these coordinates, you can think that this is a two dimensions two n-dimensional space, so you have all q1, q2 and qn in here perpendicular to all these and similarly p1, p2 and so forth all these coordinates are here. An imagined a patch a region, so a region not a patch because I am thinking of two n-dimensions, so imagine a region, here and this is going to occupy some volume, so just to give it a little bit of 3 dimensional look or whatever, now let me draw this curly line, so you imagine it, it encloses some volume.

So, right now I am not trying to even make a any connection with system or whatever, your system is given, so each point corresponds to one system, but forget about all that, that how many systems are there and all these questions just look this as a geometrical problem, we are given some volume enclosed by this, so the region let us call it I think I want to call it some or something whatever and we also tell the equations of motion.

So, each point here is going to evolve in time according to the Hamilton's equations of motion. So, it is going to go somewhere as time goes on and what we want to ask is if I start with some volume here, so if I look at all these points, all these points are going to move in time and goes to somewhere else and I want to ask how this volume is going to change with time.

So, if I wait for some time and after a while if this initial volume enclosed here was V0 or V at time 0 t0 what it becomes after some time has passed, that is what we want to ask, let us say it becomes like this. So, I am not saying the shape will be similar or same, shape changes and we want to know whether it has increased or decreased or what has happened at a different time t, so you go from t naught to t and this is what we want to ask.

And what we are really looking at now is what is called Liouville's theorem I will tell you at the end what the theorem is, VI double L Liouvilles theorem. So, I hope the question is clear and let me just create some notation. So, I want to define this entire set of coordinates q1, q2 so and so forth qn and then you have p1, p2 as pn so you have all the 2n coordinates here, let me call these collectively as x and I put a vector as x just to mean all these that it is a multi-component thing.

And then also I have my Hamilton's equations which let me write down qk dot is delta H over delta pk and you have pk dot as delta H over delta qk and if I am differentiating H with q I remember that there is a minus sign here. So, I define a vector f to be the following, so delta H over delta p1, delta H over delta p2 so and so forth, delta H over delta pn, then you have delta H over delta q1 with the minus sign, so I am what I am doing is I am just listing these right hand side parts here and here along this.

And you go further and delta H over delta qn again there is a minus sign I put here, this entire set of quantities I denote by vector f. So, this equations of motion Hamilton's equations of motion here using the notation which we have written down just now can be written as x dot is equal to delta f over sorry is equal to f. So, if I look at for example what is x1 dot it will be f1 which is delta H over delta p1.

And what is x1? It is q1, so q1 dot, q1 dot should be delta H over delta p1 which is correct, which is what you have it here and let is look at this one this is let us say x of n plus 1, x of n plus 1 would be p1 and p1 dot should be f of n plus 1 which is minus delta H over delta q1 which is correct. So, all I have done is written down Hamilton's equation in this form beyond that I have not done, there is no content in here it is just notation.

So, these are our 2n first order differential equations in time and then I can write x at any time I mean at time t naught plus dt, so I am starting with time t naught and I look at what happens at after small interval of time dt and you can write down this as x of t naught plus f of x at t naught times that small time interval dt and then you will have all the higher order corrections, this is just this equation written or solved near t naught, that is fine that is good.

So, what is contained in here is the information about how the points are going to move in time, that is what is contained in here which is same as this equation. Now, let us first before we are able to ask how things how the volume is going to change with time I should first write down the expression of the volume and write down how volume changes.

So, the volume is the following so let me here it write down here itself, so volume at, volume at t naught is what is d2nx at t naught, which is nothing but dx1, I am dropping the t naught now dx2, so and so forth, dx2n, which is just dq1 dq2 up to dqn and then dp1 dp2 up to dpn, that is what the volume is at time t0, you can put t0 here as the arguments.

And then at time t naught plus dt you will have d2xn of t naught plus dt which is again the same thing, similar thing dx2n t naught plus dt. So, that is fine, that is that was easy. Now, how are the two volumes related this one and this one? You know how they are related and the relation is this you take you go let me just show here, you see the x is going to change like this, so you have a transformation of your x from this to this one which is given by this relation. So, all you have to do is find the Jacobian of transformation.

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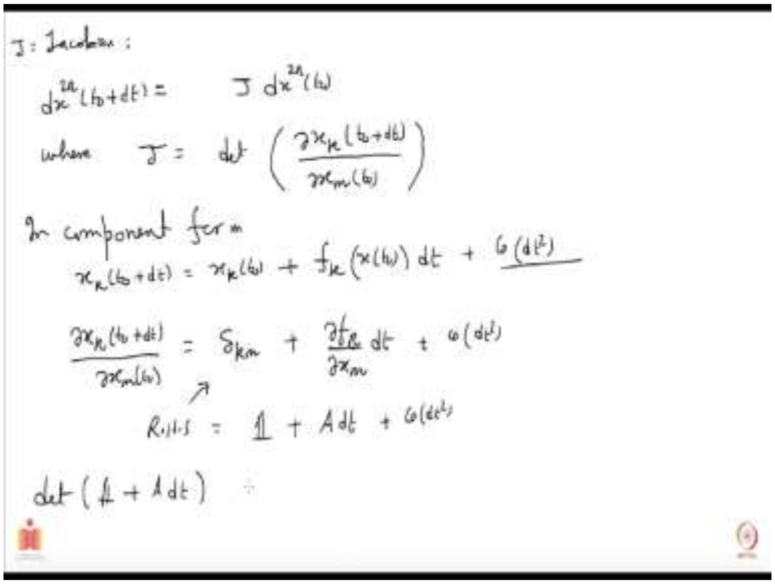

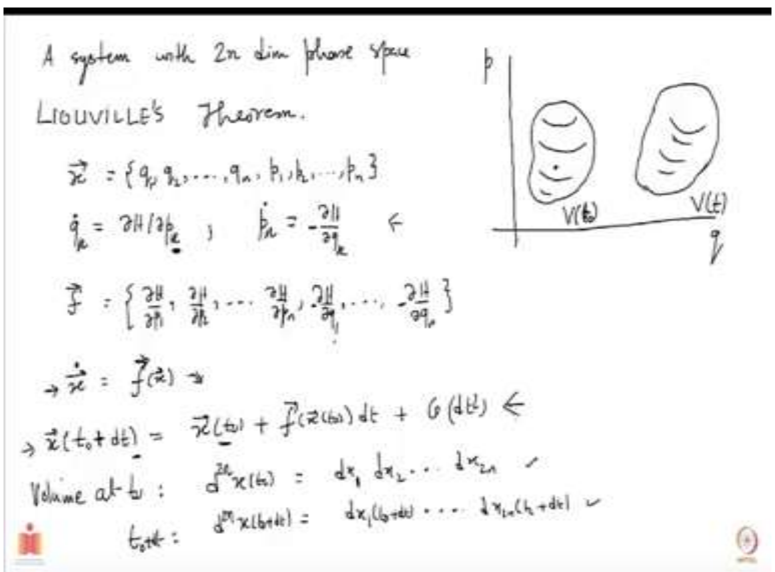

So, all you have to do is find the Jacobian and let us denote Jacobian by J, then you know that the two volumes are related by the following dx2n at t naught plus dt is equal to dx2n at t naught and the change is this, it is you have a factor of the Jacobian which comes in here and if you recall where the Jacobian is just the determinant of this matrix dxk t naught plus dt over dxn, let me put m here, so you have to just take these you have to construct the matrix which contains all these partial derivatives.

So and k runs from 1 to 2n and also runs to from 1 to 2n, so this is 2n cross 2n-dimensional matrix and you have to find the determinant of this and then multiply it here which gives you the new volume. So, now the task is to evaluate the Jacobian, the moment I know the Jacobian I know the answer and I will have to just integrate over that entire region to generate the volume, right now these are differential elements of the volume, so let me do that, let me try to evaluate the Jacobian and of course to do that you have to use this piece of information that which knows how things are changing with time.

So, let me write this equation here this vector equation x as xt naught plus f times dt in the component form. So, in component form you have x of k t naught plus dt is equal to x of k at t naught plus f of k x at t naught dt and you have higher order terms also which anyway we are going to drop because we are looking at infernal decimal dti I can drop it here itself it is unnecessary to carry this.

So, let us calculate this derivative delta xk delta xm, del xk over del xm at t naught, clearly this will give you delta km unless k is equal to n this guy is 0 only when knm are equal this is going to give you 1 plus the derivative of f with respect to xm, so delta fk over delta xm, I will drop this x of t naught for ease of writing and clarity but it is there that argument is there and you have a dt here, let me keep writing this.

Now, what you have if I want to write the same thing using matrix notation then the right hand side this side you can write as identity matrix plus let me think I want to put dt here I am that is correct, let me denote this matrix by matrix A, so the matrix A has as its components delta fk over delta xm times dt you have and you have other order dt square terms.

So, what we want to do is look at the determinant of this, so we are interested in finding out the determinant of 1 plus A dt and we are we want the result only up to dt, we are not interested in higher-order contributions. Now, that is easy, actually that is not very difficult, all you need to do is there are many ways in which you can do this, but you can use the following formula which you should be familiar with.

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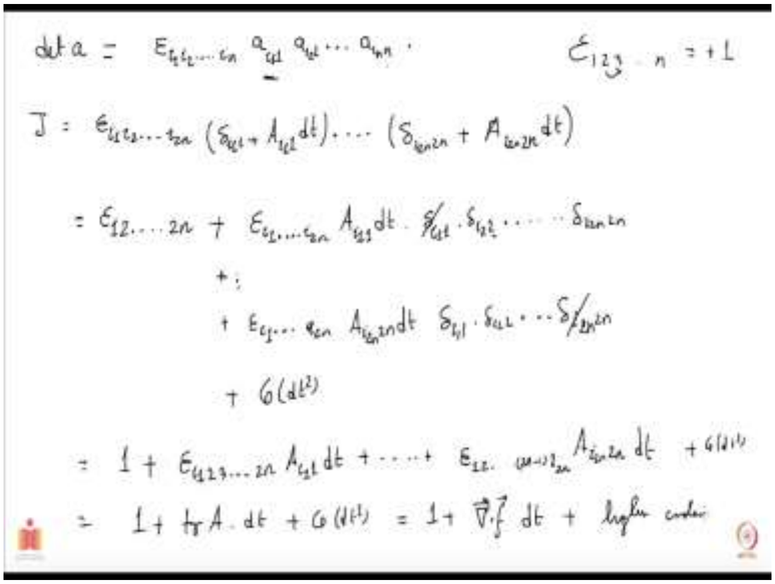

Let us say you have a as A matrix, then the determinant of that matrix can be written as epsilon i1 where epsilon is Levi-Civita tensor let me be little bit more clear epsilon i1 i2 in times ai11 ai1 i22 ain n. So, that is the result and you know the Levi-Civita tensor is or this and total in the symmetric matrix is 1 if you have i this is so epsilon 1 2 3 so and so forth n is defined to be plus 1 and whenever you interchange any two indices you pick up a minus sign that is how this option is defined and ai11 these are the matrix elements of the matrix A.

So, that is what I am going to use here. So, let us see what my jacobian is now, my J is epsilon i1 i2 and it will go up to 2n because you have two n-dimensional 2n cross 2n dimension matrix and then here you have your 1 plus Adt, so in component form this becomes a1 the first sorry corresponding to this you have to write delta i11 plus capital Ai11 dt that is what you will have here and then you have all those factors and for this last one you will have delta i2n2n plus Ai2n2n dt

Now, let us collect terms of order I mean the terms which are not proportional to dt and then the terms which are proportional to dt and then we will drop the terms which are higher powers in dt that is what I am going to do. So, the terms which do not involve any dt are going to come from multiplying this epsilon with the deltas without involving any of the As.

So, if you have this epsilon multiplying all the deltas these deltas will force the i but this delta will force the i1 to be 1 in here, so it will become epsilon 1, then i2 will be forced to 2, so it will become epsilon 1 2 and similarly this thing will be force i2 into be 2n, so one term you are going to get is epsilon 1 to n so forth up to 2n, so that is the term without any dt plus now you are going to generate the terms proportional to dt in many ways.

So, if you pick up this factor from this term and all the others are deltas then you are going to have a term proportional to dt. Similarly, if you pick from this one a delta and you pick a21 a2 ai22 and all others to be delta then you get another term which is proportional to dt, so let me write that down, so you are going to generate terms like this, let us reduce some labour in writing, so you are going to get Ai1t sorry Ai11 dt and all the others will be just delta.

So, you will have delta I am just writing delta i11 also, but it is not there I will just strike it off, i22 and similarly you proceed and go up to i2n2n and this factor is missing in this one. And these kind of terms let me if I was a bit more careful I could have, so I will just pull out this term because you see now you understand you are going to the next term which I am going to write will again have an epsilon it will have a corresponding A times dt and the same piece with this being removed and you will keep this maybe let me I should let me write down.

Plus epsilon i1 i2n and then you will have Ai2n2n dt and all these deltas, i subscript 2n2n, so this time, and the last term this will be gone and the all the other terms that you are going to generate will involve multiplication of at least two pieces of As which will bring with them at least a second power of dt, so we are going to drop all those, so this is not there, so all those other terms will be higher orders in the change in time.

And now that is easy your first term is just one which you also expect because to the lowest order there is no change right because the time you are looking at the volume change after and infinite decimal time, so you expect that Jacobian would be to the lowest order 1, so that is correct, that is good, you have got of the 1 here plus we expect something proportional to dt and let us look at here, these all these deltas will force all the except for i1 all the other indices to be equal to what you have here.

So, you will have epsilon i1 2 3 4 up to 2n and then you have ai11 dt, so you will get let me write down epsilon i11 2, sorry not 2, 2 3 so forth to 2n with Ai11 dt and all such terms are here let me I down anyway, epsilon this will be 1 2 so forth up to 2n minus 1 and then you will have i subscript 2naAnd you will have Ai2n2n dt, plus higher order terms.

Now, you see if this will be non-zero only if i1 is 1, if i1 is 2 or 3 or something it is going to be 0 and you have a summation over i1 because it is summation convention is still here, so the only possibility for this is to be 1 and that will give you a positive sign because this will be fine epsilon 123 and that will force this one to be A11.

So, it is 1 plus this will be A11 dt, next one will be A22 dt and similarly A2n2n dt. So, you have trace of A times dt, the sum of all the diagonal entries is the trace of this matrix. A. That is good, now this is just 1 plus let me write down now you remember your capital A was the matrix this one this matrix, now we are looking at the trace of this so you have delta fk over delta xk and you have a summation over k which is just gradient of sorry divergence of f.

So, I can write as divergence of f, remember divergence is del f by del x1 plus del f by del x2 which is just what is appearing as traces for you here. And dt is still there plus higher order.

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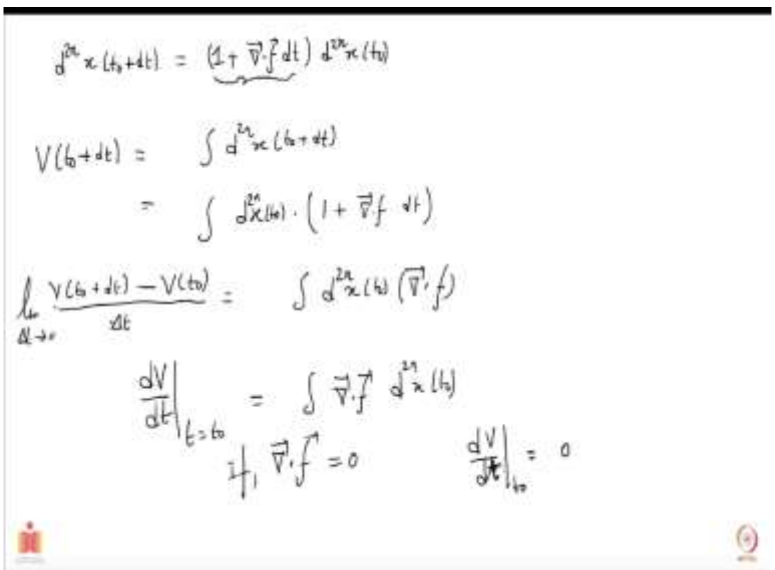

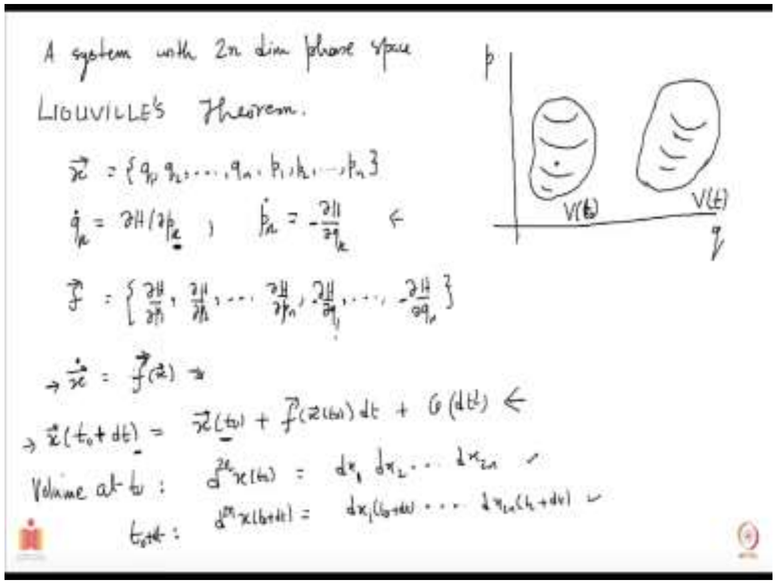

So, we are almost there we have shown that if you look at the differential volume element in this 2n dimensional space sorry t naught plus dt then it becomes 1 plus divergence of f dt, I am going to drop now all the higher-order pieces, that is how you transform. So, that is what your Jacobian is, this is your Jacobian. Now, let us look at the volume, so volume at t naught plus dt it is just obtained by integrating all the differential volume element and this as you have determined is just d2x and then d2nx at t naught and then you have your Jacobian which is 1 plus divergence of f times dt.

So, let me split it into two integrals, so one will have this one multiplied in here and this will give you the volume at time t naught, let me take it to the left hand side, it will have a minus sign then and then the second term will be proportional to dt, so if I divide by the time dt I should have written delta Tt it would have made more sense, so anyhow, so you have let me call it to make it look more appropriate, so that is what I have.

So, instead of dt let us put delta t and then put delta t to the left and take that limit. Then this will be only this piece remains here, so you have the integral over the volume element of divergence of f. So, dV over dt evaluated at time t equal to t naught is integral divergence of f times d2nx t0 Now, if so happens that your f which you define here, where is it? Yes, these f, which are on the right hand side of the differential equation, then if it so happens that the f which you have here is divergence less, so let us say I assume that divergence of f is equal to 0.

If that is true, then dV over dt at any time t naught will be 0, or at (())(48:33) all times t naught at all times will be 0, so the volume is not going to change with time if your f was divergence less. So, that is the generic conclusion for any system of ordinary differential equations but now let us look at what we have for our specific case of Hamilton's equations. So, let us see what we have for divergence of f.

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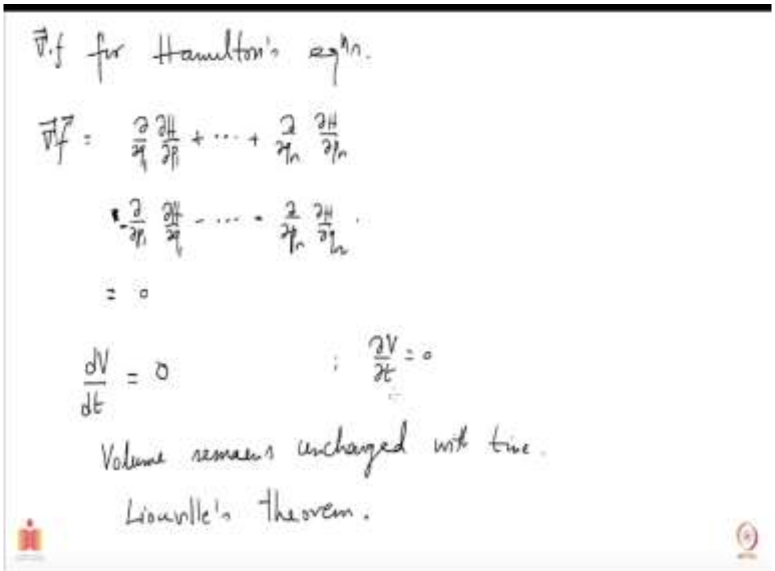

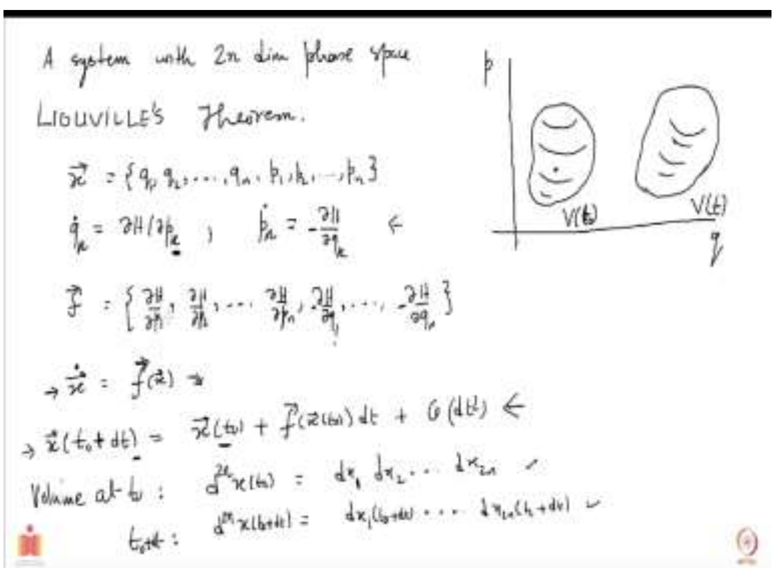

So, divergence of f for Hamilton's equations, let us see, so your divergence of f is, let us recall, delta by delta q1 that is first coordinate and then recall what del f sorry f was, f was your, where is it? delta H over delta p1, let us write it here. Similarly, you keep on going up to qn and you have delta H over delta pn, then you have more terms delta over delta p1 and let us recall what is fn plus 1 was, fn plus 1 was minus delta H over delta q1.

So, you have instead of a plus here a minus delta H over delta q1 and then you continue and del p and delta H over delta qn, that is all you have. So, this is H is being differentiated with respect to q1 and p1 and here also the same thing and the order does not matter and is the relative sign, so these two will give you 0, similarly all the other terms will also cancel pairwise and will leave you 0, so indeed for our Hamilton's equation we have divergence equal to 0 which means that whatever volume you take in the phase space it is going to evolve with time in such a manner that this time derivative is 0, meaning the volume is going to remain unchanged, remains unchanged with time.

The shape of the volume may change, it will change in general, but the volume contained all the points which move the volume which they are going to span out it will still be the same as the original volume and this is what is Louisville is theorem, that is nice cheering which is very useful in statistical mechanics when you talk about different and symbols, but here we will not even worried about defining and symbol there is no need in here.

But this is a very useful theorem which you will use for example in statistical mechanics, I think this is all that I wanted to say here I know that I can make a remark, I hope this is clear that this is the total time derivative of volume and you should contrast it with the partial time derivative. This statement where the partial time derivative is 0 is different from the total time derivative.

So, when you are talking about a total time derivative in addition to time changing the coordinates are also changing, because d over dt is partial derivative with time plus derivatives with respect to all the other coordinates, so you are really moving together with all those points as they move and you are then asking about the volume whether it changes or not.

But if you are just looking at this partial derivative, then you are staying put at one place and you are just looking at the time changing, as time passes and then of course the volume will be changing there it will be reducing for example. So, what happens there locally is different from what happens when you are flowing with the points. So, we are talking about this piece, we are not talking about just time passing at the same place. Anyhow, so this is the Louisville’s theorem which is quite useful and let us meet in the next video too carry this subject further.