Introduction to Classical Mechanics
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Lecture 58
Hamiltonian Mechanics: Hamilton's equations of motion

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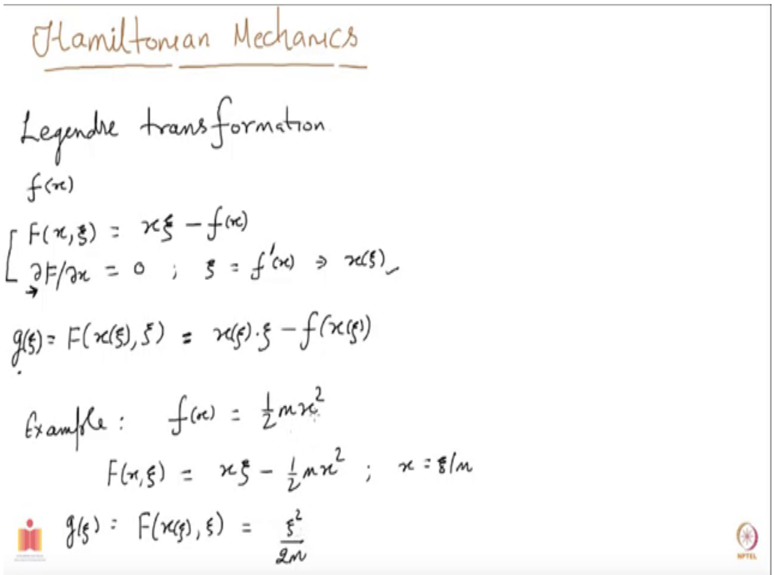

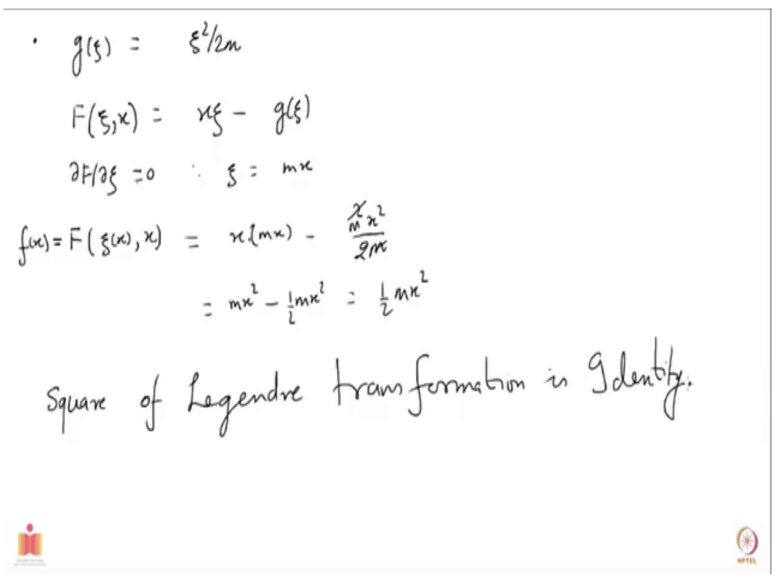

We will begin the last part of this course which is Hamiltonian Mechanics. The Hamiltonian point of view helps to gain inside into the system which is not a available in Lagrangian mechanics that is one of the reasons why one would like to do Hamiltonian mechanics. And also, the language which Hamiltonian mechanics provides is very suitable for building other kinds of formalisms for example quantum mechanics and we will see why this formalism is so useful. So, let us begin by starting with something called Legendre Transformation which you may have already encountered before. So, let me first see what a Legendre is.

Suppose you are given a function which is a function of a single variable x and I define a new function denoted by capital F and which is a function of two arguments x and psi, psi is a new variable which I am introducing. And this capital F is defined by the following. So, it is x psi minus f of x, so that is the definition.

Now we look at the stationary point of f with respect to x that is I am looking for this derivative to be 0. And this will if you take the derivative of this you will get psi minus f prime of x, so let me did here which implies psi is equal to f prime of x, you take a derivative this goes away and you have a derivative with respect to x on f.

So, you solve this one for x so the from here you will get x as a function of psi, so this solution you will get from this condition, the stationary point condition. So, you take this x which is now express in terms of psi and put back in the capital F so will get F x as a function of psi, psi so this new quantity which is only a function of psi now I call it, call as g of psi.

So this is what is called a Legendre transformation, so you have gone from the function f of x to new function g of psi through this transformation which you have here and this transformation is called Legendre transformation. So what is it; it is now x which is really express in terms of psi time you already had a psi here minus f of x of psi, so that is how Legendre transformation is defined.

Now to gain more insight into this transformation what geometrical insight I would encourage you to look into the book by Arnold and also the book by (())(04:03) these two references or any other references you can look at and you will also see that it is not true that you can always find the transformation, the transformation may not exists, so I will not go into those details but I will encourage you to have a look at these books.

Now, I will give you a simple example for a Legendre transformation. Imagine I take a function to be half m x square, now I want to arrive at g of psi. So, the first step is you construct capital F which is, so you are introducing new variable psi which the capital F is x psi minus half m x square. Now, if you take the derivative of capital F and put it equal to 0, you will get the value of x to be psi over m, so this is what I found now.

Now, you construct g of psi which is basically the capital F where x is given its value in terms of psi, so this becomes you have x is your psi times psi becomes psi square over m minus half m and here instead of x you have again psi square over m square which leaves you with psi square over 2 m. So, this is the Legendre transformation of half m x square psi square over 2 m that is the answer.

Now, this is another property of Legendre transformation which I would like to tell you is that if you do the Legendre transformation twice mean if you look at the square of the Legendre transformation it equals identity meaning if I take this g of psi and again do a Legendre transformation on it and try to find the function I will get a half m x square, let us see.

So, I have psi square over 2 m, let us start from before that let me, so g of psi is psi square over 2 m. So you what you do is we construct function F, let us follow the notation x, psi so it is a function of psi and I want to introduce just like here the function was function of x and I introduced a new variable psi, here I am have a function of psi I want to introduce a new variable let me call it x which will be now again as before x times psi minus g of psi.

And we again look at the condition del f over del psi equal to 0 because you are starting with psi. So this will give you psi equal to m x, so you can express your variable in terms of your new variable x and now let us define our Legendre transformation to be F of psi I have to express in terms of x and let us call whatever new quantity I am going to get let me just give it a name F of x because I am anticipating it.

It will come out to be f, small f but anyhow let us calculate, so this capital F was x times psi, so x and psi is what; m x from here minus g of psi, psi is m x so you get m square x square over 2 m. So, you get what, you get m x square minus m x square over 2 which is half m x square. And that is why that is what you started with half m x square, so you see that indeed if you do it twice this transformation you get back to the original thing meaning the transformation square is identity. Let me write it down, so that is the Legendre transformation for one variable. Now, if there are several variables which you want to transform.

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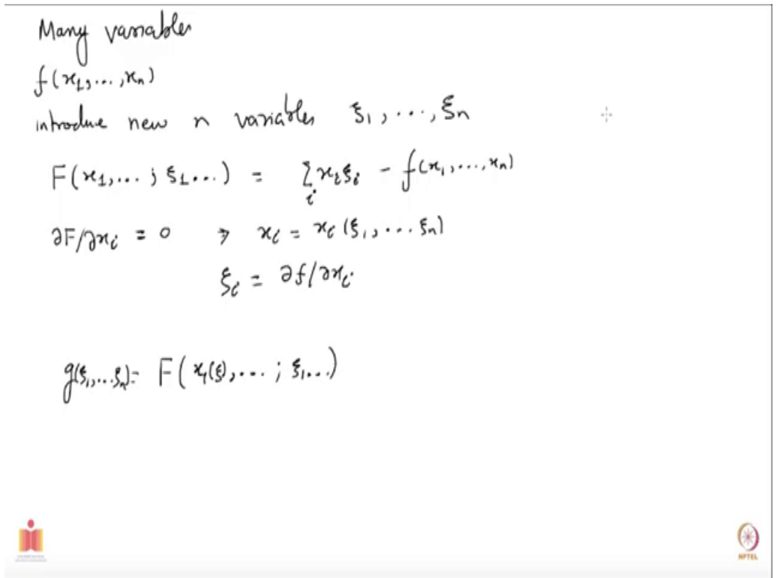

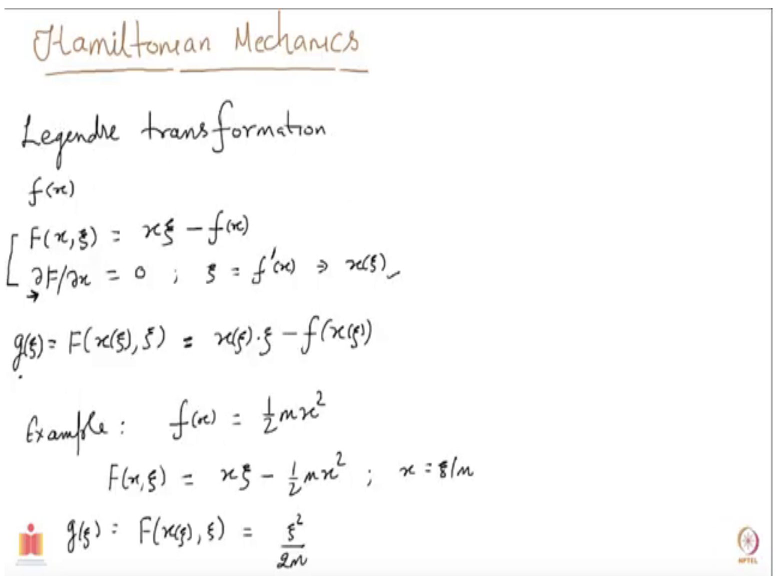

So, let us look at the case of many variables, so imagine you are given a function f of x1 to xn there are n variables, so as before we will let us go back, I will introduce psi is for each of the x and I will introduce an equivalent of capital F here. So that is what I have to do. So I have introduced new variables; new n variables psi 1 so and so forth psi n and then I again define a capital F which is a function of all these n’s and also all these psi’s, did I put it this way before; x, psi so that is good. And this will be what? Xi psi I and you sum over all the I’s minus your function and as before the condition of finding psi would be delta f over delta xi equal to 0.

So you look at the partial derivatives to be 0 so that is the, that is how you will get the stationary point of, not a stationary point with respect to x for capital F and so you then whatever you get you use it to solve for the xi in terms of the psi Is which also implies as before that your xi sorry, psi I if you take the derivative of this thing and put to 0 this will go away you will get only psi I delta f over delta xi, so from here you can get the psi I and then you can invert it to get the x.

And then again as before we (little bit more space) we take the capital F and write x1 in terms of all the psi's let me put only one psi to avoid the clutter but you understand it means all the psi and this we call now g of psi 1 to psi n and this is what provides you the Fourier transform (not Fourier) Legendre transform of this function f of x.

So, it follows the same thing there is no nothing new here and again for more details you refer to the reference which I mentioned few minutes ago. Now, what I will do is I will take the Lagrangian and I am going to do a Legendre transformation, so the first thing is your Lagrangian depends on q and q dot and time I should ask which variable I am going to change, so I am going to change the q dot.

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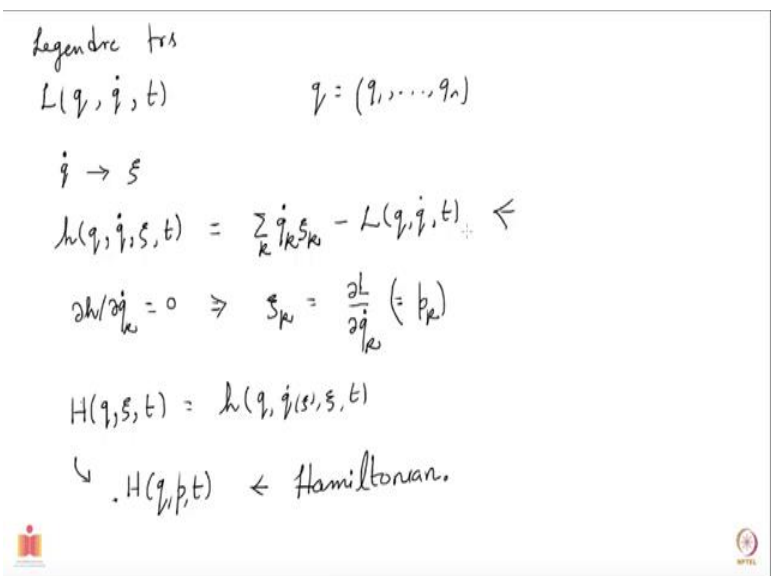

So, let us say I take my Lagrangian which is L q, q dot and t when I am writing q I mean all the q1 to qn, let us say the system is n dimensional and the q represents all of the entire set. Now, what I want to do is I want to go from q dot to psi through Legendre transformation, so am I going to define my Legendre transformation?

So, again just like you had a capital F let me introduce a capital F here or just to be more closer to the notation let me introduce small h, q I am not going to touch, q dot is going to be transform into psi so the new thing will be a function of q dot n psi and t I am not going to do anything, that is not a coordinate that is a parameter.

So, this function which is the same thing as your capital F here will be defined as q dot qk because there are several of them and psi k and you have to sum over all the k minus your Lagrangian minus the original function which is ofcourse function of q, q dot and t that is good now what you should do is take the derivatives of h with respect to these individual q dots and put them to 0 to determine your psi.

So, we put impose the condition which is part of the definition of Legendre transformation delta h over delta q k dot. Remember we are doing transformation with respect to this variable not q, q dot. So del h over del q k dot I have to put to 0 which implies that psi k because when I take the derivative only psi will remain psi k is equal to delta l over delta q k dot.

Now I will define a function which will be equivalent of this g which is just the same capital F with the coordinates replaced by the new variables psi. I mean coordinates expressed in new variables psi. So instead of calling it g I call it capital H and it will depend only on now psi because q dots will be gone but the t and q which were there before which we have not touched are still there.

And what is this? It is just your small h which q, q dot of psi, psi t just like before. So that is your Legendre transformation but now you note that del l over del q dot was just the definition of conjugate momentum with respect to the coordinate q that is what is conjugate momentum, momentum conjugate to the coordinate q, so this says p of k which means if I want to be using this piece of information then h is really a function of q p and t and this is called Hamiltonian.

So just to summarize what I have done is I have taken a Lagrangian and done a Legendre transformation and obtained a new function which is function of the coordinates the conjugate momentum and ofcourse possibly time that is good but now I want to ask before I ask let me do something more, let me do something more yes; so let us look at now the total derivative of this new function which you have obtained and which we are saying that it I am going to treat as a function of q and p. It is a function of q and p and t and I will use this relation, this definition and also this is your defining relation and write the total derivatives in two ways and then compare the coefficients, so let me do that.

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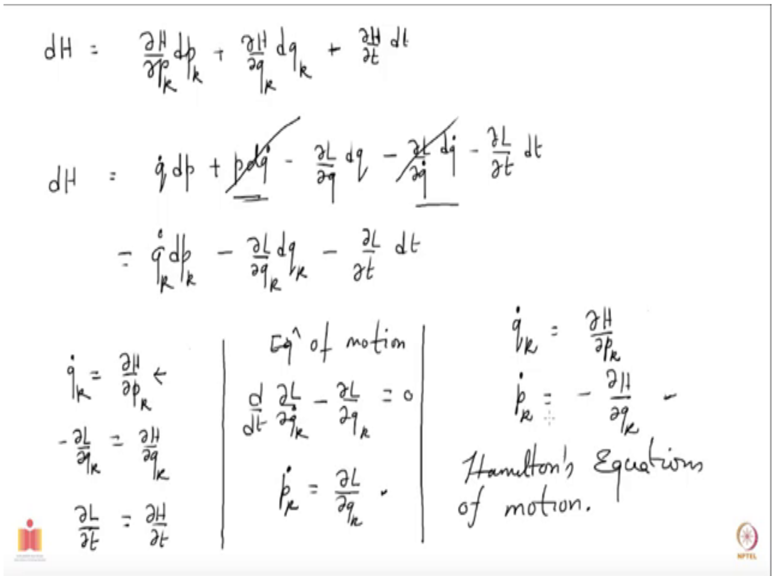

So, first I take the total derivative of h, so it is simply delta h del h over del p dp plus del h over del q dq plus del h over del t dt and you have to sum over all the coordinates and momenta, this is a summation convention implied here. But if I use this relation that h is q minus l see this small h is just the capital H it is just you have expressed the q dots in terms of psi otherwise it is the same thing. So I take this capital H and take the total derivative.

So, our dH is now let me go here yes; so I get q dot dp plus p dq dot minus del l over del q dq minus del l over del q dot dq dot and because Lagrangian we also depend explicitly on time I have del over del t dt. And the same reason why I had this term because Hamiltonian could explicitly depend on time.

So, this I have constructed using this one and remember psi k was (sorry) pk that is fine. Now you notice that p q dot p dq dot and del l over del q dot dq dot I can combine because they both have dq dots but p is del l over del q dots so that terms these two terms cancel and I get q dot dp let me put the index k minus del l over del q dq again let us put the index k minus del l over del t dt.

So, I can now just read off these coefficients and write down using the definition from which this expression I have obtained that q dot is may be q dot is delta h over delta p let us see. So, q dot is delta h over del p because this have to be the same then minus del l over del q k this p is equal to del h over del q k. Then the last term these two coefficients should of course be equal again so we have delta h over delta t is same as the delta l over delta t which means that if Lagrangian does not depend on time explicitly the Hamiltonian will also not depend explicitly on time.

Okay that is good, so all we have done till now is just Legendre transformation and we have expressed these coefficients in terms of these ones beyond that we have done nothing there is no information of the dynamics here. The dynamics is governed by the equations of motion and what we can ask is how do equations of motion look like when you are using Hamiltonian formalism, so what I am going to do is now bring in the equations of motion because let us do it and then see.

So, let us write down the equations of motion. Now, your equations of motion which are all your Euler Lagrange equation of motion which is just d over dt del l over del q dot minus del l over del q is equal to 0 for each of the coordinates but you know that del l over del q dot is p is the conjugate momentum and your dt gives you a p dot I have just put these p on the right hand side, so this is a equation of motion.

Now, what I do is (where is it?) yes here it is del l over del q k is here so I get the equation of motion in the language of Hamiltonian to be the following. Your p k dot which is del l over del q becomes minus del h over del q k. So, if you take a partial derivative of Hamiltonian with respect to q you get the p, p dot. And earlier when you took the partial derivative of l with respect to q you got p dot so that is one thing.

Second is yes, here is, second is let me write it down and then I will show you this is just this equation which we already wrote, so I am having q k dot to be del h over del p k. So, on the right hand side you have quantity is and the derivatives are with respect to the arguments of h ofcourse, so you have q and p as arguments and on the left hand side you get q dot and p dot.

And these are your Hamilton’s equations of motion, these are called so you may think that this was the equation of motion the Lagrangian formalism this is what you have and this thing has translated into this, so you may want to regard only this piece as the equation of motion and not this one but note that in this Hamiltonian formalism our q and p are on the same footing we want to treat them equally and I will show you later that we can okay let me wait for that remark.

So that is one thing, so we will treat both the equations as the equations of motion in the Hamiltonian formalism and not just this one which you might think because it comes for him here, so that is point number 1, point number 2 that your equations here if you look at this was second order differential equations, if you look at this equations but these ones are not second order differential equations, these are first order differential equations if your q and p are variables and you have only first order derivatives is involved in this equations.

Okay, so that is one thing, so our equations are now first order but now you have not n equations which you have here, you had for a system of n degrees of freedom you had n equations here, here you have a total of 2 n equations so because these are n equations and these are also n equations so a total of 2 n equations, so let me write it down.

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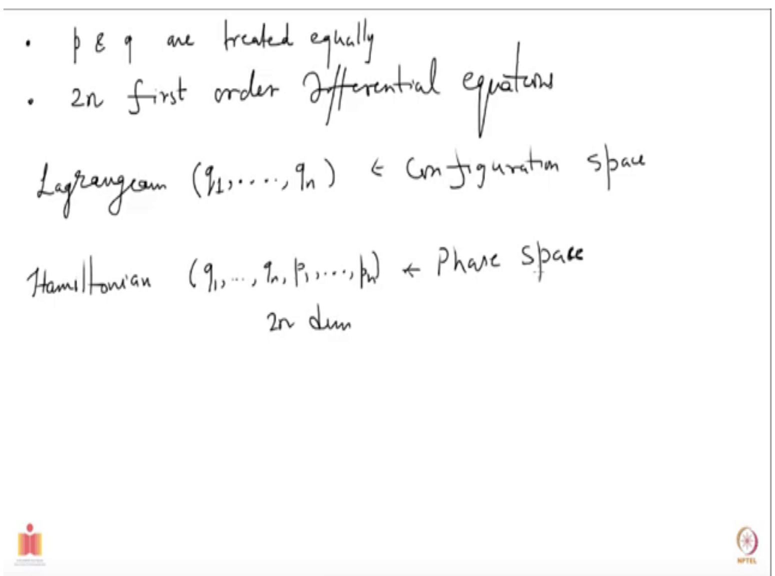

Now, we have p and q at the same footing as I am saying are treated equally, so I have now 2 n first order differential equations, so as appose to your Lagrangian mechanics where you use the configuration space of these coordinates which is n dimensional where you use configuration space in Hamiltonian mechanics you are going to use not just n dimensional thing but rather 2 n dimensional things, 2 n dimensional space.

Hamiltonian you are going to use q1 so and so forth qn and then you also have all the conjugate momenta here. So, this is a 2 n dimensional space and this is called phase space, so the system is going to move in the phase space. Let us find, now you may wonder whether we can obtain these equations of motion in the case of Hamiltonian, this Hamilton's equations of motion from a variational principle just like we could derive the Euler Lagrange equations of motion you starting from a variational principle that you take the action which is just the integral of Lagrangian over time.

And you look for extremum of this where you vary the qs their paths and then you find the extremum and from there you find the equations of motion, so the extremum is going to satisfy these equations of motion. Can a similar thing be done for a Hamiltonian case? Can we start with a variational principle and arrive directly here rather than going to all the Legendre transformations, so that is one thing you can ask and that is what we are going to do in the next video.