Introduction to Classical Mechanics
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Lecture 57
Cartesian Tensors

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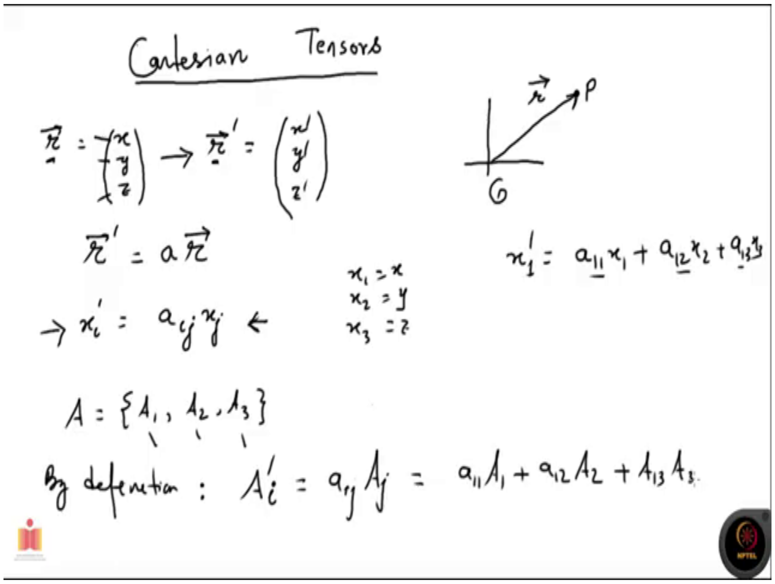

We will have a quick review of Cartesian Tensors today. And I will be specifically interested in tensors and the rotations. There are something which is what we used in for example rigid body motion. So, though it may be already familiar to many of you but some of you may not have encountered this already so I will make a quick review of this topic.

So suppose you have some coordinates system, Cartesian coordinates system and this is the origin O of the system and there is some point p here and that point p okay if I bring it closer it will be better to draw, it will be easier to draw, so here is the point p and this point p has a position vector r. Now imagine rotating this vector r by some amount so we do an orthogonal transformation and we take the point p from this location to some other location.

Now, if I do so then the components of this vector r which I denote by x, y and z let me put it as a column these numbers will change to something else so the vector r will go vector r prime, the new vector which you get after rotating this by whatever rotation you wish to do and the new components will be x prime, y prime and z prime and these numbers x prime, y prime and z prime they are related to x, y and z in a specific form.

So, if this rotation is affected by a matrix a, let me use a small a as the symbol for the matrix which is going to take us from here to there then I can say that my let us say r prime is a times r, r is the column vector, r prime is a column vector and a is the matrix. Now I want to slightly improve the notation and I will write instead xi prime is equal to aij xj so now x1 is x, x2 is y and x3 is z and aij denotes the components of the matrix a.

So, the rule which, so this is the rule or the law which takes which describes the rotation of the vector r. Now, I already know that this is something, this is a vector quantity, the vector r because we define it we already know what we mean by a vector. Now, what I am saying is that any quantity which is a set of 3 numbers, let us say you are given some quantity let me see what I have used as a symbol, let us say some quantity a which is really a set of 3 numbers.

Let us call it a1, a2 and a3. Now if I do a rotation and these 3 numbers a1, a2 and a3 they makes among themselves under rotation in exactly the same way as these components have mixed, these for the position vector then I say that a is the vector, let me say again. If the rule of mixing of these numbers is same as the rule for mixing of these number under a rotation then I say a is a vector.

So, for me the position vector is a prototype and anything which transforms in the same way would be a vector for me. Now, just hold on for a second, okay that is good. Now, this means that by definition the transformation law of rule for a under rotations would be the following. So, if this is the rule for position vector which is the prototype then the rule for is would be this, the same rule exactly the same rule.

See here what is says, it says that x1 prime is a11 x1 plus a12 x2 plus a13 x3 and whatever these matrices are these numbers are the same things will appear here. So, this will be a11 a1 plus a12 a2 plus a13 a3, so that is the definition of a vector for us. Now, let us ask why for example distance between two points is very useful quantity. Now, it is very useful quantity because that is something which does not change upon rotation. So, if you do a rotation of your coordinate system or if you take the vector itself and rotate it the length is not going to change, so let us see here.

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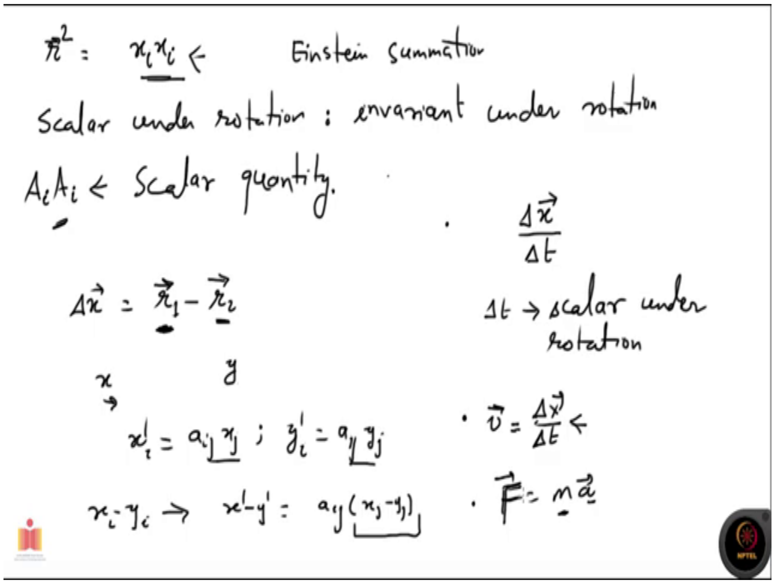

So, I am saying if you have vector r and you take a square of it which is in the notation of x will be xi, xi there is a summation over I implied so Einstein summation that imply here so if you look at this quantity this does not change under rotation such quantities are called scalars. So, these are scalars under rotations, so this quantity is a scalar under rotation. Now, you have seen already a vector, some vector we said we can define.

Now, I ask what this quantity would be; well now this quantity would also be scalar. You see it has to be scalar because the prototype is becoming a scalar, so if you have a xi, xi then the prototype tells you if you prototype vector tells you if you put two copies of the same thing and you sum over all the indices it is going to be scalar then this guy has no other option but to be scalar because the transformation law is identical.

So, this will also be scalar, if your a was a vector and you also say that if something is scalar you say it is invariant under rotation. You should relies that it does not matter what the vector ai really is it could be physically whatever quantity it is, I am making no assumption about the physical content of the vector a. No matter what it is if you take its components and arrange them in this manner meaning you put a11 a1 plus a2, a2 plus a3 this has to be a scalar.

And this is fine, okay that is good. Now, suppose you are given some vector how can you create some new vectors, okay I will give you one example. Let us say you are given a position vector or you already know for example delta x, the displacement is a vectorial quantity under rotations.

Now, why is this a vectorial quantity? Because rj sorry position vector r is vector quantity, so let me call let us define delta x to be r1 minus r2 whatever those things are, position vectors of two points.

Then the difference will also be position, will also be a vector quantity; why? Because this guy is going to transform as a vector that guy is also going to transform as a vector and the difference will also transform as a vector, so let us see why; suppose r1 I denote by x, r2 let me denote by y then under rotation this transforms as x prime is equal to aij xj this transforms as y prime will be aij yj there should be index I here.

So, clearly if I am looking at x minus y then this quantity will go to x prime minus y prime and these are individually these numbers these quantities and you can just pull out the factor aij and you get xj minus yj, so you see this piece is transforming exactly the way our vector transforms. So, we have shown that the difference of two vectors is also vector and similarly you can show that sum of two vectors is also vector.

And same way you can show that if you multiply a scalar to a vector it will transform as a vector because the scalar is not going to change at all, so it is only the vector which is going to bring aij and the rule will be the same as that of a vector transformation. So that is nice, so it is clear like if you want to produce some new vectors given from a vector you can add certain vectors if you let us say two vectors given to you subtract them or may be divide by some scalars so let us see if I am given a vector which is difference of position of two points.

Position of let us say point which is moving at two different times then I know that delta x is a vector. Now, if I divide it by the time interval, now what this quantity is? If I divide by the time interval the time interval does not change when you do a rotation, rotating the coordinates system or rotating those vectors it is not going to change the time interval that has passed. So, delta t the time interval is a scalar quantity under rotation.

If so then the argument I gave couple of minutes ago makes it evident that this ratio delta x over delta t has to be vector quantity and that is why your velocity is a vectorial quantity, now you can again take acceleration and the same argument you can use to argue that acceleration would be a vector quantity. Similarly, you can talk about force, look at force, what is force? Force is mass times acceleration.

So from the once you have established that acceleration is a vector quantity just like I did for a velocity then if you multiply acceleration with mass and mass is going to be scalar, mass is not going to change when you do a rotation, something 1 kg remains 1 kg remains so force will be a vectorial quantity, it will be vector. So that is how you can see that why these, all these quantities are vectors because they all behave in the same manner under rotations. Okay that is good, now let me talk little more about tensors.

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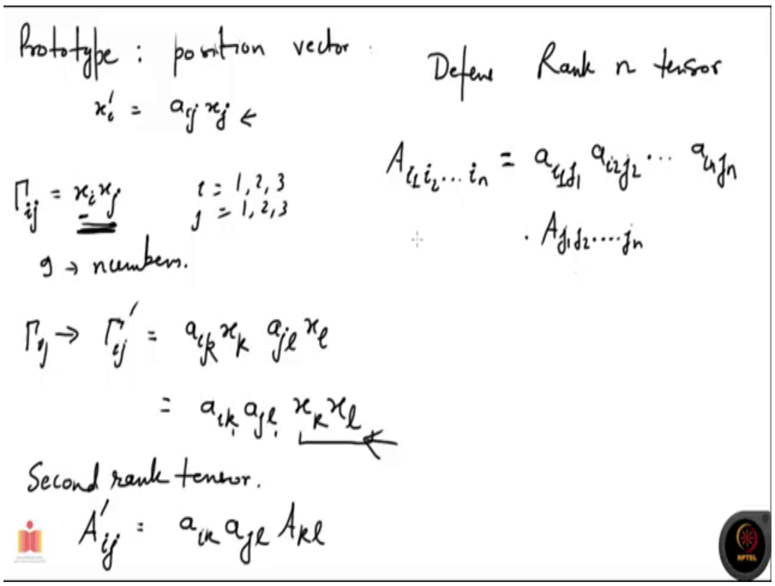

So, I talked about the prototype and the prototype vector was the position vector, what happened? position vector, now I said this is the law of transformation for a vector under rotations where a is an orthogonal matrix and these are the components. Now let us look at the following quantity. If I take xi xj let me give it a name just to make it look nice gamma aj so they are free indices on the right hand side so I should have two free indices on the left.

Now, how many numbers are these? These are a total of 9 numbers because I runs from 1, 2, 3; so I takes value 1, 2, 3; j takes values 1, 2, 3 so 3 times 3 is 9, so this is 9 numbers basically these are 9 numbers. You can think of them being arranged in a matrix, so you can think of it as a 3 cross 3 matrix. Now how do these 9 numbers or these 9 entries transform under rotation? Where it is fairly easy to figure out how the they transform because after all both of these are vectorial quantities and we know how they transform.

So, the transformation law for gamma aj is very easy to tell, so gamma ij will go to let us call gamma prime ij after rotation and what that would be? That would be the following, xi would go to xi prime and xj would go to xj prime and xi prime is aij let me put k here xk, xj prime would be ajl xl, so what is the transformation rule for gamma? It is aik ajl xk xl, so that is the transformation rule for this quantity.

Now I say any set of 9 numbers, any set if those numbers mixed under a rotation in exactly the same way as these 9 numbers are rotating, are mixing then I will say that, that quantity is a rank, is a second rank tensor, all I am saying is I have a rule here which tells me or I have a law here which tells me how these quantities transform under rotation and any quantity which will transform under rotation in the same manner I will call it to be a second rank tensor.

And the second rank because they are two indices that is the reason for using the word second rank. So, that defines for me a second rank tensor. So, as before it does not matter what that quantity physically represents if the rule of its transformation is like this it is a second rank tensor, so let me write a second rank tensor aij and let us say this guy is getting rotated I mean there is a rotation and we ask how aij transforms; it transforms as the following; aij aik ajl akl.

Here the k and l indices were contracted with these two, the same thing here k and l are contracted with these two. So that is the transformation law for second rank tensor and I can generalize it and define a rank and tensor, so by rank and tensor I mean a quantity which will have 9 indices let me denote it by ai1, ai2, sorry this is, let me remove it, ai2 ain so there are n such indices on this quantity.

So, these are 3 times 3 time so 4 3 so n times you have to multiply 3 those many elements this quantity has and the transformation what will be the transformation law for this? It will be very easy to guess because what we will do is we will put 9 n number of such pieces so xi1, xi2, xi3 so and so forth xin and the manner in which that quantity is going to transform I will take that as the definition of the transformation law for this rank and tensor and you can immediately see what that would be.

It would be ai1 j1 ai2 j2 ain jn times aj1 j2 jn so that is the transformation law for a second rank sorry rank and tensor, okay that is good let us now move on to a few (how do I go; what happened?) okay yes, so now let us look at some special tensors for the case of rotations which we are considering and all these are Cartesian tensors, let me write it down.

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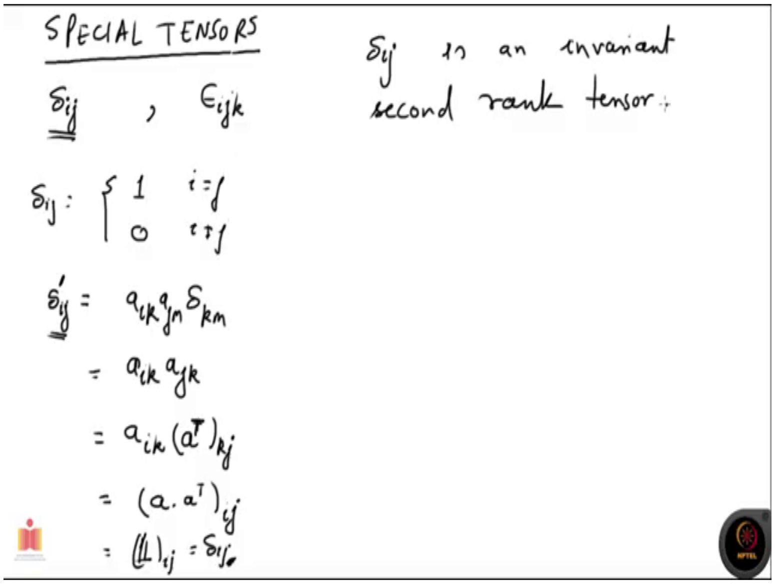

Some special and these ones which I am going to tell you they are very-very nice tensors. So first one is delta ij, that you are familiar with and the second one is also epsilon ijk I am sure you are familiar with that too. So, let us first look at delta ij, delta ij is define to be plus 1 if I is equal to j that you know already, it is 0, if I naught equal to j. So now if I a saying the delta ij is tensor then I already imply how it is going to transform under rotations which is the following.

So, let us say I am given delta ij and I do a rotation then it will become something else, it will become some new set of numbers delta prime and what will be the transformation rule? So, you have started from delta ij it will give you aik aj m delta k m so that is a transformation rule for this, okay that is nice. Now look at this it is aik aj m delta k m so you have m summed over this will force the j to become sorry this m to become k, so it will become ajk.

Now, this is let me do it slowly and carefully for you aik this is a transpose kj, all I have done is interchanged these two indices and I can interchange provided I instead of a I write a transpose because when you transpose the indices get interchanged or the entries get interchange. Now, what is this, this is just the product rule of two matrices, so this what you have in this line is a times a transpose and you are looking at I jth element of this matrix.

Now what is a times a transpose? You see we are talking about rotations, so the matrix a is an orthogonal matrix which means a times a transpose is identity, so what you have here is identity matrix and you are looking at I jth element of that which is just delta ij which means this quantity is same as that or delta ij after transformation has remained delta ij meaning delta 11 which was 1 still is 1 after the rotation, delta 12 which was 0 has not become something else it still remains 0 that is what is says. So, this is an invariant tensor, so delta ij is an invariant second rank tensor under rotations and this is why it is also very useful because this guy is not going to change.

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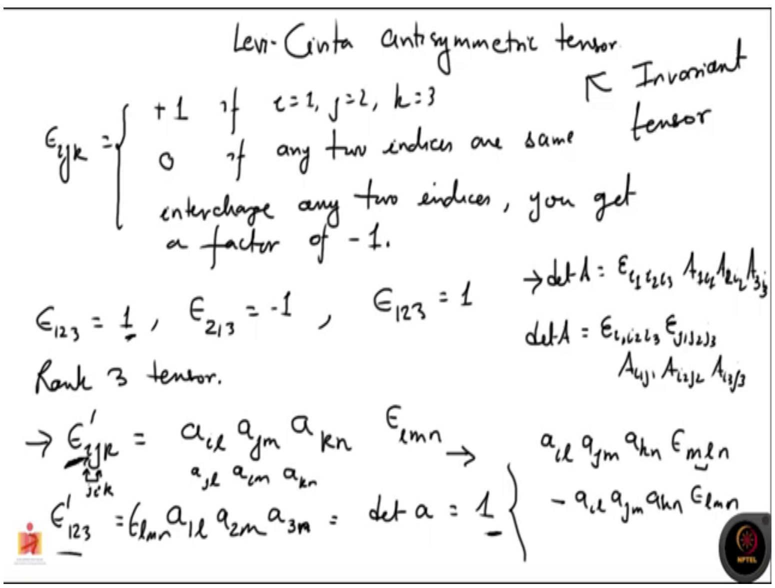

Let us look at another important very useful tensor which is lavish with a fully anti-symmetric tensor so here is the definition of it, epsilon ijk now this quantity is plus 1 if I is 1, j is 2 and k is 3, it is 0 if any two indices are same also if you interchange any two indices for example if you interchange I with j you get a factor of minus 1, so let us see if I have epsilon 1, 2, 3 that is by definition 1 this is the first point here.

If you have epsilon, if I interchange 1 and 2 then you get 2 1 3 that is minus 1 because as I said you get a minus 1 then if I interchange again let us say 2 and 1 so I go back there but it becomes 1 because you again pick up a minus 1, so from here you go back to the same thing. Now this is a tensor of rank 3 because there are 3 indices let me write it down and it is fully anti-symmetric under interchange of its indices.

Now, if I ask how does these, how these numbers epsilon 1, 2, 3; epsilon 2, 1, 3 and blah-blah they transform under rotation, they transform under rotations in the following manner. So, for each index you have to just write the rule of vector transformation that is what I will do, so how does a trans, index I trans form; index I trans forms as not j I should use something else, let me use l, ail then you have epsilon l, how does the index j transform; it transform as ij m.

How does the index k transforms? It transforms as this ijk l m n that is good, okay that is transformation rule for epsilon. Now I hope you are already aware of this I will write it down here on the side you know determinant of any matrix, not any matrix it is a square matrix can be written as this, so you can write this as epsilon i1 up in so let us say for our k we have 3 cross 3 matrix is only, so epsilon i1, i2, i3 then you have a1i1, a2i2, a3i3 and determinant of a matrix can also be written.

So here you have 1, 2 and 3 written and 1 epsilon I can write it using 2 epsilons and instead of having 1, 2, 3 you will have indices which will get contracted between the two epsilons, so you can also write it as epsilon i1, i2, i3 epsilon j1, j2, j3 then you have ai1j1, ai2j2, ai3j3 so you can use this and what we were discussing here, so let us see here what is epsilon prime 1, 2, 3. So I have done a rotation I want to know what is epsilon prime 1, 2, 3 what is the (())(31:55).

So, you have epsilon a1l a2m a3n and you have epsilon l m n, this is same what you have here these 3 indices on the epsilon are contracted with these 3 factors of a. So here epsilon 3 indices these are contracted with the indices on these a which means this is just determinant of a. Now you know already that we are talking about rotations which means the determinant of a is 1 which means epsilon prime 1, 2, 3 is same as epsilon 1, 2, 3.

Remember epsilon 1, 2, 3 was 1so even after rotation the corresponding first this component has remained unchanged that is nice. Now let us see what happens when I interchange any two indices here, let us say I interchange these two. So, let us see that see when I interchange these two it will become epsilon jik, so here you will have Ajlaim akn en epsilon lmn let me write down here, so if I interchange j and k you get jik.

So, epsilon prime jik and here you will have ajl aim akn and this will be contracted with epsilon lmn these 3 indices will be summed over with these ones. Now, you can see lmn all these are dummy they are summed over, so you can just interchange lmn here or sorry rename lmn so what I am calling as l I will call m and what I a m calling as m will be l so that will give you ail because m will become l then, it will be ajm akn which will become exactly what you have on top line here but then here epsilon will become epsilon l sorry epsilon m l n.

And you can let me write down so after doing this interchange you will get ail as I said just now ajm akn epsilon mln. Now I can switch these two but doing so will give me a minus sign, so I will get minus ail ajm akn epsilon lmn which is same as what you have here in this line except for the minus sign which means if I interchange two indices in here I pick up a minus sign, so this tensor which you have epsilon prime is also fully anti-symmetric tensor.

If this is fully anti-symmetric tensor and I have already shown that epsilon prime 1, 2, 3 is 1 then it is clear that this is also epsilon ijk, so the lavish with tensor is also an invariant tensor under rotations and that is why it is going to be very useful. So, I have shown that this is also an invariant tensor I have only talked about rotations I have not talked about inversion so I am not but you can go and read about why it is a pseudo tensor but for me it is fine all I have deal with is rotations so I just call it tensor and not make a qualification that it is a pseudo tensor.

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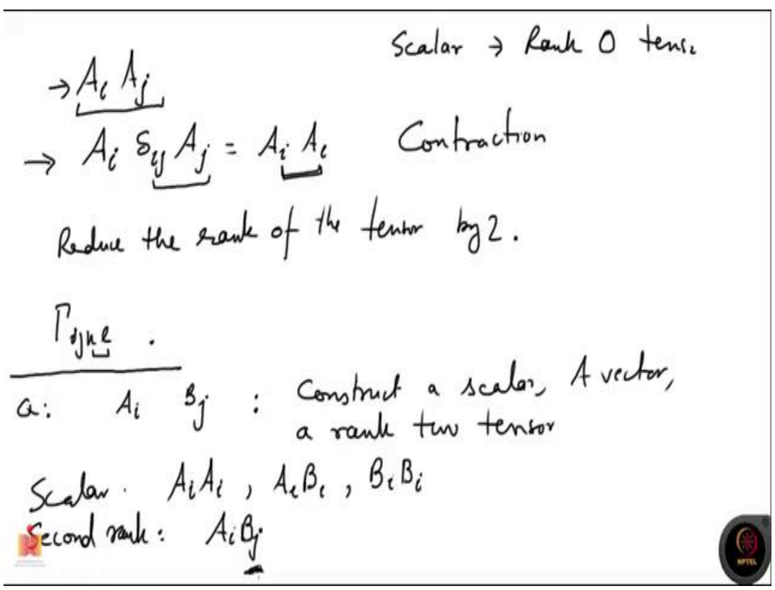

Okay, another thing so let us start with a vector Ai and I construct let us say a rank 2 tensor using the vector A and I define Aij. Now, what I can do is I can contract these two indices meaning I can make the two same which I can do like this. So, I have inserted a delta here and because all these indices are yes, because all these indices are repeated they are summed over and what you have is basically this, it is Ai and if you look at this what is that?

This is Ai again, so this is ofcourse you already aware that dot product of two vectors it is a scalar quantity, this is called, this procedure of starting with something with the rank 2 for example or whatever rank and reducing or contract or identifying two indices like here so you start from I and j and you end up with I and I this procedure is called contraction.

So, when you have contracted made a contraction you have reduced the rank of the tensor by 2. Every time you are going to contract two indices that index will disappear you see there is no index I really because this is a summed over this is dummy, these had two free indices I and j these has no index the I is dummy. So, two indices have disappeared meaning your rank of the tensor has decreased by 2.

So, in this case your rank 2 tensor has turned into rank 0 tensor I should have told you already that a scalar is also called a rank 0 tensor. So, if you had something like with several indices let us say gamma I j k l whatever and if you contract these two then from 4 rank you go to 2 rank, rank 2. So that is what contraction does, that is fine that is good let us ask one simple question now.

Suppose I give you two vectors; Ai and Bj and I ask you to construct a scalar quantity new vector quantity you already have Ai and Bj but something new and also rank 2 tensor things like that, so I am asking given 2 vectors Ai and Bj vectors A and B construct a scalar, a vector rank 2 tensor and so on. So, let us see what you will do; well scalar is easy we have already seen so scalar would be simple I will just contract the two indices so I will get Ai let me there are several possibilities so I will have Ai I can also do Ai Bi I can also make Bi Bi.

So, we can make several scalars, how about a vector? Ofcourse you have A and B as vectors something else can we do okay let us return to that in a moment but about second rank tensor that is easy you can put Ai Bj that is a rank 2 tensor. Now, if I want to have a scalar I will sorry if I want to have a vector quantity I have to have only one free index because the vector carries only one free index.

Now, here you see I have something which has 2 indices, if I could contract these two indices with something carrying 3 indices then you will be left with 1 index because 3 plus 2 is 5 and if you contract these two indices so you will be reducing each time by 2, so 5 minus 2 times 2 is 1 and it will be a vector.

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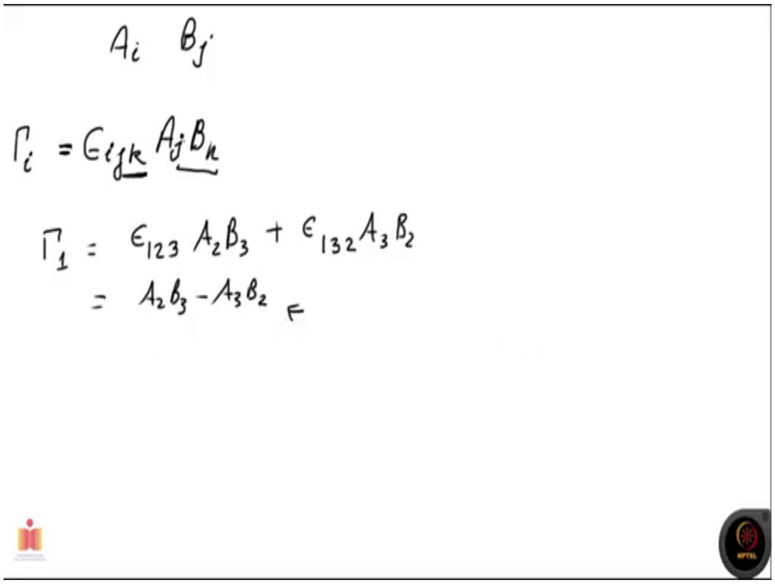

So let us see, let us see like this I am given Ai Bj I construct a rank 2 tensor Ai Bj I always have epsilon lavish with tensor at my disposal and I contract this. Let me use slightly different indices let me write Ak j A sorry Aj Bk that is what I use so as I was saying I have rank 2 tensor I can contract this with Ijk so these two will be gone, these will dummy and since summation convention will be enforce and you are left with the quantity which has only one index I and because the way we have built up everything each index transforms as a vector.

So, whatever you get here will have one index and that index has to transform as a vector. So, it is clear that this whatever this is here is a vector quantity, you see there is nothing else that can happen, it cannot transform in some other way it has to transform like a vector, I am not worrying zero vector here, it is only purely rotations. So, this is a vector quantity a new vector quantity which I have constructed. Now, let us see what this really is, let us look at what is gamma 1, so the first component.

It is epsilon 1, 2, 3 A2 B3 then I am summing over j and k so I should have epsilon 1 3 2 A3 B2, epsilon 1 2 3 is 1 so you get A2 B3 this you get by permuting this ones so you get a negative sign and you get A3 B2 and you recognize that this is just the first component of the vector product of A and B. And you are already aware that a vector product is a vector. And here you already can see that this is a special thing which is happening in 3 dimension because you have your vectors are 3 dimensional the same thing will not happen if you are in dimension higher than 3.

Okay, so I think this is roughly all I want to talk about tensors in this video and if you need to understand more and please refer to some book, I think this is sufficient for the purpose will meet in another video.