Introduction to Classical Mechanics
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Lecture 56
Calculus of variation: Several Variables

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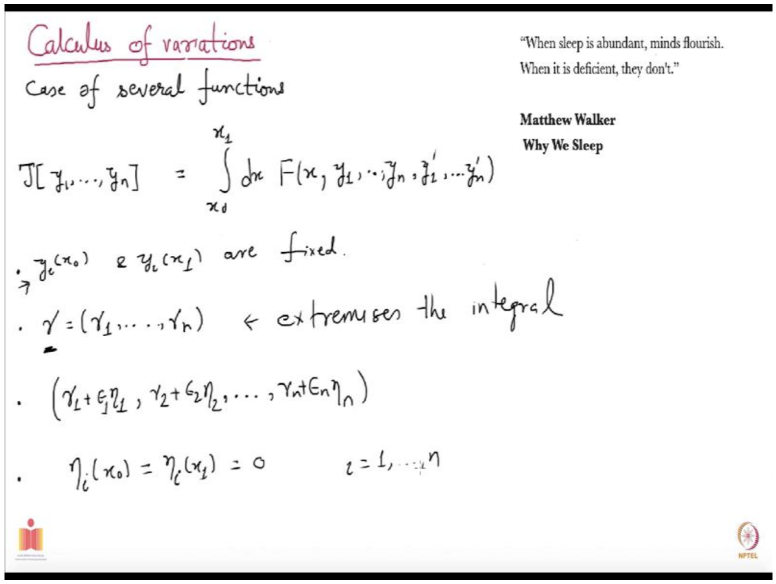

Let us, look at the case of several functions. So, imagine that you have a function F which in addition to the independent variable x depends on several functions which are denoted by y1 to yn and then it also depends on the first derivatives of these functions, it is solid, so yn prime, so it is a generalization of the case which we looked at last time where we had only one function y which was unknown and we still have our integration of, this is the integral we are looking at and we want to extremize this.

So, let us call it y1 the up to yn, so that is the functional J we have and I put square brackets to emphasize that this is a functional. As before I also fix the values of the function at x naught and x1, so all these values are fixed, let me say that yi x naught and yi x1 are all fixed. And again as before we assume that a curve gamma which now is denoted by gamma 1 to gamma n, so these are all parameterised by time.

So, you can imagine for example a circle a circle for a circle you can write x is equal to a Cos t, y is equal to a Sin t. So, that is the kind of parametrisation I am talking about. So, all this gamma 1 to gamma n are equivalent to x and y for example, if it was only two dimensional gamma 1 gamma 2 and time parameterize is this, so you get a curve in in this n dimensional space.

So, suppose this is the curve that extremizes the integral, now again I will parameterize our curves that are in the neighbourhood of gamma, the neighbouring curves of gamma as following so I have gamma 1 plus epsilon eta 1 and eta 1 would be any good well behaved function, just like in the previous case, gamma 2 plus epsilon 2 eta 2 gamma n epsilon plus epsilon n eta n, these are our new bring curves.

And again epsilons are the small parameters which will control how much you deviate from the curve gamma and also because of because of this condition that the curve has fixed values at x naught and x1, I required to impose condition that all the eta i's should vanish at the boundary, which is also the same thing as you saw in the previous case.

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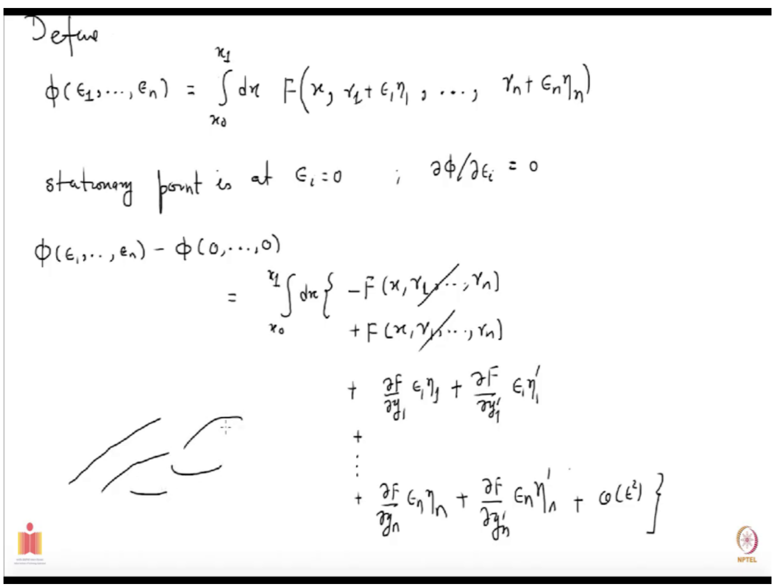

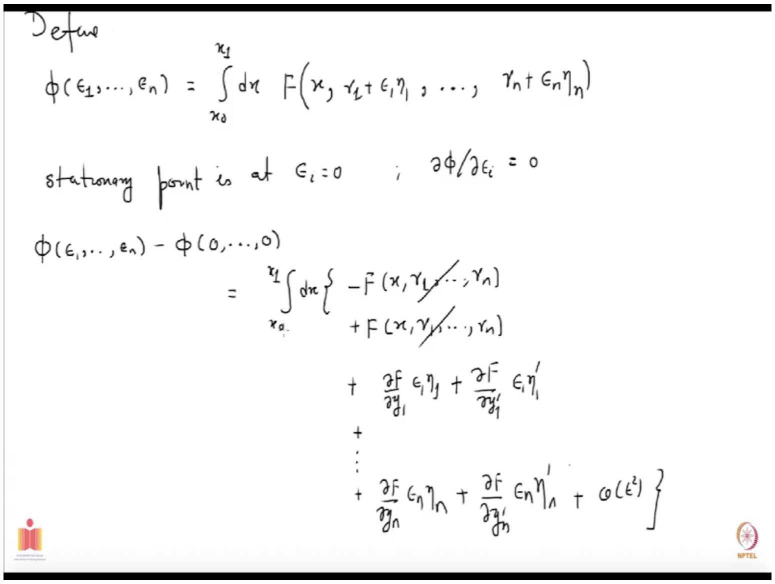

And as before I will define a function phi to be a function of all these variables epsilon 1 to epsilon n and this will be our dx x naught to x1 and then we have our function F, let me use a big round bracket and that is the independent variable and then we have gamma 1 plus epsilon 1 eta 1, so and so forth to gamma n plus epsilon n eta n.

And up by our construction the stationary point is located at epsilon 1 all the epsilons to be 0, so stationary point is at all the epsilons are 0, that is if I look at delta phi over delta epsilon i they all should vanish, they all should vanish, so it may happen that for some of the epsilons phi is taking a maximum and for some of the epsilons it is taking minimum and for some others some saddles points, so these are all possible there.

Now, again as before I will look at the derivative of phi which I can find by looking at this difference and this is equal to integral x naught to x1 dx let me write down the first minus 5 0 0, so minus the 5 0 0 would be just F of x gamma 1 to gamma n. so, let me write down that term first F of x gamma 1 to gamma n, that is good.

Now, I look at the phi epsilon and I will do a Taylor expansion about epsilon equal to 0, which will give us plus phi x gamma 1 gamma n, so these two terms will cancel but let me continue with the higher order terms and you will have delta F over delta y1 times epsilon 1 eta 1 and let me also write delta F over delta y1 prime epsilon 1 eta 1 prime, delta F over delta yn epsilon n eta n plus delta F over delta yn prime epsilon n eta n prime plus higher order terms.

So, clearly these two cancel, I notice that there are pixels it is snot as earlier days I do not know why, anyhow and again I will use integration by parts and pull out the derivative from eta prime. So, this eta prime is delta eta 1 over sorry d eta 1 over dx and I will put the derivative on this one and you will pick up a minus sign if you recall and then you will have a boundary term and that boundary term is going to vanish because your etas are 0 at the boundaries. So, that is what I am going to do and what you will have is the following.

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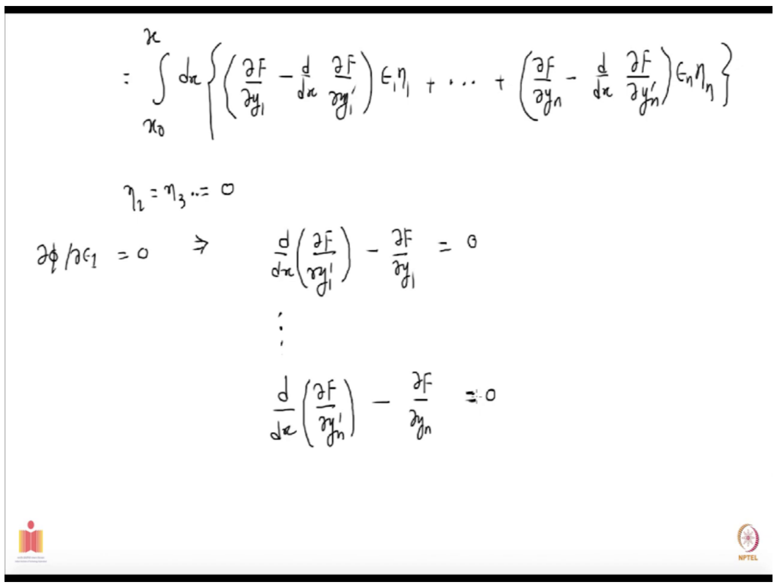

This is same as integral x naught to x dx and delta F over delta y1 minus d over dx delta F over delta y1 prime and this thing this entire thing is multiplied with epsilon 1 eta 1 plus delta F over delta yn minus d over dx delta F over delta yn prime epsilon n eta n. So, that is what you get plus the boundary terms which I have dropped and all the higher order terms have also been dropped.

So, let me put first eta 2, eta 3 and all of them to be 0 except for the eta 1, if I do so, then I am left with only this term and I can conclude that because eta 1 is arbitrary that by fundamental Lemma of calculus of variation that this piece in the square brackets has to vanish, so just as before I get the condition that d over dx delta F over delta y1 prime minus delta F over delta y1 is equal to 0.

And similarly you can put eta 2 to eta 2 be eta 2 to be non-vanishing and all other etas to be 0 and again you can find delta phi over delta epsilon 2 and you will conclude the same thing. So, continuing that way you get a set of equations for each variable, so you get this, these are your Euler Lagrange equations.

So, in general for calculus of variations when you are looking these equations are called Euler Lagrange equations, we called Euler Lagrange the case of classical mechanics, but in general they are called Euler Lagrange equations. So, clearly you can see that if you are given a classical system a mechanical system which has which is described by a set of generalized coordinates q1 to qn, then if you choose F to be the Lagrangian, then you get the same equations of motion as you got before in the in the very first few lectures of the of the course.

So, it means that we can derive the equations of motion for a mechanical system from an variation principle and that principle is called Hamilton's principle. Let me write it down.

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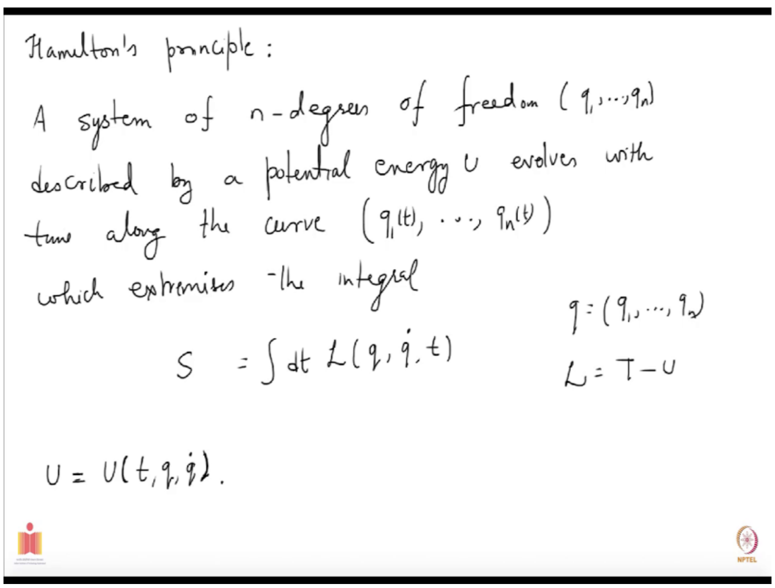

$$= \int_{x_0}^{x} dx \left\{ \left(\frac{\partial F}{\partial y_1} - \frac{d}{dx}\frac{\partial F}{\partial y_1'}\right)\epsilon_1\eta_1 + \cdots + \left(\frac{\partial F}{\partial y_n} - \frac{d}{dx}\frac{\partial F}{\partial y_n'}\right)\epsilon_n\eta_n \right\}$$

$$\eta_2 = \eta_3 \cdots = 0$$

$$\partial\phi/\partial\epsilon_1 = 0 \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right) - \frac{\partial F}{\partial y_1} = 0$$

$$\vdots$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y_n'}\right) - \frac{\partial F}{\partial y_n} = 0$$

So, a system of not looking nice, a system of n degrees of freedom, so which means that if I am denoting them by q1 to qn these are your y's, if you look here these y1 y2 yn are now called q1 q2 so and so forth to qn. So, a system of n degrees of freedom described by a potential u, potential energy u, I will specify the arguments a little later u evolves with time along the curve by the curve I mean the curve in the q1 q2 qn space along the curve q1 of t so and so forth qn of t which extremizes the integral dt.

So, t is in the place of x your x was the independent variable there and here is the t which is independent variable and then instead of f you have the lagrangian, q q dot and t, where q represents up to qn and L is given by T minus u. And this quantity is called action and denoted by s that is a standard notation.

Also note that in this derivation here we assume that the F was a function of T sorry x y's and y dots y primes, so here L will be a function of q q dot and T and if I allow u the internal energy to be a function of t q and q dots generalized velocities the duration will still go through, so I will I mean we can allow for u to depend on generalized velocities also and it can also explicitly dependent time and this will still go through.

So, this is one generalization that is possible. We will in the next video we will look at the system which have constraints that are non-holonomic, meaning I cannot write down a function, I cannot write down a set of functions relating the different variables are different coordinates and I cannot eliminate and write down the lagrangian using only independent coordinates, because the constraints I am going to take will be non-holonomic and we will see what kind of equations we get in. those cases, that will be the topic for next video.