Introduction to Classical Mechanics
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Lecture 54
Method of Lagrange Multipliers

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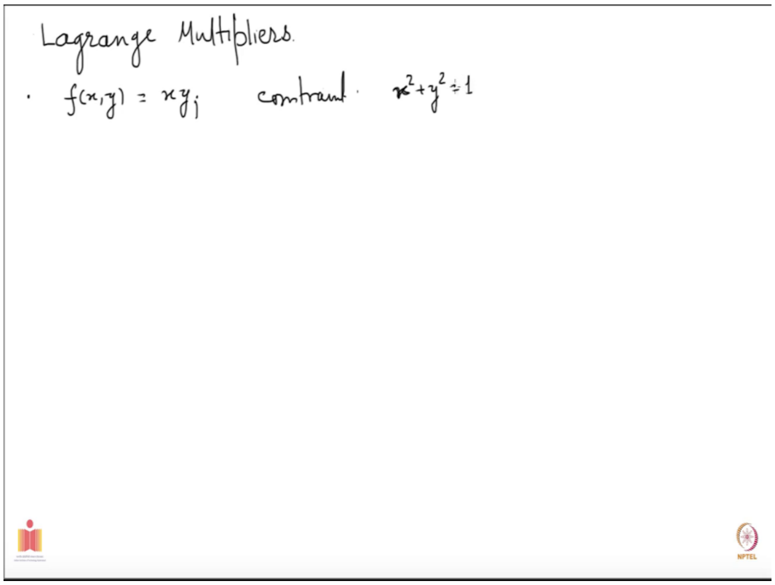

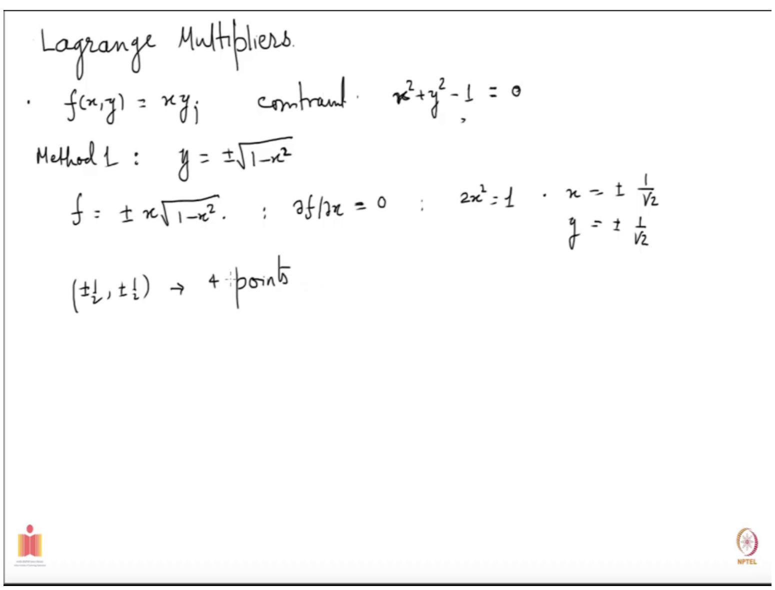

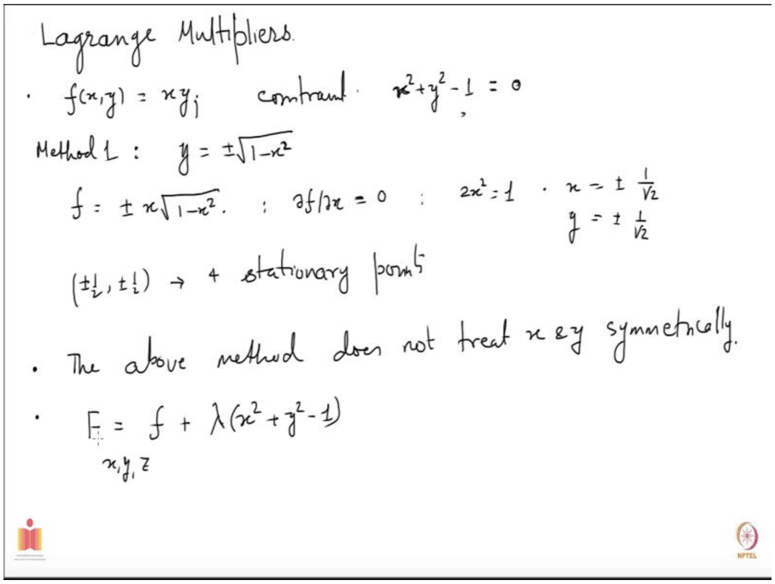

In the last video I was talking about the method of Lagrange multipliers and in this video I want to give you an example of it, so let us say I am given a function f of just function of two variables x and y and let us say the function is x times y and I want to also impose a constraint on the values that x and y can take, so let us say the constraint is that all the points I mean the domain is that of a unit circle.

So x square plus y square is equal to 1. So, let us say this is the constraint or I can put it in this form minus 1 is equal to 0, the same thing. So, if you look at the function xy it has its extremum points, you can find them out, but if you are in addition to the function given a constraint and you are asked to find the extremum points, of course you can do one thing you can eliminate y from using this relation and substitute in here.

So, you can let us say method one. So, I can write down y as 1 minus x square the whole thing in the square root and of course you have two possibilities plus and minus of this thing and with this you can write your function f to be x x times 1 minus x square, that is what you can do. And then you can try to now this is a function of one variable and now you can try to find out where this derivative is 0.

And if you do so you will find that this should satisfy 2x square equal to 1 which means that x can be plus minus 1 over square root of 2 and which corresponds to y also taking plus minus 1 over square root of 2. So, this way you can solve it and now you see that your, you get 4 points where the extremum will occur or can occur and these points are plus minus half, plus minus half.

So, for plus half you have two possibilities and for minus half you have two possibilities, so four points in total. 4 extremum points let u say or stationary points. Now, I do not wish to do it this way for one reason that in this treatment I have not treated x and y on the same footing, see I have eliminated one variable the variable y and have worked only with the variable x.

So, the treatment is not symmetric with respect to both the variables, so let me write it down the above method which is what you would will naively do which is correct, the above method does not treat x and y in a symmetric fraction. But then we saw the last time in the last video that we can have a method which will treat x and y symmetrically and the method was that you construct a function F capital F which is your original F which are looking who's extremum you are looking for, plus you take your constraint equation which is sorry x square plus y square minus 1 and multiply with the Lagrange multiplier.

And now you treat this F as a function of three variables x y and z. And now there are no constraints, these are our equations, this is our function and I should look at the partial sorry why did I write z, x y and lambda. And I should now look at the stationary points of capital F with respect to x y and lambda. So, as if there are no constraints. So, that is what I should do and let us do it this way now.

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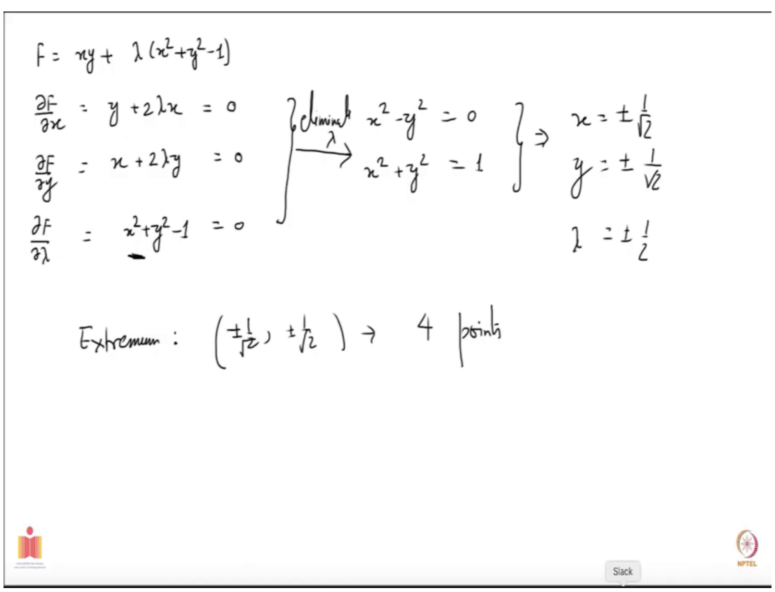

So, I have my F to be xy plus lambda x square plus y square minus 1, so we should calculate delta F over delta x which will be let me just so you will get y from the first term plus 2 lambda x from the second term and I want to equate this to be 0 and then similarly for all other variables delta F over delta y which will be x plus 2 lambda y equal to 0 and then for del F over del lambda.

And what is that? x square plus y square minus 1 is equal to 0. So, this is your original equation of constraint that you recover here. Now, I have got three questions in three variables and I should solve it and you can solve it to get that x square minus y square is equal to 0 that is one equation you will get, you will be able to reduce it to this, x square plus y square is equal to 1 that is already a constraint equation, this you will these two equations will obtained by eliminating lambda.

So, you eliminate lambda and that is what you get. And then these imply that your x is equal to plus or minus half, sorry plus or minus 1 over square root 2 and your y is also plus or minus 1 over square root 2 and lambda will come out to be plus minus half, please check the calculation that everything is correct, but one thing you can clearly see that again you have got the same result as before.

So, you have extremum or stationary points, we can be sure about that this the these are stationary points, one may has one may need to do more work to claim it those points to extremum, but anyway the stationary points are located here. So, again there are 4 extremum points 4 points. So, there are 4 such places on this unit circle, where this function will be stationary. And that is the method how you use the Lagrange multipliers, let me I have made a small one line program in Mathematica to show you this example, let me try to do this, somewhere here.

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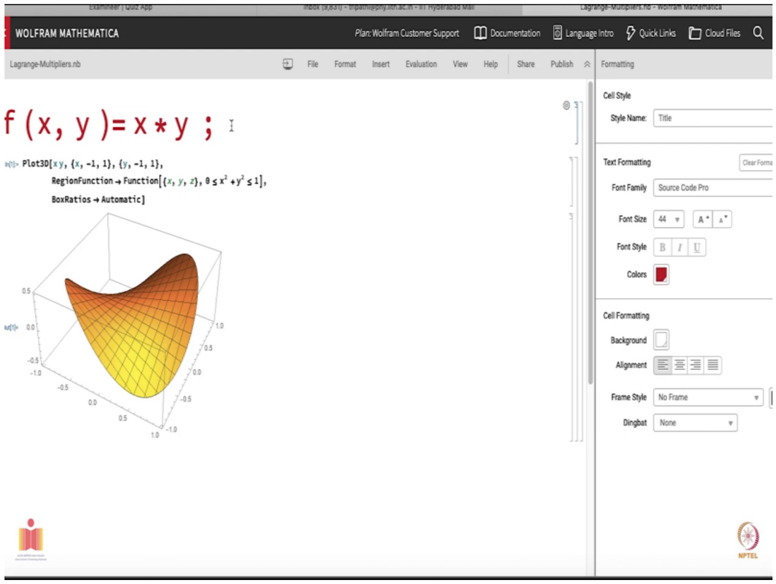

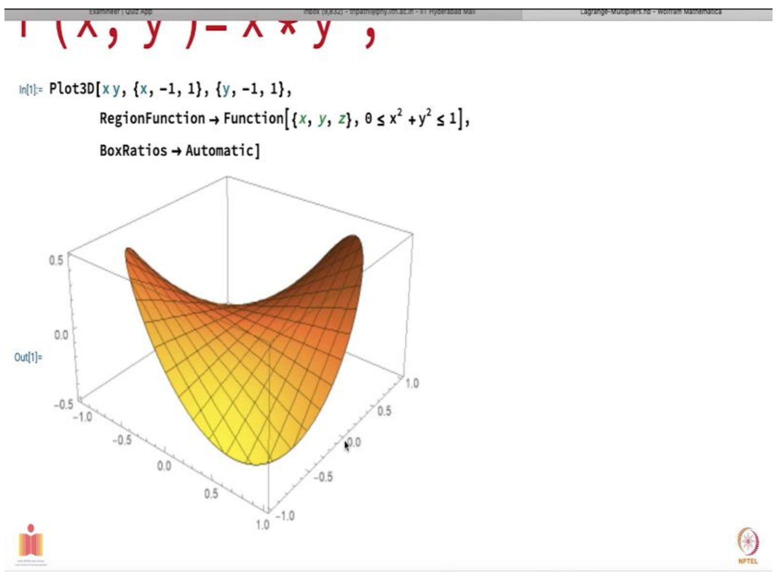

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          RegionFunction → Function[{x, y, z}, 0 ≤ x^2 + y^2 ≤ 1],
          BoxRatios → Automatic]
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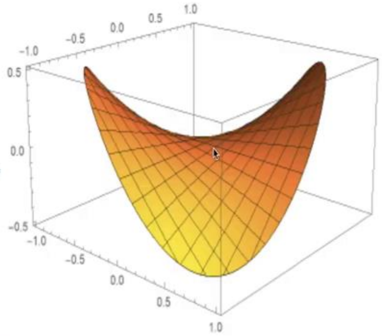
Out[1]=

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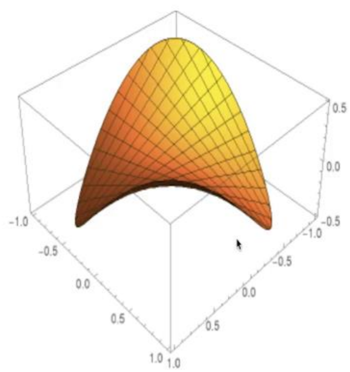

f (x, y)= x * y ,

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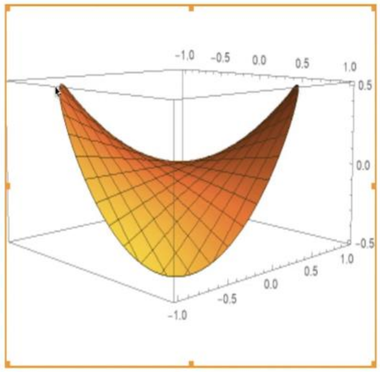

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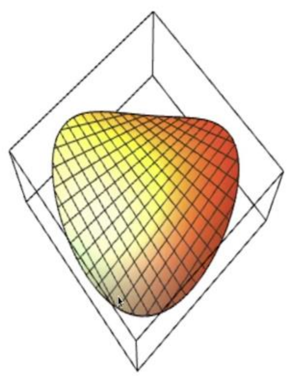

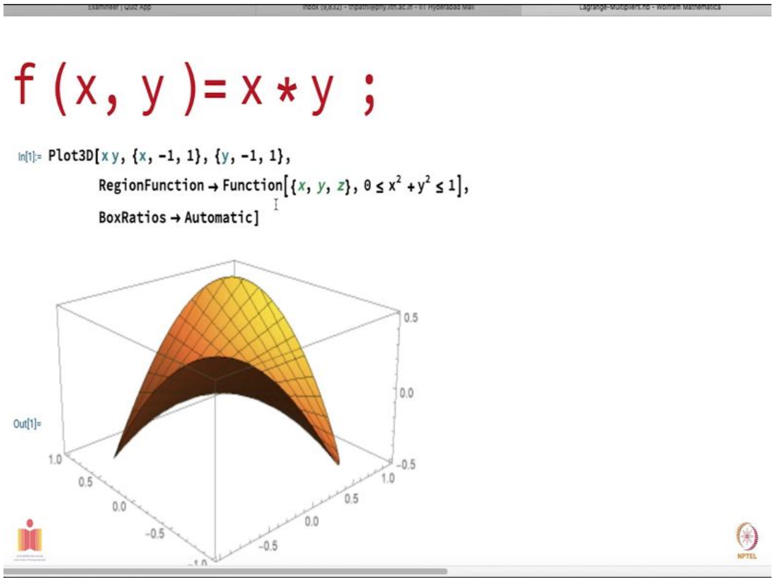

So, you can see it in your, this is nice let me so you see I have plotted x times y here and I am doing really nothing, I am just plotting, this is a comment here in the red, just the title, now I do a 3D plot, I hope you can see it, if not, let me do this. So, I have done a 3D plot, plot 3D of the function x times y and the range is from minus 1 to n for x and minus 1 to 1 for y. And I am also defining region function, so the plot will be limited to this region.

So, what is the region it? It is that x square plus y square should be less than or equal to 1 and it should be greater than 0, this was not necessary I think, but the lower limit was not necessary. So, this is what you get, let us see, this is what you get, so these are it should show you, you see these are your axis x and y axis, so origin is here, so our genesis is somewhere in the middle of this entire curve entire thing.

So, you see if you if there were no constraints origin is the place where you get a saddle point, so that is the saddle point, but now you are given a constraint that x square plus y square is equal to 1, which means I am looking at the extremum of this function at the in the region where all your points lie on the circle and that is why I have put this condition here.

So, this boundary of this this figure or this curve not curve, how do you say? This 3D figure let us say, is x square plus y square is equal to 1, so whatever you are seeing on the boundary all these points of the function correspond to x square plus y square is equal to 1, so they are on the circle and you see clearly that there are four places where this function takes extremum values.

One is here this one displays it becomes a minimum, one is here it becomes a maximum, one is here it becomes maximum and one is behind this, let me try to do this, it is there and the values are all the same, see let us see what the values are. So, remember your x was plus minus 1 over root 2 and y was plus minus 1 over root 2 and the function is x times y, so the values will be 1 over 2 which is 0.5 or minus 1 over 2 and that is what you see here.

You see let us see, this is the this box in which this entire figure is put this goes up 2.5 as you can see here, this place. So, these two points are touching 0.5, the value of the function and here it goes to minus 0.5. So, these two this one and this one has minus 0.5, so this is how you can visualize this and of course if the function has more than two variables then the same thing you will not be able to do easily. Anyway, this was decided more to encourage you to play with Mathematica when you are doing calculations and to verify them. So, let us go back to what we were doing here.

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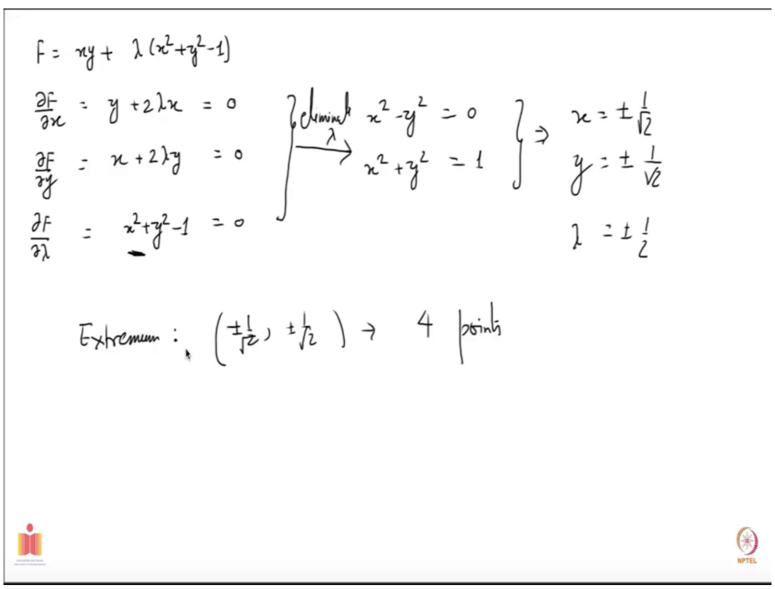

Let me give you one more exercise, this will be I will give you the results but it will be more for you to do as an exercise. So, maybe some colour.

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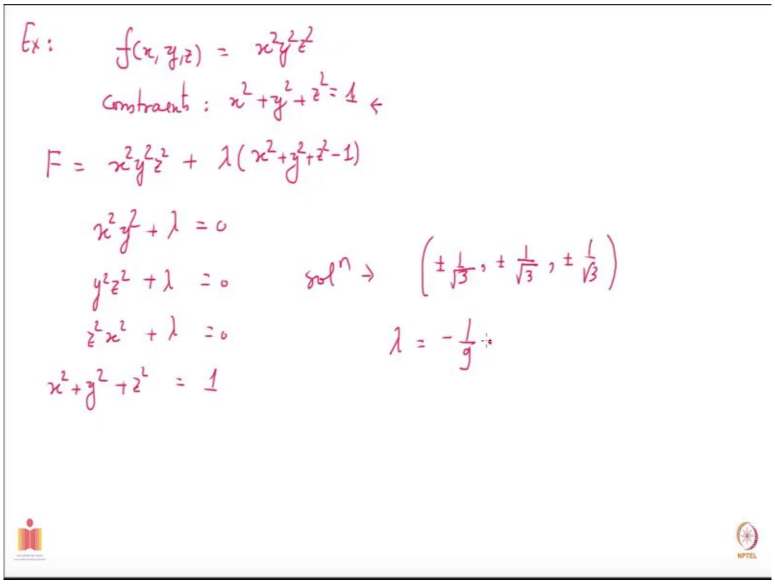

So, I take a function of three variables x square y square z square and I put a constraint that all the points are on x on a unit sphere, so I am looking for the extremum of the function f on the sphere. So, how do you do it? Let us, use the method of Lagrange multipliers I consider the function f x square y square z square and then I put my constraint x square plus y square plus z square minus 1, remember we have to also always bring minus 1 on that side so that you always have a constraint in the form phi equal to 0.

So, that is the phi here and you have a Lagrange multiplier, again put the partial derivatives to 0 and you will get four equations, in four variables, x square plus y square plus lambda equal to 0, y square times z square plus lambda is equal to 0, z Square x square plus lambda is equal to 0 and your original equation of constraint y square plus z square is equal to 1.

And you can solve these and you will find the solution to be the following, your x will be allowed to have two such values and same for y and same thing for z. So, these are all the places where you are function keep function f small f will take its extremum values, I mean together with the constraints allowed constraints and you will also find that the Lagrange multiplier is going to come out to be minus 1 over 9. I think these two examples are sufficient to appreciate the method of Lagrange multipliers and we will start with calculus of variations in the next video.