Introduction to Classical Mechanics
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Lecture 52
Rotating Frames, Euler Equations

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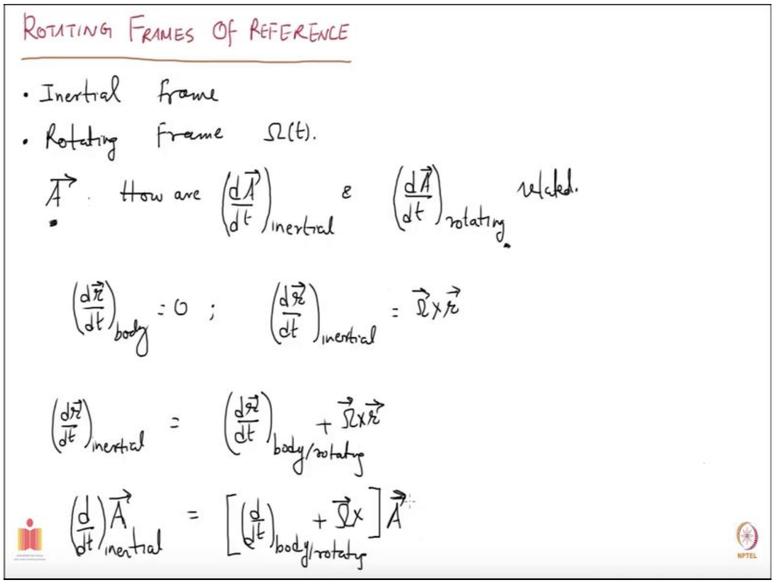

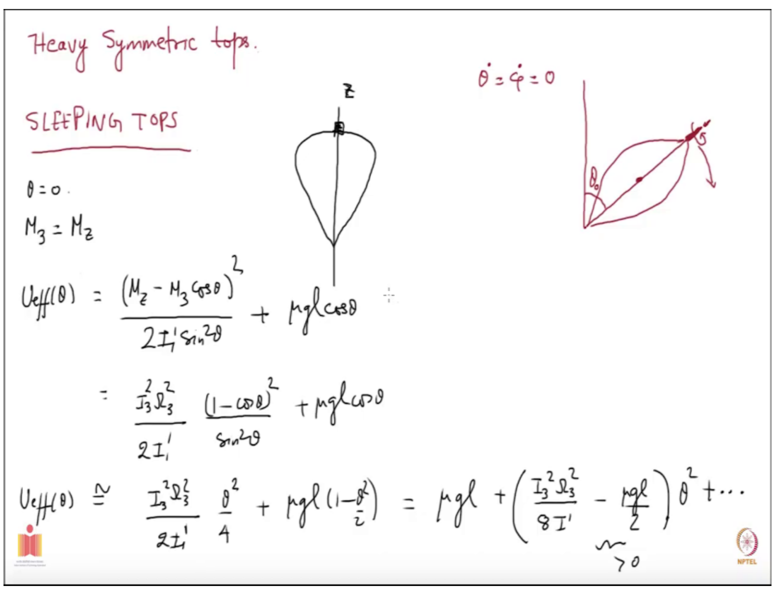

Today, I would like to write down the equations of motion which are called Euler equations of motion, they will be telling us how the angular velocities for example change with time. But before I do so, let us just pause for a minute and try to recollect or realize what we have used in figuring out the motion of all these tops that we talked about.

In all the discussion that we have had so far, never use an equations of motion. All we used was, if you recall only the conservation principles, so we were using like energy conservation or angular momentum conservation not or and also we, if you recollect wrote down our first integrals by just looking at which of the coordinates are cyclic.

So, these were the ingredients in our analysis. We never used any equation of motion in analysis till now, but today I want to write down an equation of motion, which we will use to give one example. Hopefully I will do that. So, but before I do that I should talk a bit about rotating frames of reference and how time derivatives are related into. So, the title is rotating frames of reference. So, what we do is we imagine one frame, which is inertial another frame, which is rotating with respect to this inertial frame, rotating frame, let me write frame.

You may imagine that the origin of inertial frame and origin of rotating frame are coinciding at all the time, so they are at the same place and the rotating frame is rotating with respect to the inertial frame omega, which is the angular velocity and this may depend on time as well. So, omega maybe will be a function of time for us.

And what we want to ask is, if there is a vector given to us and we are looking at its change with time, so if I am looking at the time derivative of A in the inertial frame, how is it related? How what is the relation of this time derivative with the time derivative in rotating frame. So, what we are asking is, if I am given a vector A, how are dA over dt this quantity calculated in the inertial frame and dA over the same vector, now if you are looking at the time derivative in the rotating frame however these related.

So, let me see. Now, how are we going to proceed with this? So, always whenever this question arises or such questions arise, think of a prototype of a vector, so what is the prototype of a vector? What is the basis on which vectors have been defined? The prototype is the position vector. So, imagine a position vector of point, which is let us say not changing with time in the rotating frame.

If that vector is not changing in with time in the rotating frame, it means dA over dt is 0, so you can think of some vector which is as viewed from the inertial frame is moving together with the rotating frame. If it helps to visualize you can imagine a rigid body, where body coordinates have been fixed in it and the body is rotating and you are looking at some point in the rigid body and as far as the motion of the that point any point p in the body is concerned, it is not changing with time if you are looking from the body coordinates.

So, that is the kind of thing I am imagining. So, if you look at a point, which is stationary in the rotating frame, then clearly dr over dt of the point in the body 0. But the same thing is not 0, if it is looked from the inertial frame. So, from the inertial frame it will appear of course to be rotating and you know, it will be this.

Now, what if that point we are looking at, it is not a point in the, it is not a point which is stationary in the rotating frame, but that point itself is moving. So, meaning there is a non-zero rate of change of that r. So, we have a non-zero dr over dt in the body frame and then we asked how it is related to the dr over dt in the inertial frame and obviously the answer is the following you write dr over dt the velocity in the inertial frame is equal to the time derivative in the body frame plus omega cross r.

So, that is how the prototype of a vector transforms or that is how the prototype vector has its time derivative related in the two frames. So, if you are looking at any other vector, it has to obey the same rule because it’s, it has to exactly follow what the prototype does. So, let us say you are vector is generally denoted by capital A here, then we have the relation d over dt calculated in the inertial frame.

It is too big here but, A let me put A, will be equal to d over dt evaluated in the body frame. So, that is a operator that is acting and this entire operator will act on the vector A. So, that is the general relation we have and of course the vector A, I mean vector has its meaning independent of the coordinates, the coordinate system you choose to write its components.

So, it is up to you in which components you want to break it up. So, you may choose to write vector A in either the body coordinates or rotating coordinates I have been calling, initially I call it rotating but then I started calling body, because I was imagining a rigid body which is moving. But anyway, you can write here rotating also if you wish slash rotating. So, up to you whether you want to decompose vector A in the body coordinates or the inertial coordinates or the fixed coordinates that is a separate issue and what this relation is saying is about the time derivative how they are related, that is fairly simple.

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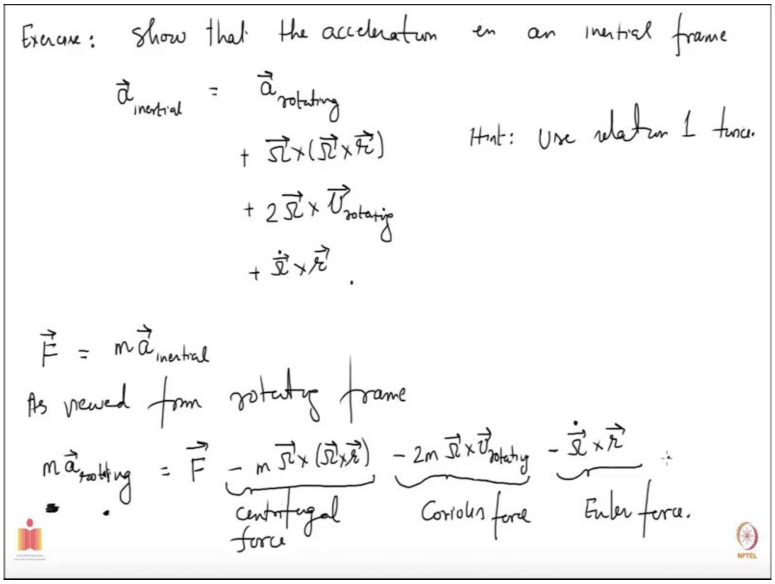

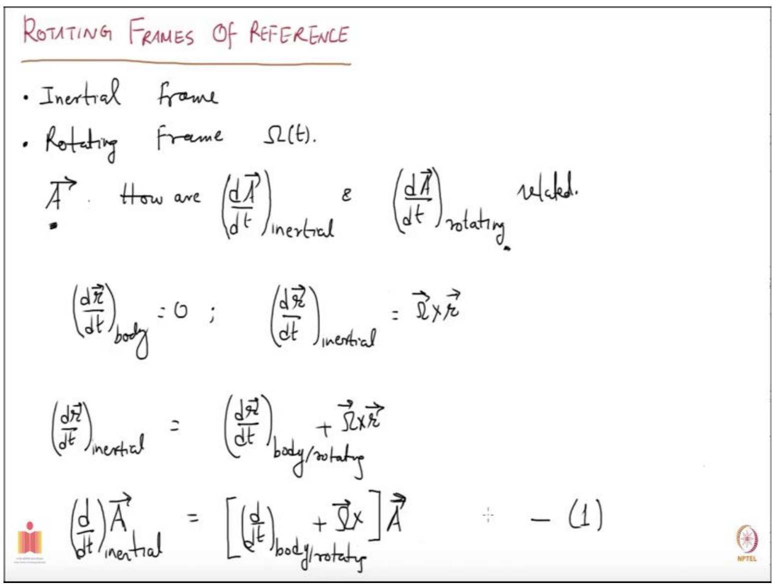

Now, I will give you an exercise to do which is again, not difficult. So, please show that if you are looking at the acceleration in an inertial frame, so if you have some point, which is moving around and you look at the acceleration of it then show that the acceleration of a point in an inertial frame, initial frame which I denote by a inertial that will turn out to be the following.

It will have one term which will equal to the acceleration of that particle in as viewed from the rotating frame plus some more pieces, one is omega cross omega cross r, omega is the angular velocity of the frame and r is the coordinate of the particle which are looking at plus 2 omega cross the velocity of the particle as found in the rotating frame plus omega dot cross r and yes, that is all.

Let us see 1, 2, 3, 4, 1, 2, 3, 4 there are 4 terms. So, this is what you will get and here is a hint to do this, what you should do is go back to this relation which we have derive let me relation number 1, use this twice. So, you take this instead of I mean take the vector a to be vector r, so it will connect your velocities in two frames and then you put the V inertial here and then you will get the acceleration inertial frame and on the right hand side you will get all these terms.

So, please do this simple exercise. So, hint was use the relation 1 twice. So, what does that all mean? It means that if you are looking at for example, some object which is moving around and experiencing some forces, then I know that the forces according to Newton's second law is this, that is the Newton's second law it is says that the forces are mass times the acceleration in the inertial frame.

And do this F you will be able to associate some physical origins like you will be able to say that it is being pulled by earth or it is being pushed around by some electro-magnetic forces. You will have always a physical origin to F, there will be some agent which will be pulling or pushing it around and that is why you will have the inertial acceleration.

But now let us ask how things appear if you view the same object accelerating in a rotating frame of reference. So, let me do this, what I all I have to do is. So, take this here expression here and multiply throughout by the mass of the particle and when I am looking from as viewed from rotating frame I will, there I will be, if I am calculating, if I am looking at the acceleration of that object and I multiplied by mass then I am looking at this quantity R O T A T I.

So, that is what you will see and this will be what m times A, so I multiply throughout and here m times inertial will be the force which is the true force on it. So, I get the force minus all these terms here multiplied with m. So, we will have minus m omega cross omega cross r, then you have minus 2m omega cross v in the rotating frame and then you have why is there a, I think should be minus let us check correct it should be minus omega dot cross r.

So, that is what you will have, is that all correct? It is all correct. So, if you are in a frame, which is rotating and you look at the acceleration of some object, then it will not match with the Newton's law, because whatever sources you will see for the force on the object will not be sufficient to get this piece because as viewed from the rotating frame, they are additional pieces. So, you can call them as fictitious forces because they do not have any physical origin in its not coming from some push or pull of some physical origin, they are fictitious because you are in the rotating frame.

So, you have to use this as your force to arrive at the left hand side and these fictitious forces are called this one is called your centrifugal force. So, this entire quantity. So, all this is from the point of view of the rotating from frame, not from the inertial frame and this is your Coriolis force.

So, it is it has a velocity dependence on this and then there is another piece, if the angular velocity itself is changing with time. If omega dot is 0 meaning angular velocity is constant then this term is absent but in general it will be there and it is called Euler force. So, that is one exercise I think this is fine, very good.

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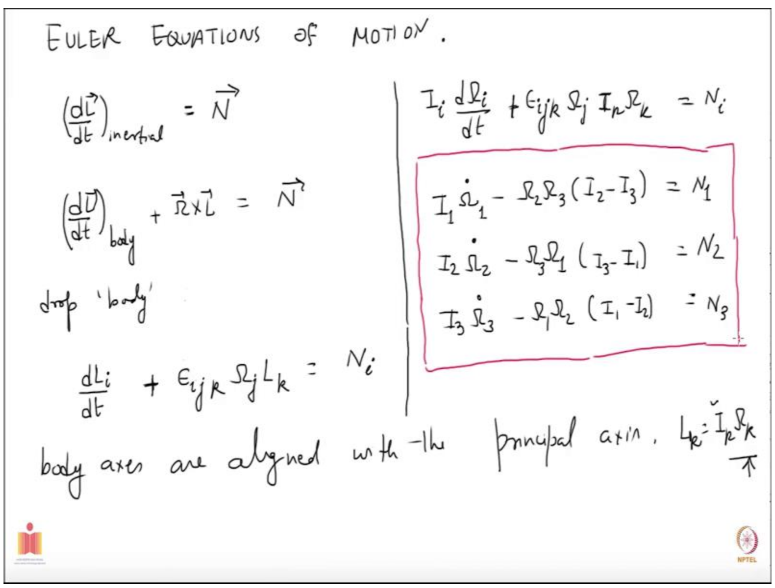

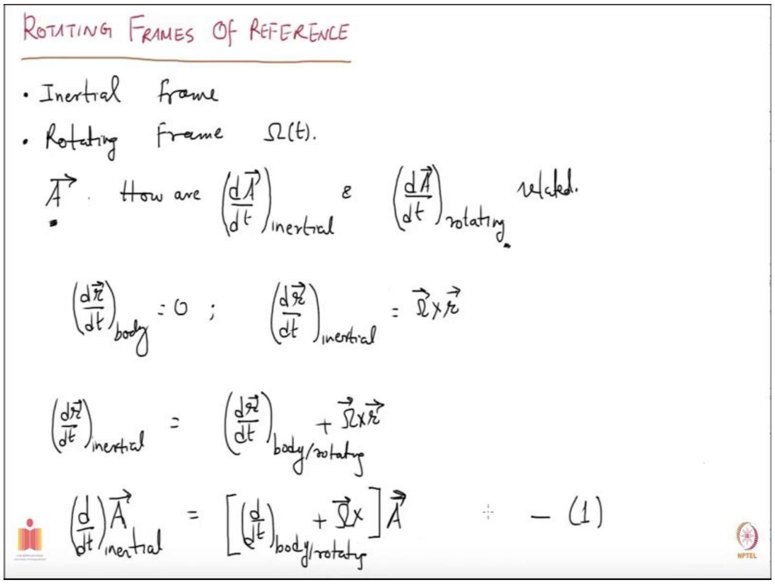

Now, what I want to do next is use this relation one, which we have derived to write down equations of motion for omega, which are called Euler equations of motion, it is not looking nice. Euler's E Q U A equations of motion that we can apply for rigid bodies. So, I am not going to use the Lagrangian formulation, formulation I will start from Newton's equations and we know that the rate of change of angular momentum is equal to the torque which is acting on that body.

So, this is being calculated in the inertial frame and this is the external torque, which is acting on it, which I denote by N and with the help of what we had earlier here this relation. So, now is for A I have used angular momentum, so I can replace it by this operator acting on the angular momentum.

So, I will write it as your N or dL over dt calculated in the body frame plus omega cross L this will be equal to the torque. Now, I will drop the reference to body frame. It will be understood that I am working body frame. So, I drop the drop body and I write down the components of this equation. So, there these are three equations. So, let me look at the ith component of this equation.

So, I get dLi over dt plus what is the cross product it is you know, you can use (())(18:37) to write it, so it is epsilon i j k omega j L k and this will be equal to Ni, is that is correct. Now, I can take the body axis to be aligned with the principal axis. So, let us say body axis are aligned with the principal axis, with the principal axis.

If that is the case, then your L is just i times omega, where i are the principal moments. Otherwise, you have the full relation L k is L k would be Ik L omega L, where you will be dealing with the full tensor. But if you choose the principle axis, then L becomes simpler. So, you have Lk is Ik omega k and remember there is no summation over k here, k is free index look in the left side k free, so there is no submission implied here, even though they are repeated. So, if I substitute this I get let me like here.

So, Lk I substitute here, so I get I of sub index I d omega over dt d omega i over dt plus epsilon i j k omega j Lk would be Ik omega k and this will be equal to Ni. So, these are set of three equations in this term also, there is no summation over I even though they are repeated because you see there is only one index I it is just because of this expression it looks like this.

And here also even though they are the k is appearing three times there is no mistake, which I hope you understand very well because of this, so it is I told you I think at some point of time that repeated indices cannot appear three times. Generally, they will mean it is a mistake but there may be occasions when they can appear. So, there is a summation over k in this, there is a there is no summation over I in this and now I just want to write it for all the three components explicitly.

So, using the anti-symmetry of epsilon i j k you can write down the following. I1 and this will be omega 1 dot plus or minus omega 2 omega 3 I2 minus I3, this is fairly easy to see and you will be able to do it, so I do not have to tell and then if you look at the second component I2 omega 2 dot, then you have omega 3 omega 1 I3 minus I1 is equal to N2 and then you have I3 omega 3 dot minus omega 1 omega 2 I1 minus I2 and this will be N3.

Please try to do the simple algebra and these are called Euler equations of motions, these are called let me put some bracket or something. So, we have obtained Euler equations of motion and we will stop the video here and I will see you in the next video.