Introduction to Classical Mechanics
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Lecture 50
Nutation and Precission of a Heavy Symmetric Top

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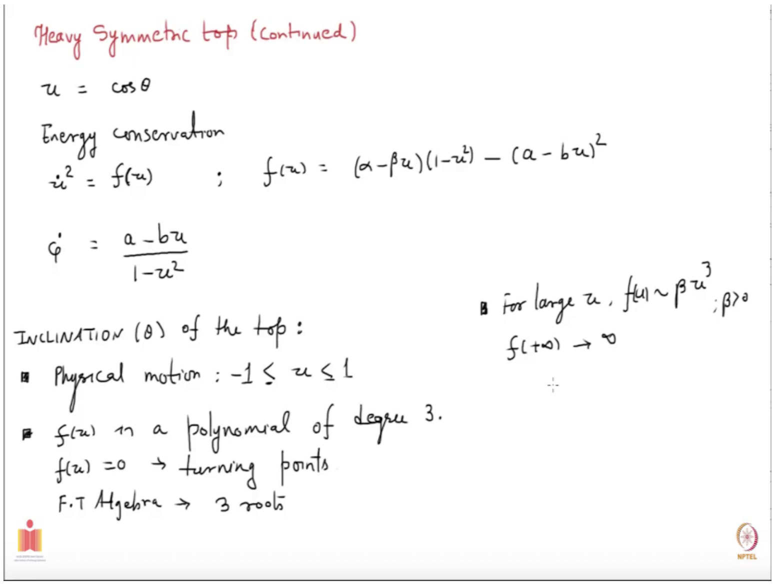

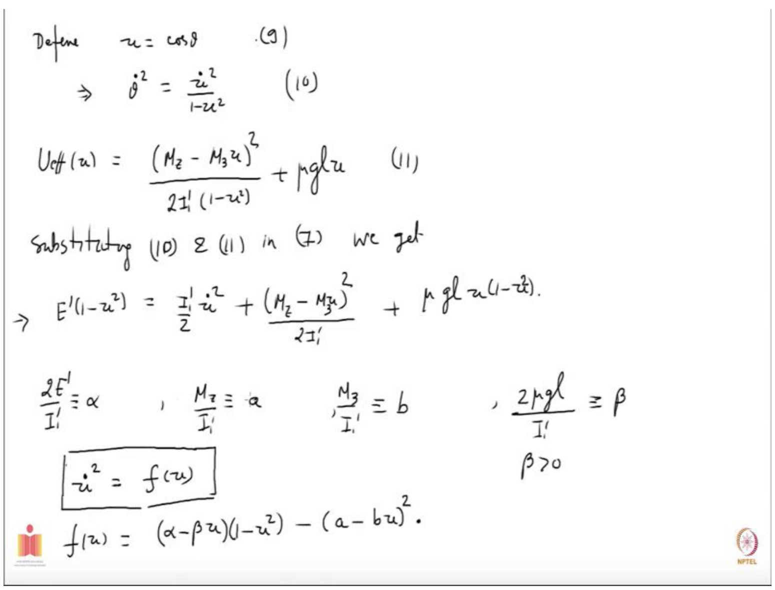

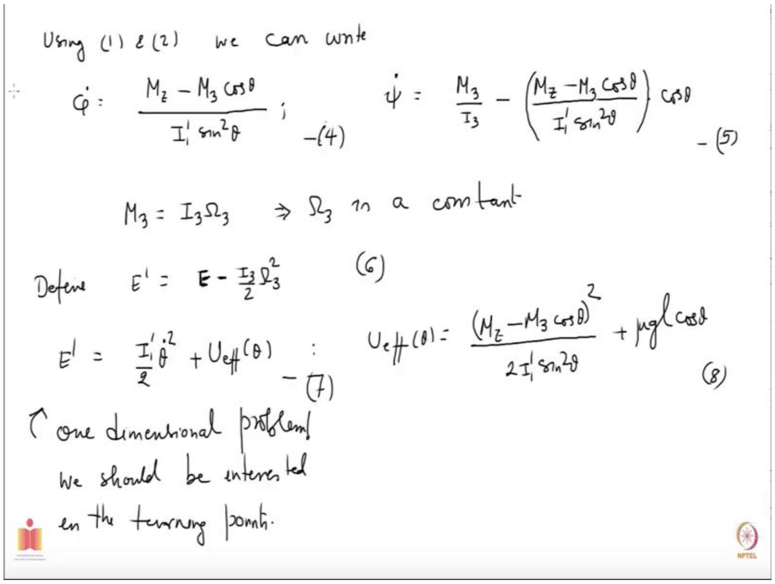

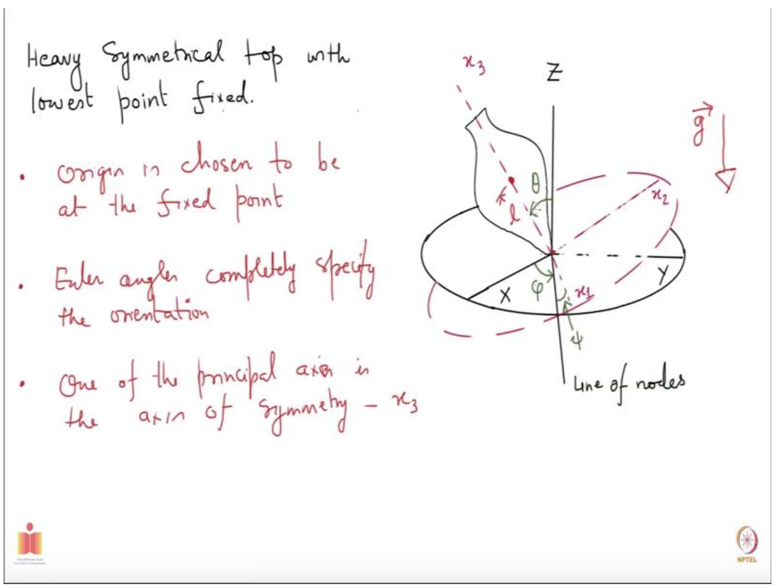

We were talking about a heavy symmetric top last time and this top was supposed to be having a fixed point, which is its lowest point as well and last time we defined a variable u, which was cos of theta, where theta gives you the inclination of the axis of the top with respect to the vertical direction. So, in this problem, you have a direction because of the presence of gravitational force, which we have taken to be uniformly pointing downwards.

So, imagine your top to be being put on the surface of the earth and you are treating the field to be uniform. So, that is what we did last time and that and then with this definition or definition of u, we could write down the energy conservation, conservation as the following equation, so we could convert our energy conservation equation to u dot square is equal to f of u and this was equal into one body, one dimensional problem and here our f of u, we found out to be, we found out it to be alpha minus beta u 1 minus u square minus a minus bu whole square, that is what we found.

Let us, go back and see somewhere here it should be. Now, also you see that beta is positive you have 2 mu g l over I1 prime. So, beta is positive. Let us return, that is good and also let us go back and see that we had written down an equation for I believe phi dot somewhere yes, so we have a phi dot that is what we wrote and this time derivative of phi can again be written down in terms of constants a b alpha and beta, if I plug them in that expression of phi dot, I will get the following, phi dot is equal to a minus bu over 1 minus u square. You can verify this so it is a, (())(03:09) algebra, that is good.

Now let us ask how the inclination of the top changes with time or I mean we will not answer how it really changes in time, but we want to know at least some information about the inclination of top which is encoded in this relation u dot square is equal to f of u. So, let us look at it, inclination theta of the top.

Remember, let me go to the figure here, this is the inclination that is the theta and this is the Z direction. So, right now we are asking about this this angle, very good. So, let us see. So, first of all the physical motion will happen in the range minus 1 to 1 because cos theta will can take values here, let us go back theta could be 0 or theta could be pi.

So, it happens between u equal to 1 and minus 1. So, that is what let us write down. So, point number one is the physical motion it happens in the range minus 121 for the variable u, which corresponds to theta equal to 0 and theta equal to pi, that is one piece of information we have. Now, another thing you note that f of u is a polynomial of degree 3.

So, you see you have a beta u which is multiplying u square which makes it beta cube and that is the highest power you get for u. So, f of u is a polynomial of degree 3, that is good and the turning points you see we remember we are having the same thing as we had in the case of two body problem. So, as far as the equations are concerned it looks the same. So, the turning points meaning the place where u dot will go to 0, is given by f of u equal to 0.

So, the equation f of u equal to 0 gives us the turning points, they are the turning points for in R here the turning points are in theta, which basically tells you what will be the maximum theta and minimum theta which are allowed for this top, what is the minimum inclination it can make and what is the maximum inclination it can make that is good. Now, f of u equal to 0 is a cubic equation and you know from the fundamental theorem of algebra that this will have three roots in general.

So, let me write it down, so your fundamental theorem of algebra tells you that there are three roots for the equation f of u equal to 0. We will try to see whether see the roots are in general complex, it is not guaranteed that your roots will be real. So, one has to and if the roots are not real then it does not directly apply to your problem because your problem is happening in the real space. So, the cos theta or the theta has to be real. So, that is why we need to think a little bit more.

Now, from whatever information we have available here, we will be able to make conclusions which are sufficient for our purpose. So, first thing we note is that may be here, if you look at f of u and take u to be very large then the cubic term will dominant, because that is the highest power you have, so it will be the leading term.

So, for large f of u, for large u f of u behaves as beta u cube and remember that beta is greater than 0. So, it basically blows up in the positive direction. So, which means if you take the limit f plus infinity you get infinity that is one information we have, which is going to be very useful. So, let me go and start drawing a diagram which will make things appear simpler.

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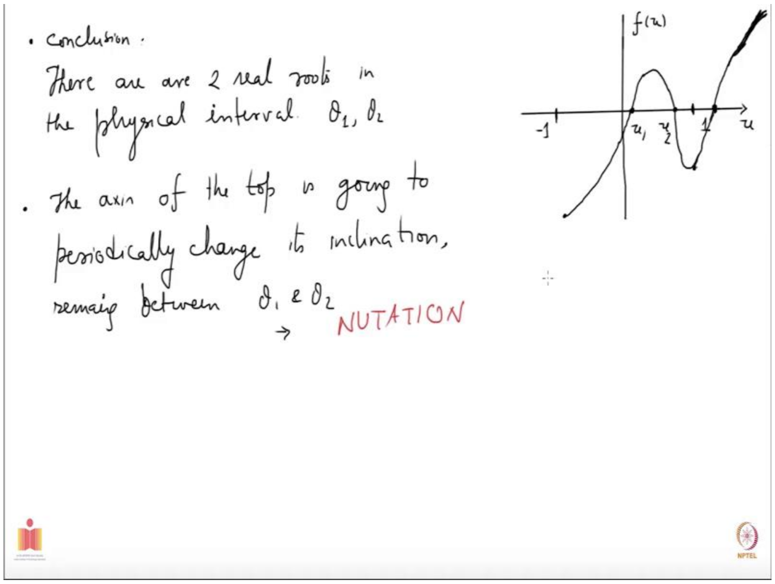

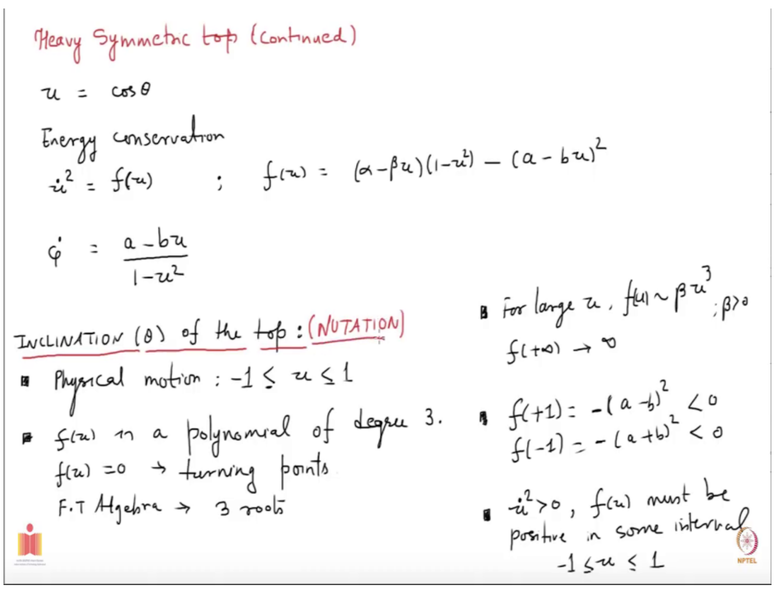

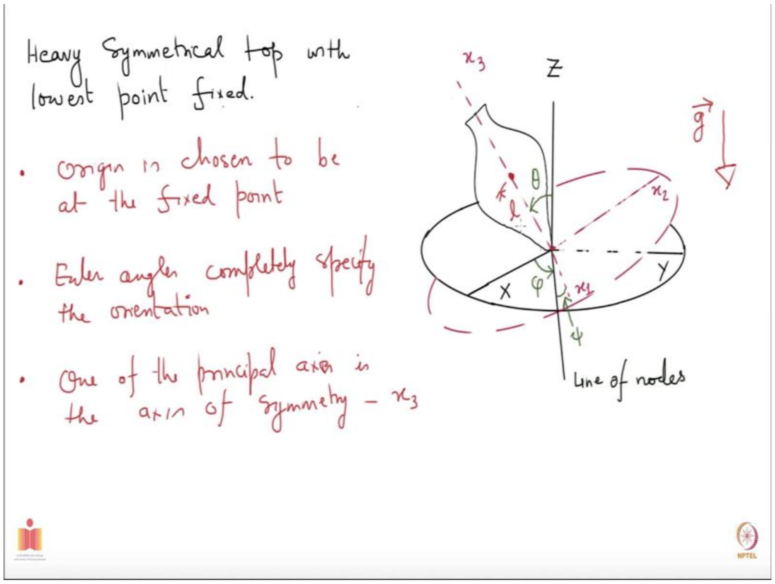

So, here maybe just a second if I take this it will be easy perfect. So, this is my u direction and let me take this here and we will make this as our f. So, Y axis or the vertical axis, I am plotting f of u and here you have u. So, what I have said is for very large values of u as you are going towards this direction, your function is going to blow up.

So, it will be something like this. It is going to blow up. So, this is just generic, do not take seriously this straight line kind of thing, it just has this behaviour. So, that is one thing we have concluded. Second let us evaluate what the values f takes at the boundaries of physical region, the physical region is bounded between minus 1 and 1.

So, in the physical region, I am let the boundaries, if you evaluate f of plus 1, this you see put plus 1 here or even if you put minus 1 because of this term it will become u square is 1 so this will go away will give you 0 which means first term will not contribute and only the second term will contribute and it will be minus of a minus b square which is negative because of this sign and f of minus 1 will also be negative because there is a minus sign, you have a square of two real numbers.

So, in either case you get a negative value that is good. So, let us go back to this figure and draw somewhere here a plus 1 and a minus 1 here. So, what we have concluded is that in this region, let me try again. So, here it should take some negative value. So, let me just put something generically.

Some negative values it will take and I am not really bothered about what the values it takes, that is another thing we have concluded. Now, third thing which is going to really fix everything for us is that because our u dot square is positive, it is a square of real numbers, so it has to be positive. Which means that f of u must be positive let me write down must be positive.

In some interval between minus 1 and 1 must be positive in some interval, interval. Because otherwise, there will be no solution means, otherwise there is no solution possible. So, it has to be positive somewhere. Because u dot square is positive. So, it has to be at least in some interval, maybe not the entire interval from minus 1 to 1 but in some interval it should be positive. Otherwise nothing makes sense. So, which means that in this interval somewhere f of u should become positive, so it should be above the above this horizontal line.

Now, my function is negative here, is goes to infinity here at very large values and then it should definitely come down to negative. Because it has to be negative at u equal to plus 1. So, at this point it has crossed the horizontal axis where f of u equal to 0, so this is first root and this I know will happen because it has to come down to here, but then I know that it has to become positive somewhere in the region.

So, it has to go up and because it has to again go down, because it has to reach this point I expect something of this sort. Which means I have three real roots of this. So, let us call this u1 which corresponds to value theta 1 and this u2 which corresponds to value theta 2. So, what I have concluded is that there are two real roots in the physical interval, see this one is not in the physical interval it is outside the range minus 1 to 1, so in the physical interval that is good and these two will correspond to values theta 1 and theta 2.

Which means that as the top moves the axis of the top is going to periodically change its inclination and it has to periodically move between theta 1 and theta 2. So, let me go back to this figure, which means as the top is moving whatever it is doing this will this axis will keep going down and up down and up between some theta 1 and some theta 2.

So, it will keep going up and down like this in addition to whatever it is going to do which we are going to talk now in more detail that is nice and this motion of the x is going up and down is called nutation. So, let me write it down that part also, so what we have concluded is that the axis of the top is going to periodically change its inclination, inclination remaining between theta 1 and theta 2 always remaining and but it will always remain between theta 1 and theta 2 these are the limits and this is called nutation.

Let me try to get some colour. This is nutation, T A T I O N, perfect. So, now you know three things about the tops one they spin about their axis of course, then you saw even in the case of freely moving top, freely moving symmetric top and freeze in space it was processing. Now, you have come across a new phenomena which is nutation. So, in the case of free top the angle was always fix it was not changing. But here you see that the angle will also change between theta 1 and theta 2 and this is called nutation that is good. Let us, talk more about now the azimuth of the axis.

So we have an equation for phi dot which tells you about the azimuth, let us go here this one. So this is giving you the azimuth of this axis of the top and this is what we want to know now. So, we are not going to solve in detail the entire motion of the top nor we are interested in, I am not so keen on knowing what will be the value of theta 1 and what will be the value of theta 2 given the parameters we describe this top.

I am fairly happy by in knowing that the motion is such that the inclination is going to change with time and it will remain between these two and let us make some study of theta dot now azimuth. So, let me see here inclination theta of the top that is what we did. Let me writehere itself nutation. Now, we go here maybe it to the next page.

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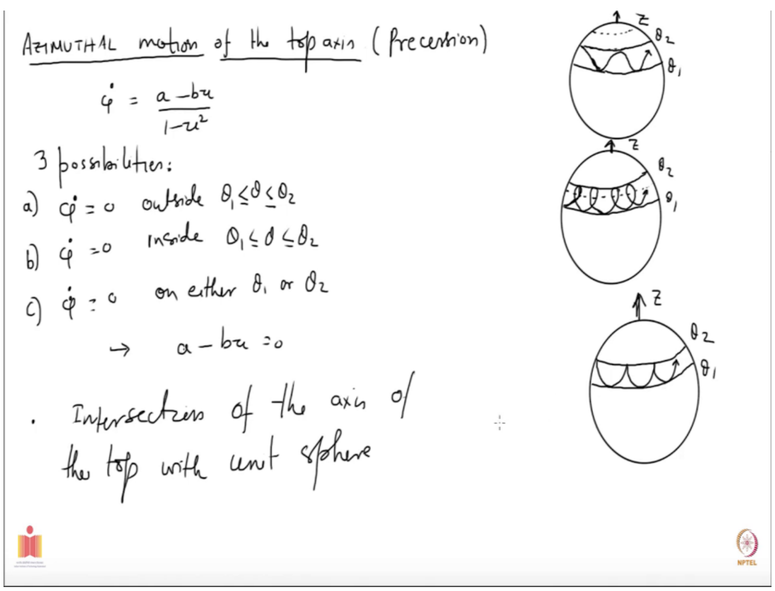
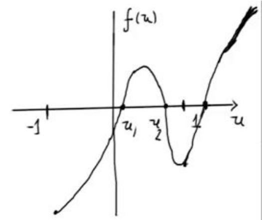


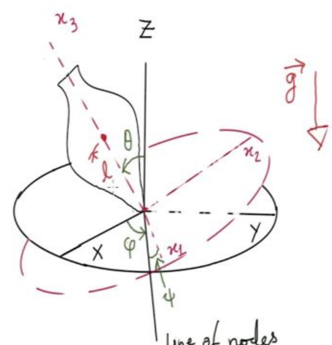
- Conclusion:

  There are are 2 real roots in the physical interval. θ_1, θ_2

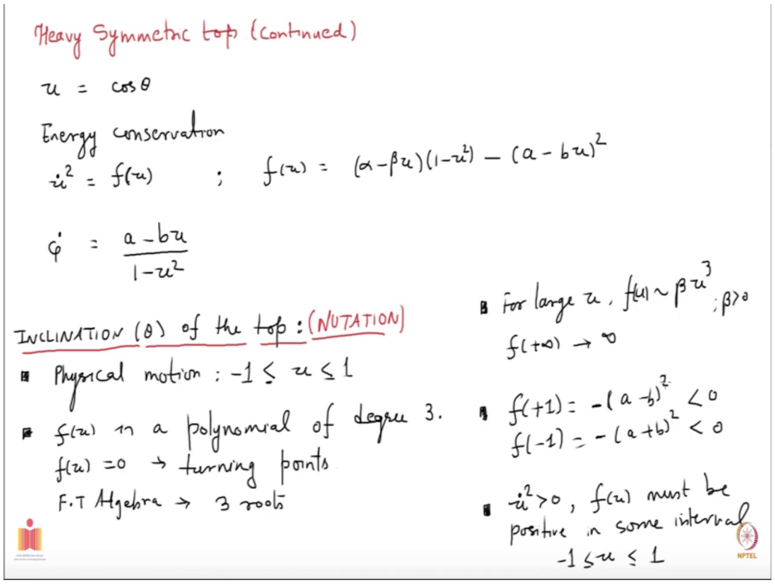


- The axis of the top is going to periodically change its inclination, remaining between θ_1 & θ_2 → NUTATION

Heavy Symmetrical top with lowest point fixed.



- Origin is chosen to be at the fixed point
- Euler angles completely specify the orientation
- One of the principal axes is the axis of symmetry – x_3



So, now we are interested in azimuthal motion of the top, I am just interested in that how its axis is moving and as you already know you have encountered this before this is called precession that was the precession. So, let us look at this or phi dot we have already written down that it is a minus bu over 1 minus u square. Now, there are three possibilities, one is that the phi dot is 0 outside theta 1 and theta 2 interval. So, outside the interval I mean the value of theta or the value of u for which phi dot is 0 lies outside the interval theta 1 theta 2. So, let me write it down, so there are three possibilities.

So, first possibility I have already described phi dot is equal to 0, outside theta 1 and theta 2 and this theta 1 theta 2 is what we talked about in this context in the case of nutation. Now, there is a second possibility that phi dot is equal to 0 inside this range and the third possibility is it is on one of these, phi dot is equal to 0 on either theta 1 or theta 2. So, these are only three possibilities and all these possibilities correspond to a minus bu equal to 0 and these you solve, this equal to 0 and you solve it and your solution gives you one of these possibilities that is what we are talking about.

Now, to geometrically which visualize what is happening, what I will do is, I will draw a unit sphere with the centre at the origin and origin is as you recall at the tip of the top here, so we are imagining that I will draw unit sphere around this taking this as a centre and then what I want to look at is where this x is x3 axis cuts the sphere. So, it will it will pierce through the sphere at some point and what we are asking is how that point moves as the top moves.

So, it will give you a visual picture of how the axis is moving around with time. So, that is what we want to describe now, very good. So, I will do the following may be here itself just a second. So, let me draw a circle, not a circle a sphere yes it is better I do not do a freehand drawing. Otherwise it will be, no not this, a circle and then because there are three cases let me draw three circles, I am not sure whether I will be able to draw them equally of equal radius, but anyhow.

Now, anyway these are all the same circle. So, even if they look different we can pretend they are all same. So, let me so you have a no what is happening. Let us, go here and remove this perfect. So, let us look at the possibility number one that theta dot is outside this interval. So, let me first draw. So, here is your direction of Z because this direction is set by the direction of your gravitational force and then let us say these are your theta 1 and theta 2 values that is good.

Now, let us say my theta which gives phi dot to be 0 is outside this interval. So, the let us say that theta corresponds to something here, theta 1 theta 2. Now, imagine that you are top axis is piercing this thing somewhere and it will be of course within this band of theta 1 and theta 2. Now let us look at possibility one.

So, as time goes on the theta 1 and theta 2, I mean the inclination will keep periodically changing between theta 1 and theta 2 that you already know, just like the case of your two body problem your, the radius factor was changing between its, how did you say that, it is limiting values of minimum R and maximum R turning points.

So, here it will be the same and then phi dot. So, phi will keep changing with time. So, the possibility one, is that it let us start from here. So, phi is increasing. So, this is the place where the axis is piercing this sphere, imaginary sphere. So, here the axis is crossing and then value keeps increasing, keeps increasing.

So, this has this kind of behaviour it keeps does not come out very nicely. So, this is how the tip of the not necessary the tip the axis of the top will carve out a shape on this unit sphere. Now, let us go to the second case where phi dot is 0 somewhere between this region. So, let me first draw this is my z, this is theta 1 and theta 2 region, theta 1 theta 2 and I am saying somewhere here I have a solution of phi dot equal to 0.

So, phi dot goes 0 somewhere here, which means let us say I start like this. So, the particle is the axis is going, it is increasing in its value of phi. So, azimuthal angle is increasing, it is increasing but here when it reaches at this point, the rate becomes 0. So let me do it again it become 0. So, it has to become like this and then because it became 0 after that it has to change sign, so it changes sign.

So, here the derivative is 0 is at this point and that is why you get this slope to be 0. If you view it with respect to theta and then it has to become negative, but then it after a while it touches the boundary and it has to come this, the same thing repeats those slope become 0 at this point then the direction of phi becomes opposite and then it goes again like this.

So, that us the way the top is going to the axis of the top is going to move it will make loops, then the third option is that phi dot is 0 either on theta 1 and theta 2. So, let me draw again, not very nice. Let us, do it again this is theta 1 this is theta 2 and let us suppose that theta 2 is the place on which phi dot is going to 0.

So, let us say the derivative goes to 0 on this on this arc. So, what is the situation now, so your tip will move like this, it returns it goes up. Now, the derivative has to vanish here, but your higher values of theta are not allowed. So, it has to return back and this is what you will get. So, it makes cusps here. So, it continues in this manner, where this was the Z axis.

So, let me write down. So, all these correspond to your a minus bu equal to 0 and what we have done here is, it is the intersection of the axis of the top, top with unit sphere. That is a nice way to visualize what the top is doing. So, this is quite nice.

So, in addition to your spinning about its axis, the top also processes which we saw earlier in the free case and then in this case with gravity being there and the top being heavy we saw that there is also a nutation and if you look at that at axis of the top, it is making this these nice kind motion and there are three possibilities and one of these is going to be realized in your, in your top.

So, I think this is really nice that we can without solving any equations. We have not solved any equation, you see we have not done any quadrature we never tried to solve theta or phi or anything with as a function of time, we did not do any integrals, we did not solve any equations of motion purely based on the first integrals, which we derived or which you obtained by the symmetries.

We looked at some cyclic coordinates and knowing those cyclic coordinates, we could arrive at, rather informative picture of what the top does as it moves around in time and of course, I have not derived what are the values of theta 1 and theta 2 as functions of, as a function of the parameters and all other details. But who cares if we, if we wish we can derive but we are at the moment not so much interested in those details. So it is nice, we will continue a little more on tops in the next video.